Resurgent Trans-series in Hopf-Algebraic Dyson-Schwinger Equations

Gerald Dunne

University of Connecticut

Higher Structures Emerging from Renormalisation
Schrödinger Institute, Vienna

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M. Borinsky & GD, 2005.04265; M. Borinsky, GD, M. Meynig, 2104.00593

(O. Costin & GD, 1904.11593, 2003.07451, 2009.01962, 2108.01145, ...)

[DOE Division of High Energy Physics]

Motivation

- Kreimer-Connes:

[perturbative] QFT renormalisation $\longleftrightarrow$ Hopf algebra stucture

$\Rightarrow$ enables perturbative computations to very high order

- Écalle: resurgent asymptotics

[perturbative] series $\longrightarrow$ [perturbative + non-perturbative] trans-series

$\Rightarrow$ non-perturbative physics encoded in perturbative physics

IDEA: use resurgent trans-series to decode non-perturbative properties of QFT from their perturbative Hopf algebra structure

comment: the connection between *perturbative* and *non-perturbative* physics can be probed most efficiently at high orders of perturbation theory

What do physicists mean by "non-perturbative"?

- in quantum mechanics and quantum field theory perturbation theory is very accurate, but it is typically divergent

- however, perturbation theory is only part of the story ("physics beyond all orders")

- e.g. Stark effect (1919 Nobel Prize): electron energy levels are modified by an applied weak electric field $\mathcal{E}$

$$\text{perturbation theory :} \quad E_{\text{pert}}(\mathcal{E}) = E_0 + E_1\mathcal{E}^2 + E_2\mathcal{E}^4 + E_3\mathcal{E}^6 + \dots$$

- this formal perturbative series is asymptotic: $E_n \sim (2n)!$

- but there is also a "non-perturbative" effect: ionization

$$E(\mathcal{E}) = E_{\text{pert}}(\mathcal{E}) + i\,\exp\left[-\frac{\text{const}}{\mathcal{E}}\right] E^{\text{ion.}}_{\text{pert}}(\mathcal{E}) + \dots$$

- resurgence $\Rightarrow$ these two aspects are intimately related

Trans-series

- an interesting observation by Hardy:

No function has yet presented itself in analysis, the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms

G. H. Hardy, *Orders of Infinity*, 1910

- deep result: “this is all we need” (J. Écalle, 1980s)

- observation: this structure matches the asymptotics of QFT Feynman diagrammatic computations

Resurgent Trans-Series

- Écalle: *resurgent functions* closed under all operations:

(Borel transform) + (analytic continuation) + (Laplace transform)

- building blocks: *trans-monomial elements*: x, $e^{-\frac{1}{x}}$, $\ln(x)$
- in physics applications: semi-classical trans-series:

$$f(x) \sim \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, x^p}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{1}{x}\right]\right)^k}_{\text{non-perturbative}} \underbrace{\left(\ln\left[x\right]\right)^l}_{\text{logarithm powers}}$$

- new: analytic continuation encoded in trans-series
- new: trans-series coefficients $c_{k,l,p}$ are highly correlated
- theorems in ODEs, PDEs, difference eqs.; evidence in QM, matrix models; being explored in QFT, string theory, ...

“Resurgence”

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle

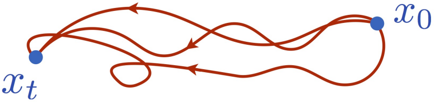
grand conjecture: resurgent trans-series solve all “natural problems”

The Feynman Path Integral

$$\text{QM}: \int \mathcal{D}x(t) \exp\left[\frac{i}{\hbar} S[x(t)]\right]$$

$$\text{QFT}: \int \mathcal{D}\phi(x) \exp\left[\frac{i}{g} S[\phi(x)]\right]$$

$$\langle x_t | e^{-i\hat{H}t/\hbar} | x_0 \rangle =$$



- Feynman path integral = generating function of Feynman diagrams
- rich combinatorial, algebraic & graph-theoretic structure
- Feynman path integral = generating function of non-perturbative saddles
- new perspective: Feynman path integral = a trans-series

- renormalization makes resurgence in quantum field theory more interesting

- recent progress for semiclassical QFT, lattice QFT and integrable QFT

- here: invoke Hopf algebra structure of the perturbative renormalization of QFT

Q1: do the Dyson-Schwinger equations contain **all** information (perturbative & non-perturbative) about a QFT?

Q2: [how] can one decode non-perturbative information?

Q3: is there a natural Hopf algebraic formulation of Écalle's "bridge equations" which relate the perturbative and non-perturbative features ?

