Classes of nonlinear PDEs related to metrics of constant curvature

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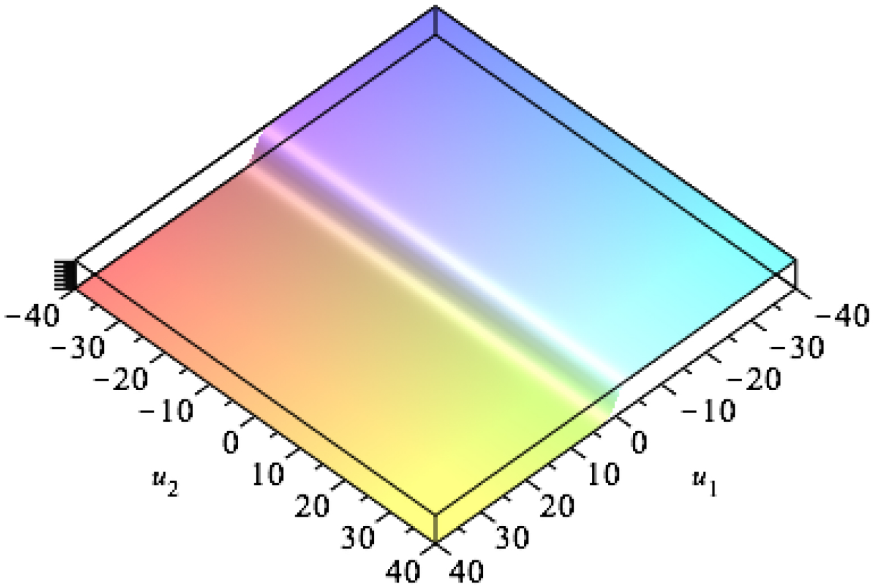
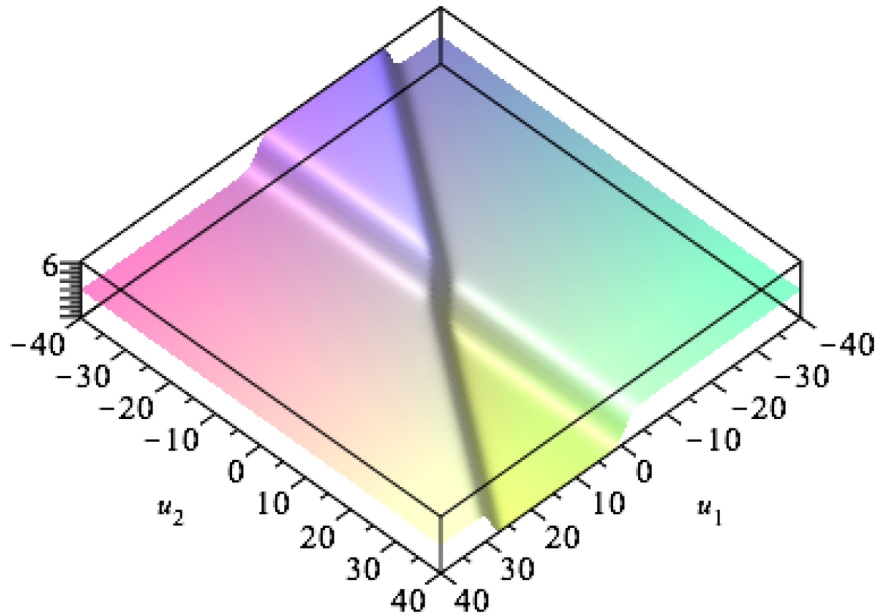
- **Surfaces in $\mathbf{R}^3$ of constant negative Gaussian curvature were associated to the Sine-Gordon equation (SG), since 1862 (E.Bour)**

