Matter Coupled Carroll Gravity

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based upon work in progress done with P. Concha, E. Rodríguez and O. Fierro

talk given at the

ESI Thematic Programme

Carrollian Physics and Holography

April 3, Vienna, Austria
Outline

Carroll Scalars
Outline

Carroll Scalars

Carroll Geometry
Outline

Carroll Scalars

Carroll Geometry

Carroll Gravity
Outline

Carroll Scalars

Carroll Geometry

Carroll Gravity

The Compensating Mechanism
Outline

Carroll Scalars

Carroll Geometry

Carroll Gravity

The Compensating Mechanism

Supersymmetry

Outlook
Outline

Carroll Scalars

Carroll Geometry

Carroll Gravity

The Compensating Mechanism

Supersymmetry

Outlook
Outline

Carroll Scalars

Carroll Geometry

Carroll Gravity

The Compensating Mechanism

Supersymmetry

Outlook
Electric Carroll Scalars

Consider the following Lagrangian describing a 4D relativistic real scalar $\Phi$:

$$\mathcal{L} = \frac{\tilde{c}^2}{2} (\partial_t \Phi)^2 - \frac{1}{2} \partial_a \Phi \partial^a \Phi - \frac{M^2}{2\tilde{c}^2} \Phi^2,$$

with mass $M$ and $\tilde{c} \equiv 1/c$. Making the redefinitions

$$\Phi = \frac{\phi}{\tilde{c}}, \quad M = m\tilde{c}^2$$

and taking $\tilde{c} \to \infty$, we obtain the following electric Carroll scalar Lagrangian:

$$\mathcal{L}_{\text{electric scalar}} = \frac{1}{2} (\partial_t \phi)^2 - \frac{m^2}{2} \phi^2$$

Under Carroll boosts: $\partial_a \phi \to \partial_t \phi \to 0$

The electric Carroll particle has non-zero energy but cannot move.
Magnetic Carroll Scalars

de Boer, Hartong, Obers, Sybesma, Vandoren (2021); Henneaux, Salgado-Rebolledo (2021)

We start from the same Lagrangian as in the electric case but written in Hamiltonian form, introducing an auxiliary field $\Pi$, and with an opposite sign of the mass term:

$$\mathcal{L} = \Pi \partial_t \Phi - \frac{1}{2\tilde{c}^2} \Pi^2 - \frac{1}{2} \partial_a \Phi \partial^a \Phi + \frac{M^2}{2\tilde{c}^2} \Phi^2$$

Making the redefinitions

$$\Pi = \pi, \quad \Phi = \phi, \quad M = m\tilde{c}$$

and taking $\tilde{c} \rightarrow \infty$ we obtain the following magnetic Carroll scalar Lagrangian:

$$\mathcal{L}_{\text{magnetic scalar}} = \pi \partial_t \phi - \frac{1}{2} \partial_a \phi \partial^a \phi + \frac{m^2}{2} \phi^2$$

under Carroll boosts: $\pi \rightarrow \partial_a \phi \rightarrow \partial_t \phi \rightarrow 0$

The magnetic Carroll particle can move but has zero energy
When do we have moving Carroll particles?

- considering two interacting Carroll particles or coupling to background fields
  
  Gomis, Longhi + E.B. (2014), see talk by Gomis

- taking a magnetic limit

  de Boer, Hartong, Obers, Sybesma, Vandoren (2021); Henneaux, Salgado-Rebolledo (2021)

- giving up Carroll boosts

  Ciambelli (2023)

- considering two interacting particles using field theory

  Ecker, Grumiller, Henneaux, Salgado-Rebolledo (2024)
  see talk by Henneaux
An Ultratension Limit of String Theory

**Carroll tensionless limit**

Isberg, Lindström, Sundborg, Theodoridis (1994); Bagchi, Chakrabortty, Parekh (2016)

Bidussi, Harmark, Hartong, Obers, Oling (2023)

