Periods, arithmetic & the Hodge structure for Calabi-Yau manifolds II

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The unreasonable effectiveness of toric geometry: bridging mathematics, computation and string theory

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geometry of families of Calabi-Yau varieties and that of their moduli spaces

arithmetic of families of Calabi-Yau varieties and modularity properties

PERIODS

physics

- black hole solutions in string theory
- scattering amplitudes

etc, etc, ...

Collaborators

- P. Candelas & F Rodríguez-Villegas 2000, 2004
- P. Candelas, D. van Straten & M Elmi 2019
- P. Candelas, D. van Straten 2021
- P. Candelas, J McGovern & P Kuusela 2021-2024
- P. Candelas & P Kuusela 2024
- P. Candelas, D van Straten, N. Gegelia in progress 2026
- P. Candelas & E Svanberg in progress 2026.

1. Calabi-Yau varieties (very brief)

2. Calabi-Yau varieties OVER FINITE FIELDS
   - recap
   - very brief

periods encode arithmetic information

- Exercise: counting points mod p
- Counting points exactly — formula in terms of periods (very briefly!)

3. The attractor mechanism & arithmetic of attractor varieties

   illustrate arithmetic strategy to study the Hodge structure at special points

4. Conclusions

① CALABI-YAU VARIETIES

Mathematical objects of interest:

$\underbrace{\text{algebraic varieties}}$ with certain special properties

set of solutions of

$$P(\varphi, \underline{x}) = 0, \quad \underline{x} \in \mathbb{A}$$

polynomials with complex coefficients φ

Calabi-Yau

· Kähler · $c_1 = 0$

(admit a Ricci-flat metric)

This talk: concerned with $d=3$

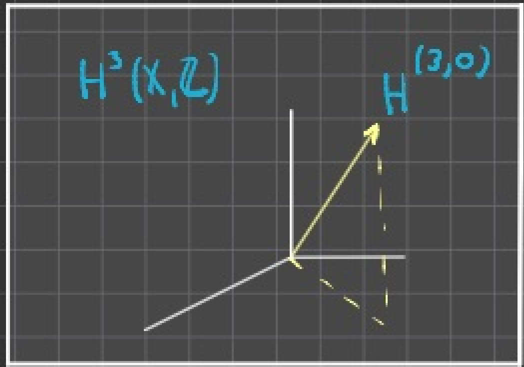
- It is a theorem that $\exists!$ (up to a constant) a nowhere vanishing $(3,0)$-form Ω which is holomorphic ($d\Omega = 0$)

so $h^{(3,0)} = \dim H^{(3,0)} = \dim H^{(0,3)} = 1$

- $H^3 = H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}$

  Hodge struture

  $\dim H^3 = b_3 = 1 + h^{(2,1)} + h^{(1,2)} + 1 = 2(1 + h^{(2,1)})$



- Vary the complex structure
  $\longrightarrow$ Ω moves inside $H^3(X,\mathbb{Z})$ (variations of the HS)
  ($h^{1,2}$ = c-dim moduli space)

Examples: very many!

- $\mathbb{P}^4[5]$ eg $\sum_{i=1}^{5} x_i^5 - 5\psi x_1 x_2 x_3 x_4 x_5 = 0 \quad (x_1, \dots, x_5) \in \mathbb{P}^4$

  $c_1 = 0$

- Verrill 1996, Hulek & Verrill 2005

(quotient of the)

Mirror of a CICY

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad h^{11} = 5$$

quotient

$h^{11} = 1$

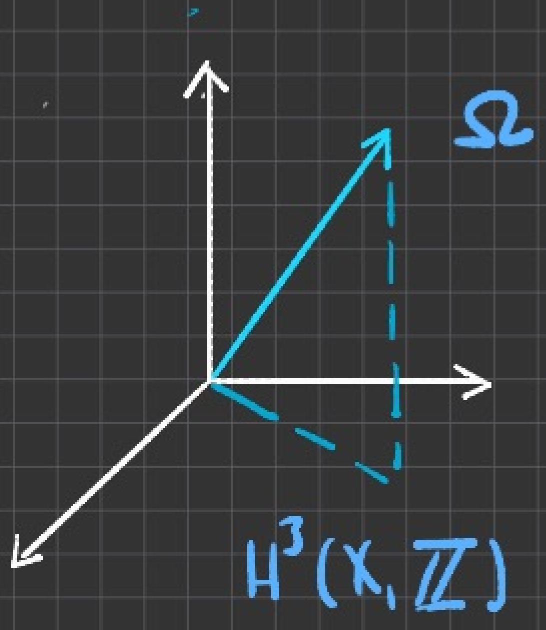
why this example? exhibits interesting arithmetic properties which have an interpretation in BH solutions of string theory (also in amplitudes)