$$u_{xt} = \sin u \qquad \textbf{or} \qquad u_{x_1x_1} - u_{x_2x_2} = \sin u$$

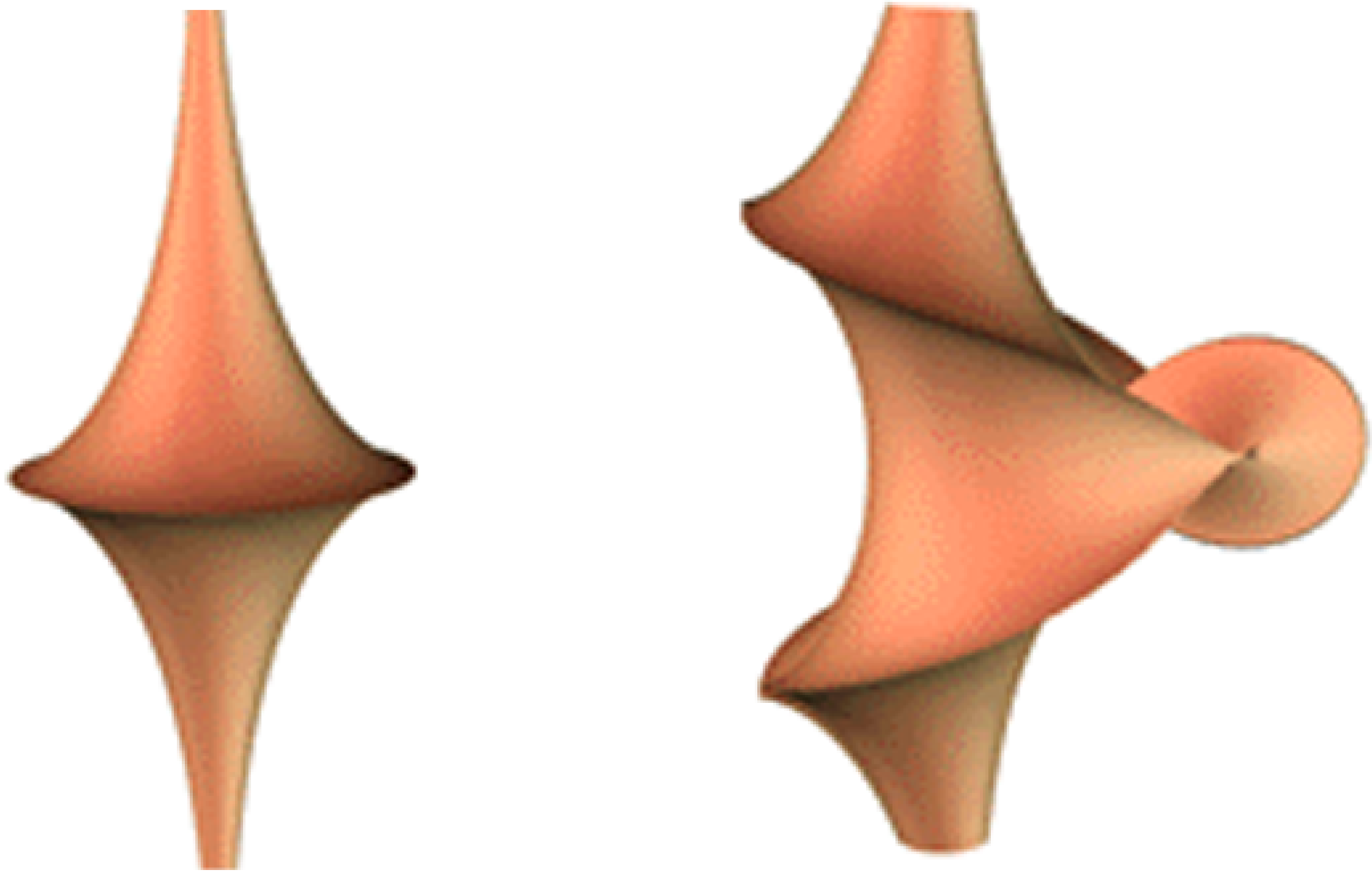
- **A geometric tranformation of such surfaces provided the analytic 1-parameter Bäcklund transformation (BT) for the SG.**
- **This is an integrable 1-parameter transformation that provides new solutions of the SG, starting with a given one.**

