Lie groups of Hopf algebra characters

ESI: Higher Structures Emerging from Renormalisation

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Lie groups and combinatorics?

Recently much interest in special Hopf algebras generated by combinatorial objects (e.g. graphs, shuffles, trees etc.)

These combinatorial Hopf algebras appear in ...

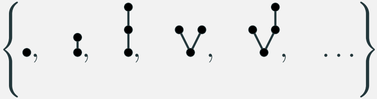
- Numerical analysis (Word series, e.g. Murua and Sanz-Serna)
- Renormalisation of quantum field theories (Connes, Kreimer)
- Control theory (Chen-Fliess series, e.g. Ebrahimi-Fard, Gray)
- Rough Path Theory (Lyons et. al.)
- Renormalisation of SPDEs (M. Hairer, Bruned, Zambotti et al.)

Common theme in these examples

Hopf algebra encodes combinatorics and "dual objects", i.e. **character groups**, carry additional relevant information

Butcher-Connes-Kreimer Hopf algebra

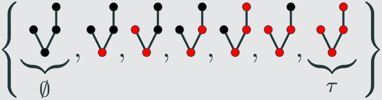
Build a Hopf algebra of rooted trees:

$\mathcal{T} :=$ 

$\mathcal{H} = \mathbb{R}[\mathcal{T}]$ polynomial algebra, graded by $|\tau| := \#$nodes in τ.

Hopf algebra has a dual notion to the product arising from disassembling trees into subtrees.

Subtrees of a tree

$\tau =$  subtrees of $\tau =$ 

(subtree nodes colored red)

Butcher-Connes-Kreimer Hopf algebra II

For a subtree $\sigma \subseteq \tau$ we get

$\tau \setminus \sigma =$ **forest** left after cutting σ from τ

e.g. $\tau \setminus$  $=$ 

Obtain a coproduct Δ turning $\mathcal{H}$ into a graded Hopf algebra.

$$\Delta(\tau) := 1 \otimes \tau + \tau \otimes 1 + \sum_{\substack{\sigma \text{ subtree of } \tau \\ \sigma \neq \emptyset, \tau}} (\tau \setminus \sigma) \otimes \sigma$$

Dualise to pass to Lie theory (Milnor-Moore theorem!)

The dual picture: Character groups

Hopf algebra characters

$\mathcal{H}$ Hopf algebra, B a commutative algebra.

A **character** is an unital algebra morphism $\phi\colon \mathcal{H} \to B$.

An **infinitesimal character** is a linear map $\psi\colon \mathcal{H} \to B$ which satisfies $\psi(xy) = \epsilon(x)\psi(y) + \psi(x)\epsilon(y)$ ($\epsilon =$counit).

Characters form a group $G(\mathcal{H}, B)$ with respect to convolution

$$\phi \star \psi := m_B \circ \phi \otimes \psi \circ \Delta.$$

Infinitesimal characters form a Lie algebra $\mathfrak{g}(\mathcal{H}, B)$ with bracket

$$[\eta, \psi] := \eta \star \psi - \psi \star \eta.$$

Why are Hopf algebra characters interesting?

Perturbative renormalisation of QFT (cf. Connes/Marcolli 2007)

Characters of the Hopf algebra $\mathcal{H}_{FG}$ of Feynman graphs are called "diffeographisms", the diffeographism group acts on the coupling constants via formal diffeomorphisms.

Regularity structures for SPDEs (Bruned/Hairer/Zambotti 2016)

For certain (singular) SPDEs (PAM, KPZ...) regularity structures allow to approximate and interpret solutions.

$\rightarrow$ Hopf algebra tailored to problem,

$\rightarrow$ ($\mathbb{R}$-valued) character group encodes recentering in the theory (= positive renormalisation).

Characters of the Butcher-Connes-Kreimer algebra

$G(\mathcal{H}, \mathbb{R})$ is the **Butcher group** whose elements correspond to (numerical) power-series solutions of ODEs (B-series).[1]

[1] $G(\mathcal{H}, \mathbb{R})$ as "Lie group" implicitely used in Hairer, Wanner, Lubich *Geometric Numerical Integration* 2006.

Infinite-dimensional structures

Calculus beyond Banach spaces

Bastiani calculus

Let E, F be **locally convex spaces** $f\colon U \to F$ is C^1 if

$$df\colon U \times E \to F, \quad df(x,v) := \lim_{h\to 0} h^{-1}(f(x+hv) - f(x))$$

exists and is continuous.To define smooth (C^∞) maps, we require that all iterated differentials exist and are continuous.

Chain rule and familiar rules of calculus apply $\to$ manifolds!

Infinite-dimensional Lie group

A group G is a (infinite-dimensional) Lie group if it carries a manifold structure (modelled on locally convex spaces) making the group operations smooth (in the sense of Bastiani calculus).

Structure theory for character groups

Theorem (Bogfjellmo, Dahmen, S.)

Let $\mathcal{H}$ be a graded Hopf algebra $\mathcal{H} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{H}_n$ with $\dim \mathcal{H}_0 < \infty$ and B be a commutative Banach algebra, then $G(\mathcal{H}, B)$ is a Lie group.