Periods

↳ there is a canonical way to give coordinates on the space of complex structures

- study the variations of the complex structure by studying how Ω varies in H^3

- the coordinates of this line are the **periods** (which then vary as we vary the complex structure)



That is,

$$\Omega(\varphi) = \boxed{z^a(\varphi)}\,\alpha_a - \boxed{\mathcal{F}_a(\varphi)}\,\beta^a$$

where

$\{\alpha^a, \beta_b\} \to$ symplectic basis of $H^3(X,\mathbb{Z})$ $\quad a,b = 0,1,\ldots,h^{2,1}$

$$\int_X \alpha_a \wedge \beta^b = -\int_X \beta^b \wedge \alpha_a = \delta_a^b\,, \quad \int_X \alpha \wedge \alpha = 0\,, \quad \int_X \beta \wedge \beta = 0$$

Let $\{A^a, B_b\}$ the (dual) symplectic basis of $H_3(X,\mathbb{Z})$

Then: $\quad z^a(\varphi) = \int_{A^a} \Omega(\varphi) \qquad \mathcal{F}_a(\varphi) = \int_{B_a} \Omega(\varphi)$

$\{z^a, \mathcal{F}_a\}$ integral basis $\qquad \Omega \to \lambda\Omega$

Bryant & Griffiths: the complex structure is completely determined by the periods $\{z^a\}$ (so $\mathcal{F}_a = \mathcal{F}_a(z)$). The set $\{z^a(\varphi)\}$ are projective coordinates on the moduli space and we have that $\dim_{\mathbb{C}} \mathcal{M}_{cs} = h^{2,1}$

Periods determine $\underbrace{\text{the geometry of the moduli space}}_{\text{special geometry}}$

- Strominger 91
- Candelas & XD 91

physics!

periods also have arithmetic content!!

Periods are calculable: they satisfy a differential eq of degree b_3 (Picard-Fuchs equation)

$$\boxed{\mathcal{L}\,\varpi = 0, \qquad \mathcal{L}\,\mathcal{F} = 0}$$

Focus today 1-parameter examples ($b_3 = 2(1+h_{21}) = 4$)

(Fuchsian)

Intuitively: $\Omega, \Omega', \Omega'', \ldots$ are all closed 3-forms but at most b_3 of these are linearly independent

$\therefore$ there must be a linear relation between the first b_3+1

ie $\exists$ an operator $\mathcal{L}$ st $\mathcal{L}\Omega =$ exact 3-form

- There is a prescription to obtain the PF eq

Dwork and Griffiths.

Gelfand, Kapranov and Zelevinskii $\rightsquigarrow$ GKZ systems

- One parameter families of CY: $b_3 = 4$

$\deg \mathcal{L} = 4$

Solutions around MUM point $\varphi = 0$

$\hookrightarrow (T-1)^4 = 0, \quad (T-1)^3 \neq 0$

$$\varpi_0(\varphi) = \sum_{n=0} a_n \varphi^n = f_0(\varphi) \quad \leftarrow \text{fundamental period}$$

$(f_0 \to 1 \text{ as } \varphi \to 0)$

$$\begin{cases} \varpi_1(\varphi) = f_0(\varphi)\log\varphi + f_1(\varphi) \\ \varpi_2(\varphi) = f_0(\varphi)\log^2\varphi + 2f_1(\varphi)\log(\varphi) + f_2(\varphi) \\ \varpi_3(\varphi) = f_0(\varphi)\log^3\varphi + 3f_1(\varphi)\log^2(\varphi) + 3f_2(\varphi)\log\varphi + f_3(\varphi) \end{cases}$$

f_i are regular series

(Frobenius basis)

Example: (mirror) quintic

$$P(x, \psi) = \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5 .$$

one parameter family

$$\mathcal{L} = \theta^4 - 5\varphi \prod_{i=1}^{4} (\theta + i/5) \qquad \theta = \varphi \frac{d}{d\varphi} \qquad \varphi \equiv (5\psi)^{-5}$$

3 regular singularities
$$\begin{cases} \varphi = 0 & \text{MUM} \\ \varphi = 1 & \text{conifold} \\ \varphi = \infty & \text{Fermat} \end{cases}$$

hypergeometric

Verrill : X_φ is $\boxed{\text{smooth}}$ for $\varphi \neq 1, 1/9, 1/25, 0, \infty$ 1-parameter case

5 singular cases (not hypergeometric !)