Soliton solutions of the SG

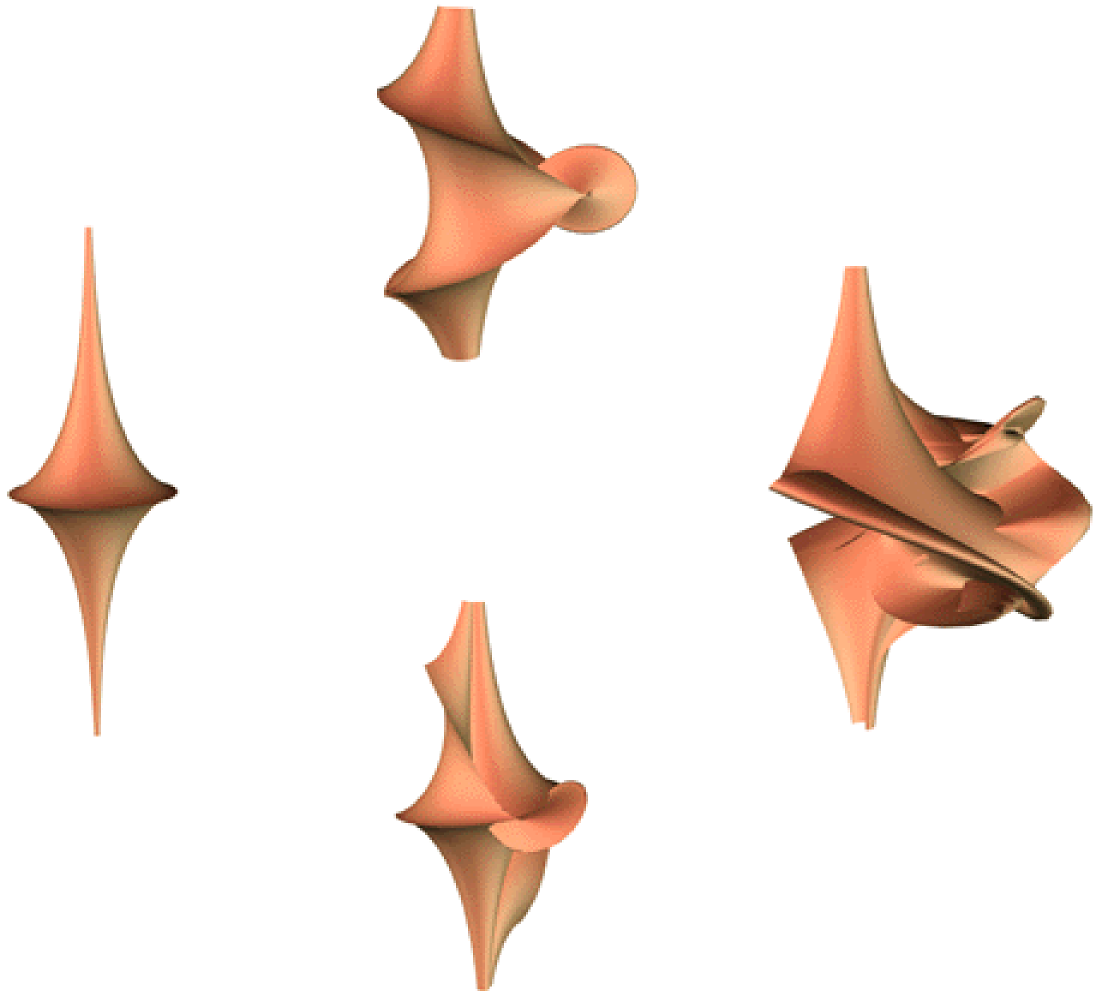




$u = 4\arctan(e^{x_1})$ $\qquad$ $u = -4\arctan(\cosh x_1/x_2)$



Corresponding surfaces:Pseudosphere and its BT $\theta = \pi/2$



Bianchi's permutability theorem

Superposition formula

Bianchi: the composition of Bäcklund transformation is commutative and provides a Superposition formula.

$$\begin{tikzcd}
 & u_1 \arrow[r, "\theta_2"] & u_{12} \arrow[r] \arrow[dr, "\theta_3"] & {} \\
u \arrow[ur, "\theta_1"] \arrow[r, "\theta_2"] \arrow[dr, "\theta_3"] & u_2 \arrow[ur, "\theta_1"] \arrow[dr, "\theta_3"] & & u_{123} \\
 & u_3 \arrow[r, "\theta_2"] & u_{23} \arrow[r] \arrow[ur, "\theta_1"] & {}
\end{tikzcd}$$

• After a first integration of BT, the superposition formula gives infinitely many new solutions algebraically for the SGE.

- **In the 70s there was a renewed interest in the SG and also in other equations that were investigated due to their applications and mainly due to the existence of soliton solutions.**
- **Solitons were first observed by Scott Russel (1834).**
- **Such solutions propagate without changing their shape and also preserve their shape after colliding with other solitons.**

Other equations with soliton solutions

For example:

Korteweg-de Vries (KdV) equation (*waves on shallow water*)

$$\mathbf{u_t = u_{xxx} + 6uu_x}.$$

Non linear Schödinger equation (*fluids,nonlinear optics*)

$$\mathbf{iq_t + q_{xx} \pm 2|q|^2 q = 0},$$

rewritten in real form by considering ($\mathbf{q = u + iv}$)

$$\begin{cases} \mathbf{u_t + v_{xx} + 2\kappa(u^2 + v^2)v = 0}, \\ \mathbf{-v_t + u_{xx} + 2\kappa(u^2 + v^2)u = 0}. \end{cases}$$

Differential equations which describe pseudo-spherical or spherical surfaces

- PDEs (or systems of PDEs) describing pseudospherical (pss) or spherical surfaces (ss) are characterized by the fact that their generic solutions provide metrics on non empty open subsets of $\mathbb{R}^2$, with Gaussian curvature $K = -1$ or $K = 1$, respectively. This is an intrinsic property.
- This concept is a generalization of the definition of differential equations describing pss, first introduced in 1986 by S. S. Chern, T. ____.

- **This definition was inspired by Sasaki's observation (1979), that a class of nonlinear differential equations, such as KdV, MKdV and SG, which can be solved by the AKNS inverse scattering method, was related to pss.**

- **Nowadays, it is known that the class of PDEs describing pss is, in fact, larger than the AKNS class.**

- **Besides the notion of differential equations describing pss, Chern- T. ____ introduced a systematic procedure of characterizing and classifying such equations.**

- Since the 80s, this procedure has been used to obtain classification results of several classes of partial differential (systems of) equations describing pss or ss, in a series of papers by:

  Beals, Castro-Silva, Cavalcante, Chern, Ding, Catalano, Gomes, Jorge, Kamran, Kelmer, Oliveira-Silva, Rabelo, Reyes, T. ____.

Remarkable properties of PDEs describing pss or ss

- **Within the universe of PDEs such equations are geometrically in two equivalence classes, due to a basic geometric fact:**

  Given two points of two Riemannian manifolds, with the same dimension and same constant (sectional) curvature, there is always an isometry between neighborhoods of those points.

$\Rightarrow$ Theoretical existence of local transformations between generic solutions of equations describing pss (resp. ss).

Characterizations of (systems) PDEs describing pss or ss

A differential equations describes pss (resp. ss)

$$\Updownarrow$$

it defines a metric ds^2 on an open subset $U \subset \mathbb{R}^2$, with constant Gaussian curvature $K = -1$, (resp.$K = 1$).

$$\Updownarrow$$

It is the compatibility condition of an associated $\mathfrak{sl}(2, \mathbb{R})$-valued (resp. $\mathfrak{su}(2)$) linear problem, also referred to as a zero curvature representation.