Nonlinear ODEs from Dyson-Schwinger Equations

• Broadhurst/Kreimer 1999/2000; Kreimer/Yeats 2006:

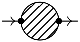
for certain QFTs the renormalization group equations can be reduced to coupled nonlinear ODEs for the anomalous dimension in terms of the renormalized coupling

• resurgence is deeply understood for (nonlinear) ODEs (Écalle, Costin, Kruskal, Ramis, Takei, Sauzin, Fauvet, ...)

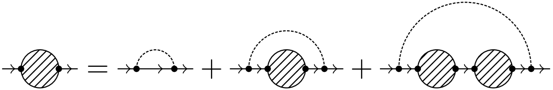
4 dimensional massless Yukawa theory

Broadhurst/Kreimer, 1999

- renormalised fermion self-energy

$\Sigma(q) :=$  $= \not{q}\, \Sigma(q^2)$

- Dyson-Schwinger equation

 $+ \cdots -$ subtractions

- anomalous dimension $\gamma(\alpha)$ ($\alpha \equiv$ renormalised coupling):

$$\gamma(\alpha) = \frac{d}{d \ln q^2} \ln \left(1 - \Sigma(q^2)\right) \Big|_{q^2=\mu^2}$$

- renormalisation group $\Rightarrow$ non-linear ODE (1st order)

$$2\gamma = -\alpha - \gamma^2 + 2\alpha\, \gamma \frac{d}{d\alpha} \gamma$$

$$\left[C(x)\left(2\,x\,\frac{d}{dx}-1\right)-1\right]C(x)=-x$$

• perturbative solution: $C(x)=\sum_{n=1}^{\infty} C_n x^n$ (OEIS: A000699)

$C_n = [1, 1, 4, 27, 248, 2830, 38232, 593859, 10401712, 202601898, \ldots]$

• combinatorics: generating function for "connected chord diagrams"

• large order asymptotics

$$C_n \sim e^{-1}\frac{2^{n+\frac{1}{2}}\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{2\pi}}\left(1-\frac{\frac{5}{2}}{2\left(n-\frac{1}{2}\right)}-\frac{\frac{43}{8}}{2^2\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)}-\ldots\right)$$

• missing boundary condition parameter ?

Écalle: formal series → trans-series : $\quad C(x)=\sum_{k=0}^{\infty}\sigma^k\,C^{(k)}(x)$

Trans-series Solution

M. Borinsky & GD, 2005.04265

- expand $C(x) = C^{(0)}(x) + \sigma\, C^{(1)}(x) + \sigma^2\, C^{(2)}(x) + \dots$
- $C^{(0)}(x) =$ previous formal perturbative series solution
- linear (in)homogeneous equations for $C^{(k)}(x)$, $k \geq 1\,(2)$

$$C^{(1)}(x) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{x}}{C^{(0)}(x)} \exp\left[-\frac{\left(C^{(0)}(x)+1\right)^2}{2x}\right]$$

$$\sim \frac{e^{-1}}{\sqrt{2\pi}} \frac{e^{-1/(2x)}}{\sqrt{x}} \left[1 - \frac{5}{2}\,x - \frac{43}{8}\,x^2 - \frac{579}{16}\,x^3 - \dots\right]$$

- note: $C^{(1)}(x) =$ (instanton factor) $\times$ (fluctuation series)
- resurgence: $C^{(1)}(x)$ expressed in terms of $C^{(0)}(x)$
- characteristic signature of resurgent structure:

$$C_n^{(0)} \sim e^{-1} \frac{2^{n+\frac{1}{2}}\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{2\pi}} \left(1 - \frac{\frac{5}{2}}{2\left(n-\frac{1}{2}\right)} - \frac{\frac{43}{8}}{2^2\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)} - \dots\right)$$

Resurgent structure

- large order asymptotics of $C_n^{(1)}$ coefficients

$$C_n^{(1)} \sim -2e^{-2}\,\frac{2^{n+\frac{3}{2}}\,\Gamma\left(n+\frac{3}{2}\right)}{2\pi}\left(1-\frac{5}{2\left(n+\frac{1}{2}\right)}-\frac{\frac{11}{2}}{2^2\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}-\dots\right)$$

- next nonperturbative solution ($\xi(x)\equiv\frac{1}{\sqrt{x}}\,e^{-1/(2x)}$):

$$C^{(2)}(x)\sim\xi(x)^2\,\frac{e^{-2}}{2\pi}\left[\frac{1}{x}-5-\frac{11}{2}x-\frac{97}{2}x^2-\dots\right]$$