$\varphi = 1, 1/9, 1/25$ conifold type

$\varphi = 0$ MUM

$\varphi = \infty$

2 CALABI-YAU VARIETIES OVER FINITE FIELDS

2.1 Zeta functions

Let X_φ be a family of algebraic varieties
st X_φ is a CY hypersurface with
defining polynomial $P(\underline{x}, \varphi)$

Let $\varphi \in \mathbb{F}_q$, $q = p^k$

The fundamental quantities of interest are

$N_k(\varphi)$ = number of solutions of $P(\underline{x}, \varphi) = 0$ over $\mathbb{F}_{p^k}$

Generating function $\rightsquigarrow$ zeta function

$$\zeta_X(T, p; \varphi) = \exp\left\{ \sum_{k=1}^{\infty} \frac{1}{k} N_k(\varphi)\, T^k \right\}$$

depends on both p & φ

properties $\rightsquigarrow$ Weil conjectures

(proven by Dwork, Deligne, Grothendieck)

$$\deg R(T) = b_3$$

$$\zeta_X(T) = \frac{\cancel{R_1(T)} \quad R(T) \quad \cancel{R_5(T)}}{(1-T)\underbrace{(1-pT)^{h^{11}}\,(1-p^2T)^{h^{11}}}(1-p^3T)}$$

smooth CY

(Picard group generated by divisors defined over the ground field $\mathbb{F}_p$)

$b_2 = b_4 = h^{11}$

1-parameter families: $h^{2,1} = 1$, $b_3 = 1+1+1+1 = 4$ so $\deg \mathcal{R} = 4$

$$\mathcal{R}(T) = 1 + a_p T + b_p p T^2 + a_p p^3 T^3 + p^6 T^3$$

$$= \det(\mathbb{1} - T\, U(\varphi)) \qquad (\text{eg, } a_p = -\mathrm{tr}\, U \ , \ \cdots)$$

$$U = \mathrm{Frob}_3^{-1} : H^3(X) \longrightarrow H^3(X)$$

map induced by the Frob automorphism on H^3

4x4 matrix constructed from the periods of Ω

$$\left.\right\} \; U(\varphi) = E^{-1}(\varphi^p)\, U(0)\, E(\varphi) \ ; \ E_{ij}(\varphi) = \vartheta^i\left(\tfrac{1}{j!}\varpi_j\right)$$

$$U(0) = \begin{bmatrix} 1 & & & \\ & p & & \\ & & p^2 & \\ p^3\gamma & & & p^3 \end{bmatrix} \qquad \gamma = \frac{\chi}{g} \zeta_p(3) \qquad \vartheta = \varphi \frac{d}{d\varphi}$$

$\boxed{\text{singular CY}}$: $\deg R(T) < 4$

Note: $N_p = p^3 + h^{11} p^2 + h^{11} p + 1 + a_p$

$$N_{p^2} = p^6 + h^{11} p^4 + h^{11} p^2 + 1 + 2p\, b_p - a_p^2$$

$\boxed{\text{Want to study how } R(T) \text{ varies with } \varphi}$

Need to compute a_p & b_p

Counting points exactly

* $\boxed{u(T) \text{ can be "quickly" computed for } \varphi \in \mathbb{F}_p \; (\varphi = 0, 1, \dots, p-1) \text{ for many primes } p}$

P Candelas, XD & D van Straten (2021) 1 parameter families

P Candelas, XD, P Kuusela (2024)
P Kuusela, M Lathwood, M Mosso Rojas, M Stepniczka, (2026)
multi-parameter families

based on the deformation method developed first by Dwork & Lauder

* formulas for point counts ← next

Many questions arise: certainly ready for substancial experimentation.

- One can construct L-functions

1 parameter families

$$L(s,\varphi) = \text{"}\prod_p \underbrace{R\,(p^{-s},\varphi)}_{\text{spinor zeta function } w=3}^{-1}\text{"}$$

What are the properties of this L-function?