The importance of the associated linear problem

This characterization implies that those equations may present properties such as:

- **Bäcklund transformations.**
- **Superposition formulae.**
- **An infinite number of conservation laws.**
- **They are natural candidates for being solved by the Inverse Scattering Method.**
- **The equations are in a certain sense "integrable".**

- **A 1-parameter Bäcklund transformations may be obtained the symmetries of the linear problem. They provide solutions of the PDE starting from a given one.**
- **Superposition formulae obtained by the permutability of Bäcklund transformations.**
- **The Inverse Scattering Method (ISM) applied to the linear problem may give solutions for a given initial condition. The ISM was introduced by Gardner, Greene, Krushkal and Miura (1967) and reformulated in terms of a Riemann-Hilbert problem by Beals and Coifman (1984).**
- **Geometry may provide conservation laws for the PDE.**

Definition. A differential equation for a real-valued function $u(x,t)$ is said to **describe a pseudo-spherical surface** (pss)

$$\Updownarrow$$

$\exists$ smooth functions f_{ij}, **depending on u and its derivatives**, such that the $1-$ forms

$$\begin{aligned}
\omega_1 &= f_{11}\,dx + f_{12}\,dt,\\
\omega_2 &= f_{21}\,dx + f_{22}\,dt,\\
\omega_3 &= f_{31}\,dx + f_{32}\,dt,
\end{aligned}$$

define a **metric** $\boxed{ds^2 = \omega_1^2 + \omega_2^2}$ on $U \subset \mathbb{R}^2$, whose Gaussian curvature is constant $K = -1$, i.e.,

$$\begin{aligned}
d\omega_1 &= \omega_3 \wedge \omega_2,\\
d\omega_2 &= \omega_1 \wedge \omega_3,\\
d\omega_3 &= \omega_1 \wedge \omega_2.
\end{aligned}$$

Examples. a) A function $u(x,t)$ satisfies the sine-Gordon equation $u_{xt} = \sin u \Leftrightarrow$

$$\begin{aligned} \omega_1 &= \frac{1}{\eta}\sin u\, dt, \\ \omega_2 &= \eta\, dx + \frac{1}{\eta}\cos u\, dt, \\ \omega_3 &= u_x\, dx \end{aligned}$$

satisfy the structure equations with $K = -1$.

b) A function $u(x,t)$ satisfies the modified Korteweg de-Vries equation (MKDV)

$$u_t = u_{xxx} + \frac{3}{2}u^2 u_x$$

$\Leftrightarrow$ the forms

$$\begin{aligned} \omega_1 &= -\eta u_x\, dt, \\ \omega_2 &= \eta\, dx + \left(\tfrac{1}{2}\eta u^2 + \eta^3\right) dt, \\ \omega_3 &= u\, dx + \left(u_{xx} + \tfrac{1}{2}u^3 + \eta^2 u\right) dt, \end{aligned}$$

satisfy the structure equations with $K = -1$.

Associated Linear Problems

- **A differential equation $\mathscr{E}$ for $\mathbf{u}(\mathbf{x},\mathbf{t})$ describes p.s.s $\Leftrightarrow$ it is the integrability condition for a linear problem for ψ**

$$\begin{cases} \dfrac{\partial \psi}{\partial x} = A\ \psi \\ \\ \dfrac{\partial \psi}{\partial t} = B\ \psi, \end{cases}$$

where A and B are the $sl(2,\mathbb{R})$ valued functions

$$A = \frac{1}{2}\begin{pmatrix} f_{21} & f_{11}-f_{31} \\ f_{11}+f_{31} & -f_{21} \end{pmatrix} \qquad B = \frac{1}{2}\begin{pmatrix} f_{22} & f_{12}-f_{32} \\ f_{12}+f_{32} & -f_{22}, \end{pmatrix}$$

and $\omega_i = f_{i1}dx + f_{i2}dt$, i.e. $\mathscr{E}$ describes p.s.s $\Leftrightarrow$ the structure equations for $K=-1$ hold $\Leftrightarrow$ $\mathscr{E}$ is equivalent to

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + [A,B] = 0.$$

- **A differential equation $\mathscr{E}$ for $\mathbf{u}(\mathbf{x},\mathbf{t})$ describes a p.s.s $\Leftrightarrow$ it is the integrability condition for the linear problem**

$$\begin{cases} \dfrac{\partial \phi}{\partial x} = \bar{A}\phi \\ \dfrac{\partial \phi}{\partial t} = \bar{B}\phi, \end{cases}$$

where

$$\bar{A} = \begin{pmatrix} 0 & f_{11} & f_{21} \\ f_{11} & 0 & f_{31} \\ f_{21} & -f_{31} & 0 \end{pmatrix} \qquad \bar{B} = \begin{pmatrix} 0 & f_{12} & f_{22} \\ f_{12} & 0 & f_{32} \\ f_{22} & -f_{32} & 0 \end{pmatrix}$$

and $\omega_i = f_{i1}dx + f_{i2}dt$, **i.e.** $\mathscr{E}$ **describes pss** $\Leftrightarrow$ **the structure equations for** $K=-1$ **hold** $\Leftrightarrow$ $\mathscr{E}$ **is equivalent to**

$$\frac{\partial \bar{A}}{\partial t} - \frac{\partial \bar{B}}{\partial x} + [\bar{A}, \bar{B}] = 0.$$

One has similar linear problems for PDEs that describe ss.