Taking a worldsheet electric Carroll limit of the Polyakov action together with
\[ T = T_e / \tilde{c}, \]
we obtain in the limit that \( \tilde{c} \to \infty \):
\[ S_{\text{tensionless}} = - T_e \int d^2 \xi \, e \, \tau^\alpha \tau^\beta \, \partial_\alpha X \cdot \partial_\beta X \]

We have effectively sent
\[ T \to 0 \, , \, \tilde{c} \to \infty \, , \, \text{with} \quad T_e \equiv \tilde{c} T \text{ fixed} \]

**Carroll ultratension limit**

cp. to Blair, Lahnsteiner, Obers, Yan (2023); Hohm, Townsend \+ E.B., work in progress

Taking a worldsheet magnetic Carroll limit together with \( T = \tilde{c} T_m \), we obtain
\[ S_{\text{ultratension}} = T_m \int d^2 \xi \, e(\pi \cdot \tau^\alpha \partial_\alpha X + e_1^\alpha e_1^\beta \partial_\alpha X \cdot \partial_\beta X) \]

where we have been sending
\[ T \to \infty \, , \, \tilde{c} \to \infty \, , \, \text{with} \quad T_m \equiv \frac{T}{\tilde{c}} \text{ fixed} \]
Outline

Carroll Scalars

Carroll Geometry

Carroll Gravity

The Compensating Mechanism

Supersymmetry

Outlook
Cartan Formulation of Lorentzian Geometry

Figueroa-O’Farrill, van Helden, Rosseel, Rotko, ter Veldhuis (2023)

The solder form $E_\mu^A$ transforms under infinitesimal local Lorentz transformations with parameters $\Lambda^{AB} = -\Lambda^{BA}$ as

$$\delta E_\mu^A = -\Lambda^A_B E_\mu^B$$

A metric-compatible spin-connection $\Omega_\mu^{AB} = -\Omega_\mu^{BA}$ with torsion $T_{\mu\nu}^A$ satisfies the following first Cartan structure equations:

$$T_{\mu\nu}^A = 2\partial_{[\mu} E_{\nu]}^A - 2\Omega_{\mu}^{AB} E_{\nu}^b$$

In Riemannian geometry we have

1. All spin-connection components can be solved for in terms of the Vierbeine $E_\mu^A$ and the torsion tensors $T_{\mu\nu}^A$:

$$\Omega_\mu^{AB} = \Omega_\mu^{AB}(E, T)$$

2. Each torsion tensor component contains a spin-connection field
Carroll Geometry as a Limit

Using a first-order formulation, the Carroll solder forms ($\tau_\mu, e_\mu^a$) and Carroll spin-connections ($\omega_\mu^{ab}, \omega_\mu^{0a}$) can be obtained by making the following redefinitions:

$$E_\mu^0 = \frac{1}{\tilde{c}} \tau_\mu,$$

$$E_\mu^a = e_\mu^a,$$

$$T_{\mu\nu}^a = t_{\mu\nu}^a,$$

$$\Omega_\mu^{ab} = \omega_\mu^{ab},$$

$$\Omega_\mu^{a0} = \frac{1}{\tilde{c}} \omega_\mu^{a0},$$

$$T_{\mu\nu}^0 = t_{\mu\nu}^0$$

After taking $\tilde{c} \to \infty$ in the first Cartan structure equations we find that $^1$

1. The torsion tensor components $t_{0(a,b)}$ do not contain any spin-connection component

2. The spin-connection components $\omega^{(a,0b)}$ can not be solved for in terms of the solder forms $(\tau_\mu, e_\mu^a)$ and the torsion tensors $t_{\mu\nu}^0, t_{\mu\nu}^a$

The torsion tensor components $t_{0(a,b)}$ are called intrinsic torsion tensors. Setting them to zero leads to geometric constraints

$^1$ We define $X_0 \equiv \tau^\mu X_\mu$, $X_a \equiv e_a^\mu X_\mu$, $X_{0a} \equiv \tau^\mu e_a^\nu X_{\mu\nu}$, $X_{ab} \equiv e_a^\mu e_b^\nu X_{\mu\nu}$
Setting boost-invariant combinations of the intrinsic torsion tensors equal to zero leads to four Carroll geometries:

**Carroll 1:** all intrinsic torsion tensors are non-zero: **no constraints**

**Carroll 2:** $t_{0a}^a = 0$: the 3-form $\Omega \equiv \epsilon_{abc} e_\mu^a e_\nu^b e_\rho^c$ is closed: $d\Omega = 0$

**Carroll 3:** $t_{0\{a,b\}} = 0$: the vector $\tau^\mu$ is a **conformal Killing vector** with respect to the spatial metric $h_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab}$ or **zero extrinsic curvature**

**Carroll 4:** all intrinsic torsion tensors are zero: **both constraints**
### Outline

- Carroll Scalars
- Carroll Geometry
- **Carroll Gravity**
- The Compensating Mechanism
- Supersymmetry
- Outlook
Magnetic Carroll Gravity


Taking the Carroll limit, along with $G_N = G_C/\tilde{c}$, of the EH action with zero torsion

$$S_{1\text{st-order EH}} = \frac{1}{16\pi G_N} \int d^4x \, EE_A^\mu E_B^\nu R_{\mu\nu}^{AB}(\Omega)$$

leads to the following first-order magnetic Carroll gravity action:

$$S_{1\text{st order Carroll grav.}} = \frac{1}{16\pi G_C} \int d^4x \, e \left( e^a_\mu e^b_\nu R_{\mu\nu}^a(J)^{ab} + 2\tau^{\mu}_\nu e^a_\nu R_{\mu\nu}^a(G)^{0a} \right)$$

In this action the spin-connection components $\omega^{(a,0b)}$ only occur linearly. They are therefore independent Lagrange multipliers leading to the geometric constraints:

$$t_{0a,}^a = t_0^{\{a,b\}} = 0 : \text{ Carroll 4 geometry}$$

Hansen, Obers, Oling, Søgaard (2021); Henneaux, Salgado-Rebolledo (2021)
Campoleoni, Henneaux, Pekar, Pérez, Salgado-Rebolledo (2022)

Solving for the other spin-connections leads to 2d-order Carroll gravity
Taking the limit of $\Omega_{\mu}^{AB}$ in a first-order formulation and then pass to a second-order formulation is not the same as taking the limit directly in a second-order formulation:

$$\Omega_{\mu}^{AB}(E) \rightarrow \zeta^2 t_0^{(a,b)} + \omega_{\mu}^{ab}(e), \omega_{\mu}^{a0}(e, \tau)$$

When taking the Carroll limit this leads to electric Carroll gravity

$$S = \frac{1}{16\pi G_C} \int d^4 x e \left( t_0^{(a,b)} t_0^{(a,b)} - t_0^a t_0^b \right)$$

Applying a Hubbard-Stratonovich transformation, the sub-leading terms are given by 2d-order magnetic Carroll gravity

Hartong (2015); Hansen, Obers, Oling, Søgaard (2021)

The electric Carroll gravity action has no first-order formulation!
Conformal Carroll Gravity I

Applying a generalized Hubbard-Stratonovich transformation one may obtain two different electric Carroll gravity theories:

1. **Electric Carroll Gravity (ECG)** is invariant under Carroll transformations:

   \[ S_{\text{ECG}} = \frac{1}{16\pi G_C} \int d^4 x t_{0a}^a t_{0b}^b \]

2. **Conformal Carroll Gravity (CCG)** is invariant under conformal Carroll transformations:

   \[ S_{\text{CCG}} = \frac{1}{16\pi G_C} \int d^4 x e t_0^{\{a,b\}} t_0^{\{a,b\}} \]
Outline

Carroll Scalars

Carroll Geometry

Carroll Gravity

The Compensating Mechanism

Supersymmetry

Outlook
Relativistic Conformal Gravity

The relativistic conformal algebra consists of translations $P_A$, Lorentz transformations $M_{AB}$ plus additional dilatations $D$ and special conformal transformations $K_A$.