- continues to all orders $\Rightarrow$ all-orders summation

$$C(x)=\left[\exp\left(\sigma\,\xi(x)f(x,y)\,\frac{\partial}{\partial y}\right)\cdot y\right]_{y=C^{(0)}(x)}$$

$$\text{generating function}:\quad f(x,y)\equiv\frac{1}{\sqrt{2\pi}}\,\frac{x}{y}\,\exp\left[-\frac{1}{2x}\,y(y+2)\right]$$

- note: no logarithms, just powers of x and $e^{-1/(2x)}$

Resurgence in the 6 dim. Scalar ϕ^3 Theory

Borinsky, GD, Meynig, 2104.00593

- physically more interesting quantum field theory

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{g}{3!}\,\phi^3 \qquad , \qquad a := \frac{g^2}{(4\pi)^3}$$

- asymptotically free; $d = 6$ critical dimension
- conformal field theory
- known non-perturbative physics: Lipatov instanton
- renormalon-like bubble-chain diagrammatic structures
- multi-component fields $\rightarrow$ percolation, Lee-Yang edge singularity, critical exponents, ...
- β function & anomalous dims computed to 4 loops (Gracey 2015); now 5 loops (Borinsky et al, 2021)

- Broadhurst/Kreimer: 3rd order ODE for anomalous dimension, with quartic nonlinearity

$$a = 8a^3\gamma(a)\left(\gamma(a)^2\gamma'''(a) + \gamma'(a)^3 + 4\gamma(a)\gamma'(a)\gamma''(a)\right)$$
$$+4a^2\gamma(a)\left(2(\gamma(a)-3)\gamma(a)\gamma''(a) + (\gamma(a)-6)\gamma'(a)^2\right)$$
$$+2a\gamma(a)\left(2\gamma(a)^2 + 6\gamma(a) + 11\right)\gamma'(a) - \gamma(a)(\gamma(a)+1)(\gamma(a)+2)(\gamma(a)+3)$$

- formal perturbative solution:

$$\gamma_{\rm Hopf}(a) := \sum_{n=1}^{\infty}(-1)^n \frac{A_n}{6^{2n-1}}a^n$$
$$\sim -\frac{a}{6} + 11\frac{a^2}{6^3} - 376\frac{a^3}{6^5} + 20241\frac{a^4}{6^7} - 1427156\frac{a^5}{6^9} + \dots$$

$$A_n = \{1, 11, 376, 20241, 1427156, 121639250, 12007003824, \dots\}$$

- no known combinatorial interpretation of A_n: OEIS A051862

Large Order Behavior

- Dyson-Schwinger equation factorizes ($G(x) := \gamma(-3x)$):

$$\left[G(x)\left(2\,x\,\frac{d}{dx}-1\right)-1\right]\left[G(x)\left(2\,x\,\frac{d}{dx}-1\right)-2\right]\left[G(x)\left(2\,x\,\frac{d}{dx}-1\right)-3\right]G(x)=3x$$

- formal perturbative series: $G^{\text{pert}}(x) \sim 6\sum_{n=1}^{\infty}\frac{A_n}{12^n}\,x^n$

- factorially divergent large order behavior

$$A_n \sim S_1\,12^n\,\Gamma\left(n+\frac{23}{12}\right)\left(1-\frac{\frac{97}{48}}{\left(n+\frac{11}{12}\right)}-\frac{\frac{53917}{13824}}{\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)}-\frac{\frac{3026443}{221184}}{\left(n-\frac{13}{12}\right)\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)}-\dots\right)+\dots\quad,\quad n\to\infty$$

- 3 missing boundary condition parameters ?

Écalle: formal series → trans-series

- perturbative series $G^{\rm pert}(x)$ has no b.c. parameters

$$G(x) \sim G^{\rm pert}(x) + \sigma\, G^{\rm non-pert}(x)$$

- $G^{\rm non-pert}(x)$ is *beyond all orders* in perturbation theory:

$$G^{\rm non-pert}(x) \sim x^{\beta}\, e^{-\lambda/x}\,(1+O(x))$$

- linearize equation for $G^{\rm non-pert}(x) \longrightarrow 3$ solutions

$$\vec{\lambda} = (1,2,3) \qquad , \qquad \vec{\beta} = \left(-\frac{23}{12}, +\frac{1}{6}, -\frac{11}{4}\right)$$