Examples

- **Sine-Gordon equation**

$$u_{xt} = \sin u$$

is the compatibility condition for the linear problem for V

$$\frac{\partial V}{\partial x} = \frac{1}{2}\begin{pmatrix} \eta & -u_x \\ u_x & -\eta \end{pmatrix} V,$$

$$\frac{\partial V}{\partial t} = \frac{1}{2\eta}\begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix} V.$$

- **The non linear Scrödinger (NLS$^-$) equation describes pss**

$$\begin{cases} \mathbf{u_t + v_{xx} - 2(u^2+v^2)v = 0}, \\ \mathbf{-v_t + u_{xx} - 2(u^2+v^2)u = 0}. \end{cases}$$

It is the compatibility condition of the linear problem for V

$$\frac{\partial V}{\partial x} = \begin{pmatrix} -v & u-\eta \\ u+\eta & v \end{pmatrix} V,$$

$$\frac{\partial V}{\partial t} = \begin{pmatrix} 2\eta v - u_x & -2\eta u + v_x + 2\eta^2 + u^2 + v^2 \\ -2\eta u + v_x - 2\eta^2 - u^2 - v^2 & -2\eta v + u_x \end{pmatrix} V.$$

- **Changing the red sign, we get NLS$^+$ that describes ss.**

- **The linear problems are determined by the functions** f_{ij}**.**

Conservation laws (Chern, T. __, 1986); (Cavalcante, T. __, 1988)

Geometric properties of a surface with constant $K=-1$ imply

Proposition. **For any differential equation that describes p.s.s., with $\omega_i = f_{i1}dx + f_{i2}dt$, there is an integrable system for ϕ**

$$\begin{aligned}\phi_x &= f_{31} + f_{11}\sin\phi + f_{21}\cos\phi,\\ \phi_t &= f_{32} + f_{12}\sin\phi + f_{22}\cos\phi.\end{aligned}$$

For any such solution ϕ, there is a closed form

$$\mathscr{C} = (f_{11}\cos\phi - f_{21}\sin\phi)\,dx + (f_{12}\cos\phi - f_{22}\sin\phi)\,dt$$

If f_{ij} are analytic functions of a parameter η at zero, then $\phi(x,t,\eta)$ and $\mathscr{C}$ are analytic in η. Infinite number of Conservation laws may be obtained by considering the coefficients of η in $\mathscr{C}$.

Classes of PDEs that describe pss or ss

- A particular case $f_{21} = \eta$, a parameter and f_{11} and f_{31} independent of η (AKNS)
- Chern, T. ____ (1986) considered the problem of classifying the evolution equations of the form
$$u_t = F\left(u, u_x, \ldots, \frac{\partial^k u}{\partial x^k}\right),$$
which describe p.s.s., under the assumption that $f_{21} = \eta$ (no assumption on f_{11} or f_{31}). Ex: Burgers, KdV, MKdV, etc.
- Jorge, T. ____ (1987) classification problem, with $f_{21} = \eta$, for equations of type $u_{tt} = F(u, u_x, u_{xx}, u_t)$.
- Rabelo (1989) classified classes of equations of type $u_{xt} = F$.

- **Among so many classes of differential equations describing pss, in particular, the equation**

$$u_{xt} = u + (u^3)_{xx}.$$

was obtained by Rabelo in his thesis (1989). Nowadays it is known as the "short pulse equation".

- **In 2004, it appeared in non linear optics.**

- **Schäfer-Wayne showed that it describes the propagation of ultra-short light pulses in silica optical fibers. Important for trasmission of data.**

The short pulse equation is contained in the family of equations

$$\{\mathbf{u_t} - [\alpha \mathbf{g}(\mathbf{u}) + \beta]\mathbf{u_x}\}_{\mathbf{x}} = \varepsilon \mathbf{g}'(\mathbf{u})$$

where $\mathbf{g}(\mathbf{u})$ satisfies $\quad \mathbf{g}'' + \mu \mathbf{g} = \theta, \quad \varepsilon = \pm 1$ and $\mu, \alpha, \beta, \theta$ are real constants, describes pss.

- **By considering such a function g and appropriate values for the constant α, β, θ and μ, one gets the sine-Gordon equation, the sinh-Gordon equation, the Liouville equation, and the short pulse equation.**
- **Beals, Rabelo, T.____ (1989) The associated linear problem was used to apply the ISM to solve the equation only for $\varepsilon = 1$.**

- **Kamran, T. ____ (1995) gave a complete characterization of the evolution equations of type $u_t = F(u, u_x, \cdots, \partial^k u/\partial x^k)$. which describe p.s.s.(no restriction on $\mathbf{f_{ij}}$).**

- **Reyes (1998) considered differential equations of type $\mathbf{u_t = F(x, u, u_x, \cdots}$**

- **The complete characterization results (Kamran, T.____) can be used for:**

1. **Check if a given differential equation describes a pss.**
2. **Generate huge classes of differential equations that describe pss. (with potencial applications to engineering or physics).**

Fifth order evolution equations **Gomes V.P. (JDE 2010)**

$$u_t = u_{xxxxx} + G(u, u_x, u_{xx}, u_{xxxx})$$

with the assumption

$$f_{21} = \mu_2 f_{11} + \eta_2, \qquad f_{31} = \mu_3 f_{11} + \eta_3.$$

- **The main results divide the classification into four theorems.**
- **The differential equations involve arbitrary differential functions depending on u and u_x (infinite dimensional).**
- **The associated linear problems depend on the parameters μ and η.**

Examples of equations obtained by choosing the arbitrary functions

1. Kaup-Kupershmidt equation

$$u_t = u_{xxxxx} + 5uu_{xxx} + \frac{25}{2}u_x u_{xx} + 5u^2 u_x$$

2. Sawada-Kotera equation

$$u_t = u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x$$

3.