- functional equation
- formula for the conductor
- modularity $\longleftrightarrow$ paramodular forms of $\Gamma(N) \subseteq Sp(4)$ for 1-parameter cases

Langlands programme

[G Tornaria, A Pacetti, J. Voight, V. Golyshev ....

P Candelas, XD, N Gegelia, D van Straten, in progress]

2.2 AN EXERCISE: COUNTING POINTS MOD P

- Fermat's little theorem $\quad a^p \equiv a$

$$\Rightarrow \quad a\,(a^{p-1}-1) \equiv 0 \quad \Longrightarrow \quad a^{p-1} \equiv \begin{cases} 1 & a \neq 0 \\ 0 & a = 0 \end{cases}$$

Then: $$1 - P(\underline{x},\varphi)^{p-1} \equiv \begin{cases} 1, & \forall\, \underline{x} \text{ st } P(\underline{x},\varphi) = 0 \\ 0, & \forall\, \underline{x} \text{ st } P(\underline{x},\varphi) \neq 0 \end{cases}$$

$\therefore$ $$\nu_p(\varphi) \equiv \sum_{\underline{x} \in \mathbb{F}_p^n} \left(1 - P(\underline{x}, \varphi)^{p-1}\right)$$

n-varibles

Chevalley, Warning 1935

$\vdots$

P. Candelas, XD, F Rodriguez-Villegas 2000, 2004

$\vdots$

E. Panzer, F Brown . . .

(mirror) quintic:

$$\nu_p(\varphi) \equiv \sum_{\underline{x} \in \mathbb{F}_p^5} \left\{ 1 - \left(\sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 \cdots x_5 \right)^{p-1} \right\}$$

expand (multinomial ...)

$$\equiv \sum_{m=0}^{[p/5]} a_m \varphi^m \equiv {}^{[p/5]}\varpi_0(\varphi)$$

truncated fundamental period

$$\sum_{a \in \mathbb{F}_p} a^n \equiv \begin{cases} 0\,, & p-1 \nmid n \\ -1\,, & p-1 \mid n \end{cases}$$

- result extends to all hypersurfaces in toric varieties and $h^{21} \geq 1$

- Want to count points exactly !

- Igusa 1958 class of elliptic curves
- mirror quintic (P Candelas, XD, F Rodriguez Villegas)
- many other examples eg S.Kadir 2-parameter CY ~2004
  F Rodriguez V + D Roberts HGM in 2024

K3 surfaces
- C Doran, T Kelly, A Salerno, S Sperber, J Voight, U Whitcher 2018, 2020
- R Davis, J Dukes, T Gomis-Ribeiro, E Orvis, A Salerno, L Sturman, U. Whitcher 2025

F Beukers & M Vlasenko; M Kerr 2020-2025

- In progress with P. Candelas & E Svanberg

2.3

COMPUTING THE NUMBER OF POINTS EXACTLY

$\nu_p(\varphi)$ is a definite number: we seek to compute it exactly

We can expand

$$\nu_p(\varphi) = \nu_p^{(0)}(\varphi) + p\, \nu_p^{(1)}(\varphi) + p^2\, \nu_p^{(2)}(\varphi) + p^3\, \nu_p^{(3)}(\varphi) + \dots$$

with $0 \leq \nu_p^{(i)} \leq p-1$

and evaluate $\bmod p^2$, $\bmod p^3$, ...

This leads to p-adic analysis!

Counting points exactly:

In progress with P Candelas & E Svanberg: schematically

primitive part of N_p

truncated f-series

$$= \sum_{\underline{\ell}, \underline{j} = \underline{0}}^{\infty} \frac{h_{\underline{\ell}, \underline{j}}}{\underline{\ell}!} \left(\frac{p}{1-p} \underline{\vartheta} \right)^{\underline{\ell} + \underline{j}} {}^{[p-2]}f^{\underline{\ell}}(\varphi)$$

$\underline{\ell}, \underline{j} = (\ell_1, \ldots, \ell_n)$ $\qquad \underline{\vartheta} = \vartheta_1 \vartheta_2 \cdots \vartheta_n \qquad \vartheta_i = \varphi_i \frac{\partial}{\partial \varphi_i}$

mirror quintic

$$= {}^{[p-2]}f_0(\varphi) + \left(\frac{p}{1-p}\right) {}^{[p-2]}f_1'(\varphi) + \frac{1}{2}\left(\frac{p}{1-p}\right)^2 {}^{[p-2]}f''(\varphi) + \cdots$$

[also in F Beukers & M Vlasenko; M Kerr]

③ THE ATTRACTOR MECHANISM

(Ferrara, Kallosh 95, ... Greg Moore 98 .....)