- recall *generic* trans-series solution:

$$G(x) = G^{\rm pert}(x) + \sum_{\vec{k}\geq 0, |\vec{k}|>0} \sigma_1^{k_1} \sigma_2^{k_2} \sigma_3^{k_3}\, x^{\vec{k}\cdot\vec{\beta}}\, e^{-\vec{k}\cdot\vec{\lambda}/x}\, \mathcal{F}_{\vec{k}}(x)$$

- trans-series parameters, σ_j: 3 "missing" b.c. parameters
- non-generic case: three resonant $\lambda = 1, 2, 3$

Trans-series Analysis

- exponentially graded trans-series

$$G(x) \sim G_{(0)}(x) + e^{-\frac{1}{x}} G_{(1)}(x) + e^{-\frac{2}{x}} G_{(2)}(x) + e^{-\frac{3}{x}} G_{(3)}(x) + e^{-\frac{4}{x}} G_{(4)}(x) + \ldots$$

- $G_{(0)}(x) = G_{\rm pert}(x)$, the formal perturbative series
- equation for $G_{(1)}(x)$ is *linear* and *homogeneous*

$$G_{(1)}(x) = \sigma_1\, x^{-23/12}\, \mathcal{F}_{(1,0,0)}(x) = \sigma_1\, x^{-23/12} \sum_{n=0}^{\infty} a_n^{(1,0,0)} x^n$$

$$\{a_n^{(1,0,0)}\} \quad = \quad \left\{ -1, \frac{97}{48}, \frac{53917}{13824}, \frac{3026443}{221184}, \frac{32035763261}{382205952}, \ldots \right\}$$

- we have seen the $a_n^{(1,0,0)}$ before !

$$A_n^{\rm pert} \sim S_1\, 12^n\, \Gamma\left(n + \frac{23}{12}\right) \left(1 - \frac{\frac{97}{48}}{\left(n + \frac{11}{12}\right)} - \frac{\frac{53917}{13824}}{\left(n - \frac{1}{12}\right)\left(n + \frac{11}{12}\right)} - \ldots \right)$$

- large-order/low-order resurgence relations

Large-Order Behavior of First Fluctuation Series

- large order behavior for $\mathcal{F}_{(1,0,0)}(x) = \sum_{n=0}^{\infty} a_n^{(1,0,0)} x^n$:

$$a_n^{(1,0,0)} \sim 24\, S_1 \Gamma\left(n + \frac{23}{12}\right) \left[1 - \frac{\frac{49}{12}}{\left(n+\frac{11}{12}\right)} - \frac{\frac{13235}{3456}}{\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)} - \frac{\frac{43049}{3456}}{\left(n-\frac{13}{12}\right)\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)} - \frac{\frac{2496477497}{23887872}}{\left(n-\frac{25}{12}\right)\left(n-\frac{13}{12}\right)\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)} - \dots\right]$$

- note: no new Stokes constant
- what is the physics of these subleading coefficients ?

Logarithmic Trans-series Terms

$$G(x) \sim G_{(0)}(x) + e^{-\frac{1}{x}} G_{(1)}(x) + e^{-\frac{2}{x}} G_{(2)}(x) + e^{-\frac{3}{x}} G_{(3)}(x) + e^{-\frac{4}{x}} G_{(4)}(x) + \dots$$

- the equation for $G_{(2)}(x)$ is linear and inhomogeneous

$$G_{(2)} \sim \left(\frac{1}{x^{23/12}}\right)^2 \left[-2\sigma_1^2 \frac{1}{x} \mathcal{F}_{(2,0,0)}(x) + x^4 \left(\sigma_1^2 \frac{21265}{2304} \log(x) + \sigma_2\right) \mathcal{F}_{(0,1,0)}(x)\right]$$

- $\mathcal{F}_{(0,1,0)}(x)$ = second linearized solution ($\lambda = 2$, $\beta = \frac{1}{6}$)
- note appearance of $\log(x)$ term multiplying $\mathcal{F}_{(0,1,0)}(x)$
- $\mathcal{F}_{(2,0,0)}(x)$ is a new fluctuation series

$$\mathcal{F}_{(2,0,0)}(x) \sim 1 - \frac{49}{12} x - \frac{13235}{3456} x^2 - \frac{43049}{3456} x^3 - \frac{2496477497}{23887872} x^4 - 0.x^5 - \frac{3315185066507813}{247669456896} x^6 - \dots$$