$$u_t = u_{xxxxx} - 2uu_{xxx} - u_x u_{xx} - u_x$$

4. The fifth order KdV equation

$$u_t = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x$$

Fouth order evolution equations

Catalano-Ferraioli, T. ____ (JDE 2014)

Differential equations that describe pseudospherical surfaces were considered of type

$$u_t = u_{xxxx} + G(u, u_x, u_{xx}, u_{xxx})$$

with associated 1-forms $\omega_i = f_{i1}\,dx + f_{i2}\,dt, \quad i = 1, 2, 3,$ **where**

$$f_{p1} = \mu_p f_{11} + \eta_p, \qquad p = 2, 3.$$

- **Classification results into 4 large types of equations involving arbitrary functions (infinite dimensional).**

Some examples (choosing the arbitrary functions):

1. The fourth order member of Burgers hyerarchy.

2. $$u_t = u_{xxxx} - uu_{xxx} - \frac{3}{2}u^3 u_x.$$

3. $$u_t = u_{xxxx} - 2uu_{xx} - u_x^2.$$

4. $$u_t = u_{xxxx} + u_{xxx} + uu_x + u^2/2\,.$$

5. A modified Kuramoto-Sivashinsky equation.

$$u_t = u_{xxxx} + m_1 u_{xxx} + m_2 u_{xx} - uu_x + m_0 u^2,$$

where $m_0 \neq 0$. The limiting case, when $m_0 \to 0$, provides the Kuramoto-Sivashinsky equation.

Castro-Silva, T.__ (JDE 2015) Classification results for equations of type

$$u_t - u_{xxt} = \lambda u u_{xxx} + G(u, u_x, u_{xx}), \quad \lambda \in \mathbb{R},$$

that describe ss or pss.

Very large families of equations are contained in this class.

Examples:

- **The class of equations**

  $$u_t - u_{xxt} = m_1 \psi + m_2 \psi_x, \qquad m_1, m_2 \in \mathbb{R} \setminus \{0\},$$

  where $\psi(u, u_x) \neq 0$ is an arbitrary differentiable function of u, u_x (infinite dimensional).

- **The family of equations**

$$u_t - u_{xxt} = \lambda(uu_{xxx} + u_x u_{xx} - 2uu_x - m_1 u^2 + m_1 uu_{xx}) + -m_2(u - u_{xx}) + m_1\psi + \psi_x,$$

where $\lambda m_1^2 + m_2^2 \neq 0$ **and** $\psi(u, u_x)$ **is an arbitrary differentiable function (infinite dimensional), describes pss.**

Choosing $\lambda = 1$, $m_1 = 2$, $m_2 = 0$ **and** $\psi = u_x^2 - 2uu_x + u^2$, **this family reduces to the Degasperis-Procesi equation,**

$$u_t - u_{xxt} = uu_{xxx} - 4uu_x + 3u_x u_{xx}.$$

- **The three-parameter family of equations**

$$u_t - u_{xxt} = \lambda(uu_{xxx} + 2uu_{xx} - 3uu_x - m_2 u_x) + m_1 e^u u_x(u_x^2 + u_{xx} + 2u + m_2),$$

where $\lambda^2 + m_1^2 \neq 0$ **and** $m_2 \in \mathbb{R}$**, describes pss.**

  - **When** $\lambda = 1$ **and** $m_1 = 0$**, it reduces to Camassa-Holm equation,**

$$u_t - u_{xxt} = uu_{xxx} + 2uu_{xx} - 3uu_x - m_2 u_x,$$

  - m_2 **is related to the critical shallow water wave speed.**
  - **When** $m_2 = 0$**, the equation has the so called peakon solutions**

Remark

The classification results are constructive in the sense that:

- **each PDE (or family of PDEs) that describes pss or ss is presented with the corresponding functions f_{ij} explicitly given.**
- **This provides explicitly the one (or more) parameter linear problem whose integrability condition is the PDE.**

Quasilinear second order partial differential equations

Catalano, Castro-Silva, T. ____ (JDE 2020)

Classification results for second order equations of parabolic, hyperbolic or elliptic type

$$u_{tt} = A(u, u_x, u_t)u_{xx} + B(u, u_x, u_t)u_{xt} + C(u, u_x, u_t),$$

describing pss or ss, provide large families of such equations.

Some particular examples included in this classification

1. For any differentiable functions h and ℓ of $u(x,t)$

$$u_{tt} = (h(u))_{xx} + (\ell(u))_{xt}$$

describes pss.

2.

$$u_{tt} = \frac{2}{u^2+m}[\delta u_{xt} - u(u_t^2+1)]$$

describes pss or ss (with distinct associated linear problems). It generalizes the short pulse equation ($m = 0$).

3.

$$u_{tt} = m^2 u_{xx} + m(u^p)_x - (u^p)_t,$$

where p is an integer, describes pss or ss.

4. A 1-parameter family of equations that reduces to the constant astigmatism equation

$$u_{tt} = \frac{u_{xx}}{u^2} - 2\frac{u_x^2}{u^3} - 2$$

describes pss or ss.

SYSTEMS of PDEs describing pss or ss

Ding, T. ____ (JDE 2002)

Classification results gave families of first order systems:

1.

$$\begin{cases} u_t = (2up)_x + 4vp, \\ v_t = (2vp)_x - 2up. \end{cases}$$

where $p = p(u, v) \neq 0$ arbitrary diff. describes ss.