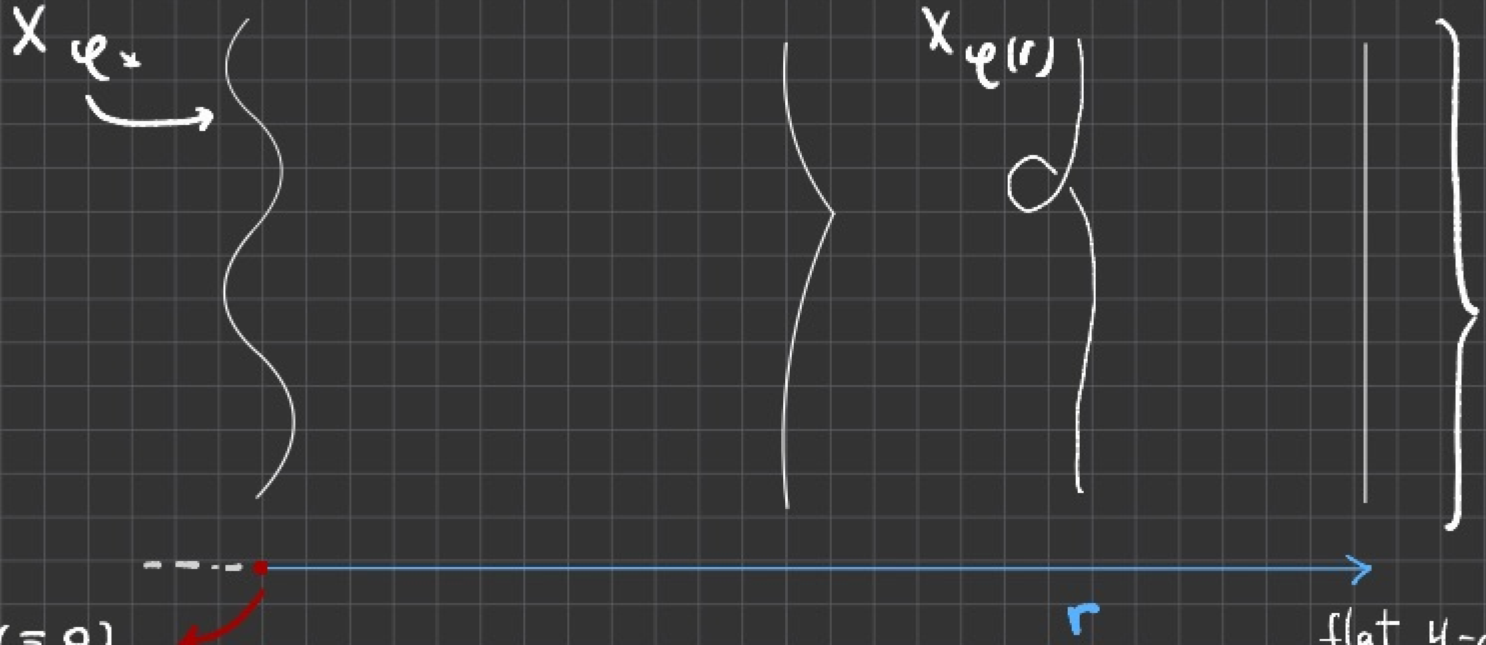
Physics : supersymmetric black hole solutions of type IIB superstrings

10 dimensional

include { Einstein's equations for gravity
+
Maxwell equations for electromagnetism

10 dim space-time: a CY, $X_{\varphi(r)}$, at each point of 4 dim space-time

4 dim space-time = spherically symmetric asymptotically flat charged BH



CY with complex structure φ varying with r

horizon $(r = 0)$

(coordinate singularity)

$u(r) \to -\infty$

flat 4-dim space-time

$r \to \infty$ $(u(r) \to 0)$

last surface visible from infinity

(surface separating interior & exterior of the BH)

$$ds^2 = e^{2u(r)}dt^2 + e^{-2u(r)}d\underline{x}^2, \quad \underline{x} = (x, y, z)$$

metric dependent only on the radial coordinate r

Type IIB supergravity contains gravity but also extra $U(1)$ gauge fields (b_3 of them)