Lie theoretic properties of $G(\mathcal{H}, B)$

- $(\mathfrak{g}(\mathcal{H}, B), [-,-])$ is the Lie algebra of $G(\mathcal{H}, B)$
- $\exp\colon \mathfrak{g}(\mathcal{H}, B) \to G(\mathcal{H}, B), \psi \mapsto \sum_{n=0}^{\infty} \frac{\psi^{\star n}}{n!}$ is the Lie group exponential
- $G(\mathcal{H}, B)$ is a Baker-Campbell-Hausdorff Lie group
- If B is finite-dimensional, $G(\mathcal{H}, B)$ is the projective limit of finite-dimensional groups

The infinite dimensional picture

Infinite-dimensional Lie-theory admits pathologies not present in the finite dimensions, e.g.

- a Lie-group may not admit an exponential map
- the Lie-theorems are in general wrong

The situation is better for the class of “regular” Lie-groups.

Regularity for Lie-groups

Differential equations of “Lie-type” can be solved on the group and depend smoothly on parameters

Regularity for Lie-groups

Setting: G a Lie-group with identity element $\mathbf{1}$,
$\rho_g \colon G \to G, x \mapsto xg$ (right translation)
$v.g := T_{\mathbf{1}}\rho_g(v) \in T_g G$ for $v \in T_{\mathbf{1}}(G) =: \mathbf{L}(G)$.

G is called **regular** (in the sense of Milnor) if for each smooth curve $\gamma \colon [0,1] \to \mathbf{L}(G)$ the initial value problem

$$\begin{cases} \eta'(t) & = \gamma(t).\eta(t) \\ \eta(0) & = \mathbf{1} \end{cases}$$

has a smooth solution $\mathrm{Evol}(\gamma) := \eta \colon [0,1] \to G$, and the map

$$\mathrm{evol} \colon C^\infty([0,1], \mathbf{L}(G)) \to G, \quad \gamma \mapsto \mathrm{Evol}(\gamma)(1)$$

is smooth.

Theorem (Bogfjellmo, Dahmen, S.)

Let B be a commutative Banach algebra and $\mathcal{H} = \bigoplus_{n \in \mathbb{N}_0} \mathcal{H}_n$ a graded Hopf algebra with dim $\mathcal{H}_0 < \infty$. Then $G(\mathcal{H}, B)$ is regular in the sense of Milnor.

Why ist this interesting?

Numerical analysis (Murua/Sanz-Serna)

Lie type equations on the Butcher group and related groups are used in numerical analysis (word series).

Why care about regularity?

Time ordered exponentials in CK-renormalisation

Consider the *time ordered exponentials*

$$\mathbf{1} + \sum_{n=1}^{\infty} \int_{a \leq s_1 \leq \cdots s_n \leq b} \alpha(s_1) \cdots \alpha(s_n) \mathrm{d}s_1 \cdots \mathrm{d}s_n$$

for $\alpha \colon [a, b] \to \mathfrak{g}(\mathcal{H}_{FG}, \mathbb{C})$ smooth.

$\to$ negative part of Birkhoff decomposition of a smooth loop arises as an exponential of the β-function of the theory.

However: Time ordered exponentials are solutions to Lie type equations on $G(\mathcal{H}_{FG}, \mathbb{C})$

Why is this not good enough?

Topology of $G(\mathcal{H}, B)$ is very coarse...

- Impossible to control behaviour of series
- Too simple representation theory of these groups

However, there is no other "good" topology on $G(\mathcal{H}, B)$.

To fix this, pass to a subgroup of "controlled characters".

Groups of controlled characters

For the Butcher-Connes-Kreimer algebra consider

$$G_{ctr}(\mathcal{H},\mathbb{R}) := \left\{ \varphi \in G(\mathcal{H},\mathbb{R}) \middle| \substack{\exists C,K>0 \text{ s.t. } \forall \tau \text{ tree)} \\ |\varphi(\tau)| \leq CK^{|\tau|}} \right\}$$

'Lie group of controlled characters'.
$\rightarrow$ limits growth by an exponential in the degree of the trees.
$\rightarrow$ leads to locally convergent series

- Geometry of the group of controlled characters much more involved (i.e. interesting)
- Lie theory for controlled groups...
- ... analysis usually requires combinatorial insights.
- Techniques are not limited to the weights $\omega_n(k) := n^k$.

Advantages of the subgroup of controlled characters

Given a (combinatorial)[2] Hopf algebra and weights $\{\omega_n\}_{n\in\mathbb{N}}$ adapted to the combinatorial structure, then the group of controlled characters...

- Controls (local) convergence behaviour
- is (in all known cases) a regular Lie groups
- depends crucially on combinatorial structure and grading

[2] A Hopf algebra is combinatorial if its algebra structure is a (possibly non-commutative) polynomial algebra and there is a distinguished choice of generating set (e.g. trees for the Butcher-Connes-Kreimer algebra.

Thank you for your attention!

More information:

Bogfjellmo, S.: The geometry of characters of Hopf algebras, Abelsymposium 2016: "Computation and Combinatorics in Dynamics, Stochastics and Control"

Dahmen, S.: Lie groups of controlled characters of combinatorial Hopf algebras, AIHP D 7 (2020).

Dahmen, Gray, S.: Continuity of Chen-Fliess Series for Applications in System Identification and Machine Learning, arXiv:2002.10140