2.

$$\begin{cases} u_t = 2h, \\ v_t = h_x - uh. \end{cases}$$

where $h = h(u, v) \neq 0$ arbitrary diff. describes pss

3.

$$\begin{cases} u_t = (u^2 + v^2)u_x + u^2 + v^2 + 1, \\ v_t = 2uvu_x + (u^2 + 3v^2)v_x. \end{cases}$$

Motivated by the following second order examples:

- **The nonlinear Schrödinger equation;**
- **the Heisenberg ferromagnet model;**
- **the Schrödinger flow of maps into $\mathbf{S}^2 \subset \mathbf{R}^3$ ss;**
- **the Schrödinger flow of maps into $\mathbf{H}^2 \subset \mathbf{R}^{2+1}$, pss.**

Classification results for second order systems of type

$$\begin{cases} u_t = -v_{xx} + H_{11}(u,v)u_x + H_{12}(u,v)v_x + H_{13}(u,v), \\ v_t = u_{xx} + H_{21}(u,v)u_x + H_{22}(u,v)v_x + H_{23}(u,v). \end{cases}$$

provided

$$\begin{cases} u_t = -v_{xx} + \delta\gamma[(u^2+v^2)u]_x + \alpha u_x + \delta\sigma(u^2+v^2)v + \beta v, \\ v_t = u_{xx} + \delta\gamma[(u^2+v^2)v]_x + \alpha v_x - \delta\sigma(u^2+v^2)u - \beta u, \end{cases}$$

where $\delta = 1$ (resp. $\delta = -1$), α, β, γ and $\sigma \in R$ are such that $\sigma \geq 0$ if $\gamma = 0$.

- **This is a 4-parameter family of equations. Particular choices of the parameters reduce to the well known systems.**

Some recent results for SYSTEMS Kelmer, T. ____ (JDE 2022)

- Classification results for systems of type

$$\begin{cases} u_{xt} = F\left(u, u_x, v, v_x\right), \\ v_{xt} = G\left(u, u_x, v, v_x\right), \end{cases}$$

describing pss and ss. These results involve parameters and arbitrary differentiable function $\psi(u_x, v_x)$. A particular case of an infinite family of systems provides

Example. The Konno-Oono coupled integrable dispersionless system

$$\begin{cases} u_{xt} = -2vv_x, \\ v_{xt} = 2vu_x, \end{cases}$$

describes pss.

- **Explicit examples of conservation laws**

The Pholmeyer-Lund-Regge type system

$$\begin{cases} u_{xt} = 2uvu_x - u, \\ v_{xt} = -2uvv_x - v, \end{cases}$$

describes pss and the first two closed forms are given by

$$[-2u(u+v)_x - 1]\,dx + [2u^2v^2 - 2u_t(u+v)]\,dt;$$

$$[(u^3 + u^2v - u_t)(u_x + v_x) + u^2]\,dx + [(u+v)(u^2v - u_t + \tfrac{1}{3}u^3)_t - u_t^2]\,dt;$$

$etc.$

$\ldots$

$\ldots$

For the Konno-Oono coupled system

$$\begin{cases} u_{xt} = -2vv_x, \\ v_{xt} = 2vu_x, \end{cases}$$

the first two closed forms are given by

$$\sqrt{u_x^2 + v_x^2}\, dx;$$

$$-\frac{1}{4}\frac{(u_x v_{xx} - u_{xx} v_x)^2}{[u_x^2 + v_x^2]^{5/2}}\, dx + \frac{u_x}{\sqrt{u_x^2 + v_x^2}}\, dt;$$

$\cdots$

$\cdots$

$\cdots$

Bäcklund transformations and Superposition for elliptic equations

- **The classical theory of surfaces in $\mathbb{R}^3$ with Gaussian curvature -1, was extended to surfaces in $\mathbb{R}^3_1$ (Lorentzian space).**
- **In particular, the surfaces in $\mathbb{L}^3$ with Gaussian curvature -1, correspond to solutions of elliptic sine-Gordon and elliptic sinh-Gordon equations.**
- **These equations also describe several physical phenomena.**
- **The geometric theory provided Bäcklund transformations and superposition formulae.**

- **F. Kelmer, L. A Rodrigues, T. ____ (2022).**

Bäcklund Transformations from solutions of the elliptic sine-Gordon equation (ESG) into solutions of the elliptic sinh-Gordon (ESHG) equation and viceversa.