So: the BH has electric & magnetic charges

$$Q = \begin{pmatrix} q_a \\ p^b \end{pmatrix} \qquad a, b = 0, 1, \ldots, h^{2,1}$$

these must all be integers

Let $\Gamma = p^a \alpha_a - q_a \beta^a \in H^3(X, \mathbb{Z})$

charge vector

(α_a, β^b) symplectic basis of $H^3(X, \mathbb{Z})$

Black hole solutions which preserve supersymmetry need to satisfy 1st order differential eqs for $U(r)$ & $\varphi(r)$. These are the attractor equations which represent a non-linear dynamical system on the $\mathbb{C}$-structure moduli space with flow parameter $\rho = 1/r$.

Given a choice of $\Gamma \in H^3(X, \mathbb{Z})$ one can prove using the attractor eq that

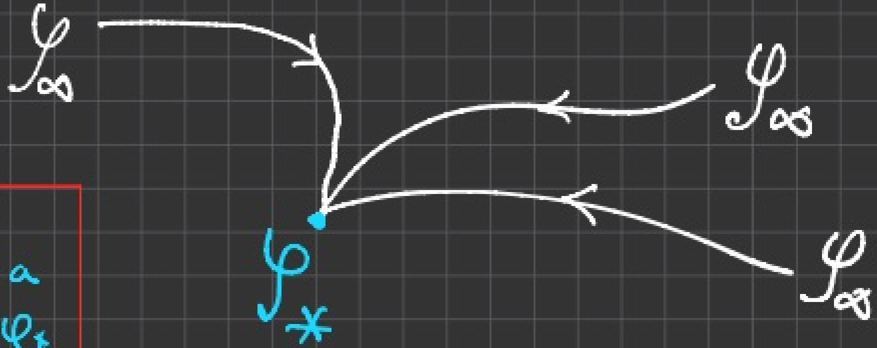
- the $\mathbb{C}$-structure parameters flow to a value $\varphi_* = \varphi(r=0)$ where $|Z_\gamma| = |e^{K/2} \int_\gamma \Omega|$ reaches a minimum, and it is independent of the starting value $\varphi_\infty = \varphi(r=\infty)$

95: Ferrara + Kallosh

. . . . .

98: G. Moore
conjectures on the arithmetic nature of attractor varieties

$\hookrightarrow X_{\varphi_*} = X_*$



φ evolves smoothly to a fixed point φ_* at $r=0$

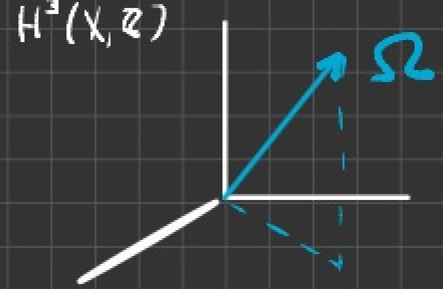
▶ Moreover,

the Q-structure at an attractor point $\varphi = \varphi_*$ is st

$$\boxed{\begin{gathered} \Gamma = p^a \alpha_a - q_a \beta^a \quad \in \quad H^{(3,0)} \oplus H^{(0,3)} \\ \text{ie} \quad \Gamma^{(1,2)} = \Gamma^{(2,1)} = 0 \end{gathered}}$$