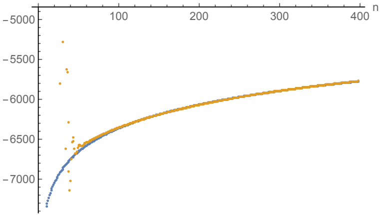
recall:

$$a_n^{(1,0,0)} \sim 24\, S_1 \Gamma\left(n + \frac{23}{12}\right) \left[1 - \frac{\frac{49}{12}}{\left(n + \frac{11}{12}\right)} - \frac{\frac{13235}{3456}}{\left(n - \frac{1}{12}\right)\left(n + \frac{11}{12}\right)} - \frac{\frac{43049}{3456}}{\left(n - \frac{13}{12}\right)\left(n - \frac{1}{12}\right)\left(n + \frac{11}{12}\right)} - \frac{\frac{2496477497}{23887872}}{\left(n - \frac{25}{12}\right)\left(n - \frac{13}{12}\right)\left(n - \frac{1}{12}\right)\left(n + \frac{11}{12}\right)} - \dots\right]$$

- resurgently related to the large order growth of $a_n^{(1,0,0)}$

- corresponding $\log(n)$ in the large-order growth of $a_n^{(1,0,0)}$

$$a_n^{(1,0,0)} \sim 24 S_1 \Gamma\left(n+\frac{23}{12}\right)\left[1-\frac{\frac{49}{12}}{\left(n+\frac{11}{12}\right)}-\frac{\frac{13235}{3456}}{\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)}\right.$$
$$-\frac{\frac{43049}{3456}}{\left(n-\frac{13}{12}\right)\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)}-\frac{\frac{2496477497}{23887872}}{\left(n-\frac{25}{12}\right)\left(n-\frac{13}{12}\right)\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)}$$
$$\left.+\frac{d-2\,\frac{21265}{2304}\log(n)}{\left(n-\frac{37}{12}\right)\left(n-\frac{25}{12}\right)\left(n-\frac{13}{12}\right)\left(n-\frac{1}{12}\right)\left(n+\frac{11}{12}\right)}+\ldots\right]$$



- higher exponential orders $\longrightarrow$ higher powers of $\log(n)$

Logarithmic Large-Order Growth

- suppose you didn't know the ODE, but you just had the original formal perturbative series coefficients A_n^{pert}
- subleading large-order growth of A_n^{pert} defines the new series coefficients $a_n^{(1,0,0)}$
- subleading large-order growth of $a_n^{(1,0,0)}$ tells you about the logs!

"Multi-Instanton" Trans-series from Dyson-Schwinger Equation

$$G(x) \sim G_{(0)}(x) + e^{-\frac{1}{x}} G_{(1)}(x) + e^{-\frac{2}{x}} G_{(2)}(x) + e^{-\frac{3}{x}} G_{(3)}(x) + e^{-\frac{4}{x}} G_{(4)}(x) + \dots$$

- higher exponential orders $\longrightarrow$ higher powers of $\log(x)$
- trans-series for anomalous dimension of 6 dim ϕ^3 QFT

$$G(x) \sim \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} c_{n,k,l}\, x^{n+1-k} \left(\frac{e^{-1/x}}{x^{23/12}} \right)^k (\log(x))^l$$

- the ϕ_6^3 trans-series has the form of a semi-classical trans-series constructed via an infinite sum of non-perturbative instantons, including their fluctuations and interactions

- $\Rightarrow$ this non-perturbative physics is indeed encoded in the perturbative Hopf algebraic formalism

Summary: Resurgence in the 6 dimensional Scalar ϕ^3 Theory

- 3rd order ODE with 4th order non-linearity $\Rightarrow$ much richer non-perturbative structure than the 4 dim Yukawa model

- "resonant resurgence": both large-order behavior and trans-series structure reveal logarithmic effects characteristic of interactions between instantons and anti-instantons

- large order/low order resurgence relations

- non-perturbative information encoded in the formal perturbative series

Conclusions

perturbative Hopf algebra renormalisation

resurgent $\Downarrow$ analysis

non-perturbative completion

$$G(x) \sim G_{(0)}(x) + e^{-\frac{1}{x}} G_{(1)}(x) + e^{-\frac{2}{x}} G_{(2)}(x) + e^{-\frac{3}{x}} G_{(3)}(x) + e^{-\frac{4}{x}} G_{(4)}(x) + \ldots$$

- multi-component fields ? (Gracey, 2015; Giombi et al; Borinsky et al 2021)
- relation with instantons and renormalons ?
- more general Dyson-Schwinger equations ? (Bellon/Rossi 2020)
- 2d σ models, Chern-Simons, SUSY, QED, QCD, ... ?
- big question: does there exist a "natural" Hopf algebraic non-perturbative trans-series structure ?