Imposing the conformal curvature constraints $\mathcal{R}_{\mu\nu}^A(P) = E^\nu_B \mathcal{R}_{\mu\nu}^{AB}(M) = 0$ allows one to solve for $\Omega_{\mu}^{AB}$ and the special conformal gauge field $f_\mu^A$ while the dilatation gauge field $b_\mu$ transforms with a shift under $K_A \rightarrow b_A$ cancels out!

We wish to couple a compensating scalar field $\Phi$ to conformal gravity as a first step towards constructing general matter couplings:

$$
\delta \Phi = -\Lambda D \Phi \quad \rightarrow \quad D_A \Phi \equiv (\partial_A + b_A)\Phi \quad \rightarrow \\
\delta D_A \Phi = -2\Lambda D D_A \Phi + \Lambda_A^B D_B \Phi + \Lambda_{K A} \Phi \quad \rightarrow \\
D^A D_A \Phi \equiv \partial^A D_A \Phi + 2b^A D_A \phi - \Omega_A^A \partial_A \Phi - f_A^A \Phi
$$

$$
\phi \partial^A \partial_A \phi \quad \Leftrightarrow \quad \Phi D^A D_A \Phi \sim f_A^A \Phi^2 \quad \Leftrightarrow \quad \Phi = 1 \quad \text{or} \quad E^A_\mu \rightarrow \Phi E^A_\mu \quad \Leftrightarrow \quad R(M)
$$

m matter \quad \Leftrightarrow \quad \text{matter} + \text{geometry} \quad \Leftrightarrow \quad \text{geometry}
Conformal Carroll Gravity II

P. Concha, E. Rodríguez and O. Fierro, work in progress; see also Lovrekovic (2022)

cp. to Baiguera, Oling, Sybesma, Søgaard 2022)

The conformal Carroll algebra consists of time/space translations \( H/P_a \), spatial rotations/Carroll boost transformations \( J_{ab}/G_{0a} \) plus additional dilatations \( D \) and vector/singlet special conformal transformations \( K_a/K \)

After imposing the curvature constraints

\[
\mathcal{R}_{\mu\nu}(H) = t_{\mu\nu}^c(a) = \mathcal{R}_{\mu b}^{ab}(J) = \mathcal{R}_{\mu a}^{0a}(G) = 0
\]

except for \( t^0\{a,b\} \neq 0 \) we find that

The gauge fields \((\tau_\mu, e_\mu^a)\) are independent,

The gauge fields \((\omega_\mu^{ab}, \omega_\mu^{0a})\) are dependent except for \( \omega^{(a,0b)} \),

The gauge fields \( b_a \) are independent but can be shifted away by \( K_a \),

\( b_0 \) is solved for by \( b_0 = t_{0a}^a; \) instead \( \omega_a^{,00} \) is shifted away by \( K \),

The gauge fields \((g_\mu^a, f_\mu)\) can both be solved for in terms of \( R(J) \) and \( R(G) \) but only a combination of the two gives Magnetic Carroll Gravity
Coupling CCG to an Electric Scalar

We wish to couple, for \( m = 0 \), an electric Carroll scalar to CCG:

\[
L_{\text{electric scalar}} = -\frac{1}{2} \phi \partial_t \partial_t \phi \quad \text{with} \quad \delta \phi = -\lambda_D \phi
\]

We first replace

\[
\partial_t \phi \rightarrow D_0 \phi \equiv \partial_0 \phi + b_0 \phi
\]

Now use that \( \delta b_0 = \partial_0 \lambda_D \) → \( \delta D_0 \phi = -2\lambda_D D_0 \phi \) →

\[
D_0 D_0 \phi = (\partial_0 + 2b_0)(\partial_0 + b_0)\phi
\]

Now gauge-fix \( \phi = 1 \), use that \( b_0 = t_{0a}^a \) and we obtain Electric Carroll Gravity:

\[
S_{\text{ECG}} \sim \int d^4 x b_0^2 = \int d^4 x t_{0a}^a t_{0b}^b
\]