$$\alpha_{x_1x_1} + \alpha_{x_2x_2} = \sin\alpha$$

$$\alpha_{x_1x_1} + \alpha_{x_2x_2} = \sinh\alpha$$

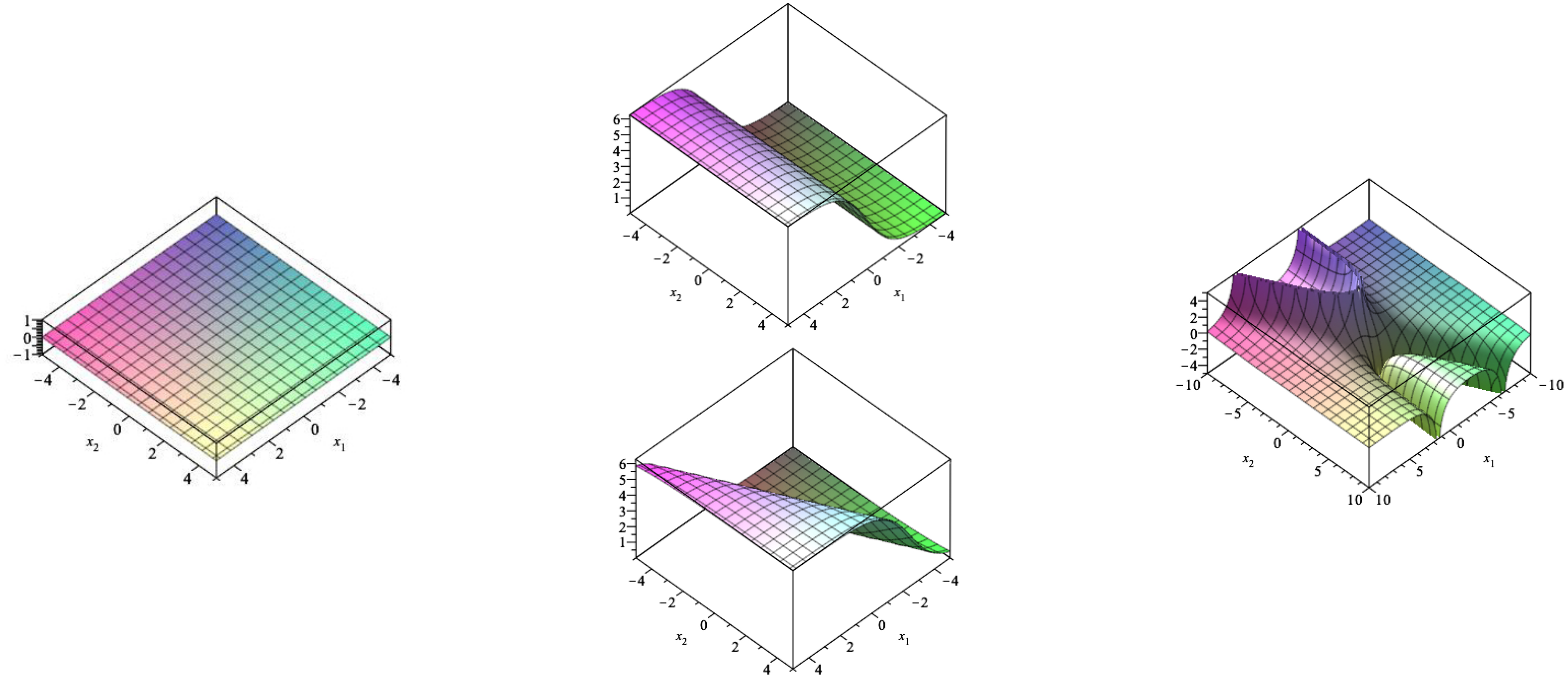
Explicit solutions can be obtained by applying these transformations.

• **F. Kelmer, T. ____ (2023) Superposition formulae (composition of Bäcklund transformations) for the elliptic sine-Gordon equations and sinh-Gordon equations.**

Unusual compared to Bianchi's formulae for the SGE.

$$
\begin{array}{ccccccc}
\textit{ESHG} & & \textit{ESG} & & \textit{ESHG} & & \textit{ESG} \\
 & & \alpha_1 & \xrightarrow{\phi_2} & \alpha_{12} & & \longrightarrow \\
 & \overset{\phi_1}{\nearrow} & & \overset{\phi_1}{\nearrow} & & \overset{\phi_3}{\searrow} & \\
\alpha & \xrightarrow{\phi_2} & \alpha_2 & & & & \alpha_{123} \\
 & \overset{\phi_3}{\searrow} & & \overset{\phi_3}{\searrow} & & \overset{\phi_1}{\nearrow} & \\
 & & \alpha_3 & \xrightarrow{\phi_2} & \alpha_{23} & & \longrightarrow
\end{array}
\qquad
\begin{array}{ccccccc}
\textit{ESG} & & \textit{ESHG} & & \textit{ESG} & & \textit{ESHG} \\
 & & \alpha_1 & \xrightarrow{\phi_2} & \alpha_{12} & & \longrightarrow \\
 & \overset{\phi_1}{\nearrow} & & \overset{\phi_1}{\nearrow} & & \overset{\phi_3}{\searrow} & \\
\alpha & \xrightarrow{\phi_2} & \alpha_2 & & & & \alpha_{123} \\
 & \overset{\phi_3}{\searrow} & & \overset{\phi_3}{\searrow} & & \overset{\phi_1}{\nearrow} & \\
 & & \alpha_3 & \xrightarrow{\phi_2} & \alpha_{23} & & \longrightarrow
\end{array}
$$

• **We need the superposition formula to produce new solutions of the same equation.**



Explicit solutions. Start with the trivial sol $\alpha = 0$ of the ESHG.

Apply BT with distinct parameters get solutions of the ESG.

Superposition formula provides a solution of the ESHG defined on the complement of two curves.

Some generalizations to higher dimensions

Ablowitz, Aminov, Barbosa, Beals, Campos, Dajczer, Ferreira, T.___, Terng, Tojeiro

- **Systems of PDEs whose solutions are related to n-dimensional submanifolds M^n of constant sectional curvature K contained in space forms ($\mathbb{R}^N$, $\mathbb{S}^N$ or $\mathbb{H}^N$) of curvature $\bar{K}$ and $K \neq \bar{K}$.**
- **Their intrinsic version in terms of metrics defined on open subsets of $\mathbb{R}^n$ with constant sectional curvature.**
- **Bäcklund Transformations, Superposition formulae.**
- **ISM applied to the systems.**

THANK YOU!

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