Electric Carroll Scalar \( \leftrightarrow \) Electric Carroll Gravity
Coupling CCG to a Magnetic Scalar

We now couple, for $m = 0$, a magnetic Carroll scalar to CCG:

$$\mathcal{L}_{\text{magnetic scalar}} = \pi \partial_t \phi + \frac{1}{2} \phi \partial^a \partial_a \phi$$

we first replace $\partial_t \phi \rightarrow D_0 \phi \equiv (\partial_t + b_0) \phi$ and $\partial_a \phi \rightarrow D_a \phi \equiv (\partial_a + b_a) \phi$

Now use that $\delta b_a = \partial_a \lambda_D + \lambda_K a$ →

$$\delta D_a \phi = \lambda^b a D_b \phi + \lambda^0 a D_0 \phi - 2 \lambda_D D_a \phi + \lambda_K a \phi \rightarrow$$

$$D^a D_a \phi = (\partial^a + 2 b^a) D_a \phi - \omega^a_{\ a^0} b D_b \phi - \omega^a_{\ a} D_0 \phi - g^a_{\ a} \phi$$

After partially differentiating in the first three terms, the non-invariance under Carroll boosts is cancelled by assigning $\delta \pi = \lambda^0 a D_a \phi$

The non-invariance of the fourth term under $K$ due to $\delta \omega^a_{\ a^0} = \lambda_K$ is cancelled, after partial differentiating, by adding the term $f_{0} \phi$

Now gauge-fix $\phi = 1 \rightarrow$ magnetic Carroll Gravity with $t_{0a} = t_{0 \{a,b\}} = 0$
Outline

Carroll Scalars

Carroll Geometry

Carroll Gravity

The Compensating Mechanism

Supersymmetry

Outlook
Carroll Fermions

Campoleoni, Fontanella, Mele, Rosseel + E.B. (2023)
See talk by Lea Mele

**Electric Carroll Fermions**

Electric limit is straightforward but electric Carroll fermions do not transform under internal Carroll boosts

**Magnetic Carroll Fermions**

Instead of using a first-order formulation one uses projected fermions and gives the two different projections different scaling weights such that, after taking the limit, one obtains two Carroll fermions $\psi_\pm$ that transform under internal Carroll boosts as a reducible but undecomposable transformation:

$$\psi_- \rightarrow \psi_+ \rightarrow 0$$

There is a non-minimal formulation and a minimal formulation where the fermion kinetic term contains a $\Gamma_5$ or $\Gamma_*$ such that the Dirac operator squares to a tachyonic Klein-Gordon operator.
Warning: supersymmetry rules can contain divergences!

**Electric Supersymmetry**

The commutator of two electric supersymmetry rules in a Carroll Wess-Zumino multiplet gives a time translation

Bagchi, Grumiller, Nandi (2022)

**Magnetic Supersymmetry**?

One can define a Carroll limit of the action plus transformation rules corresponding to a 4D hypermultiplet yielding a finite tachyonic action that is invariant under certain fermionic transformation rules.

However, these fermionic transformation rules do form a supersymmetry algebra!

The solution might be that we start from the non-minimal formulation with complex fields and take a real slice afterwards.


The problem also reminds a bit of constructing de Sitter supergravity that requires the use of nilpotent supermultiplets.
Outline

Carroll Scalars

Carroll Geometry

Carroll Gravity

The Compensating Mechanism

Supersymmetry

Outlook
Summary and Discussion

- we took the first step in constructing general Carroll matter couplings. A natural extension is to include interactions and supersymmetry.

- we pointed out a possible ultra-tension limit of string theory.

- we discussed a puzzle with defining magnetic supersymmetry.

- one may generalize to extended objects and p-brane Carroll limits.

- the analogous story for Galilei Gravity is not completely obvious. What is Electric Galilei Gravity?

  for tachyons, see Batlle, Gomis, Mezincescu, Townsend (2017)