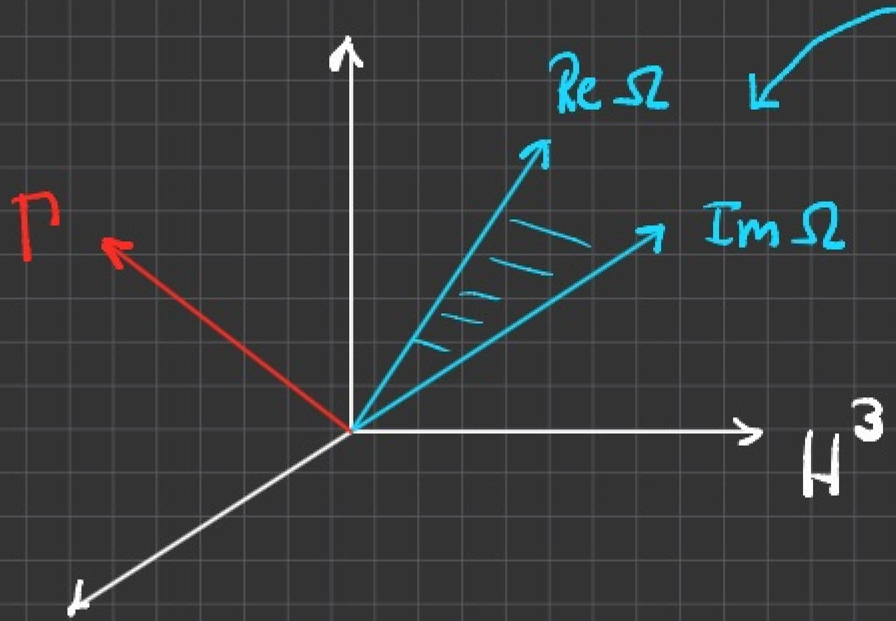
proof: exercise in special geometry

rank 1 attractors



Recall: Ω defines a line in $H^3(X,\mathbb{C})$

Ω moves inside $H^3(X,\mathbb{R})$ as we change the CS



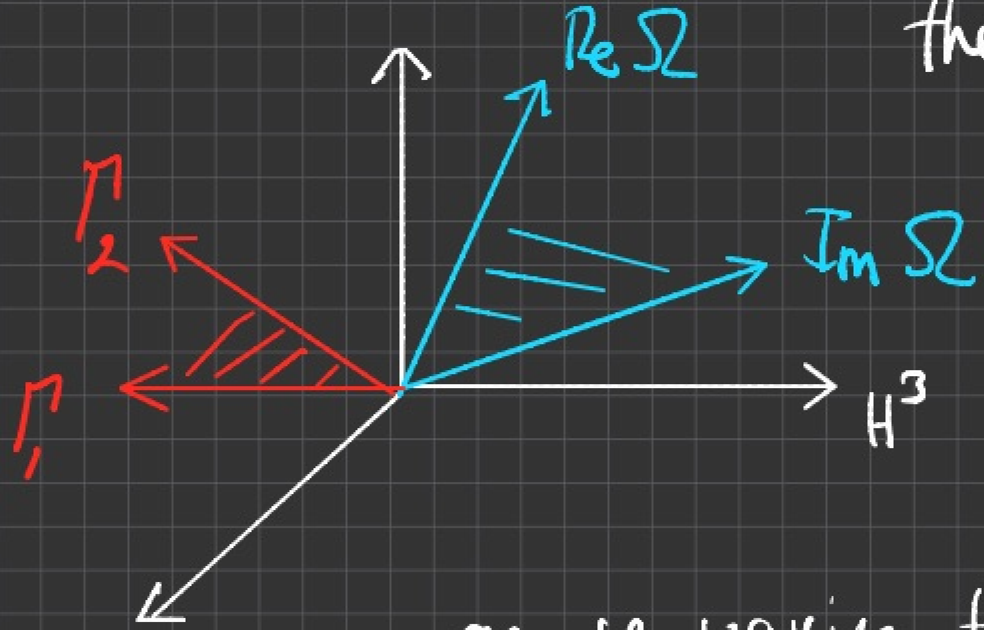
Consider $V_{\mathbb{R}}(\varphi)$ = plane spanned over $\mathbb{R}$ by $\mathrm{Re}\,\Omega$ & $\mathrm{Im}\,\Omega$

$V_{\mathbb{R}}(\varphi)$ moves with φ

OTOH: Inside $H^3(X,\mathbb{R})$ we have a lattice of vectors $\Gamma \subset H^3(X,\mathbb{Z})$ which are fixed

rank 1 A.P: φ st $V_{\mathbb{R}}(\varphi)$ contains the line Γ

$\boxed{\text{rank 2}}$ attractors



at an attractor point of rank 2 there are $\boxed{\text{two}}$ vectors in $H^3(X,\mathbb{Z})$

Γ_1, Γ_2 st $\Gamma_{1,2} \in H^{(3,0)} \oplus H^{(0,3)}$

as φ varies the plane $V_{\mathbb{R}}(\varphi)$ moves and at a rank 2 attractor point $\varphi = \varphi_*$ the plane $V_{\mathbb{R}}(\varphi)$ coincides with the plane generated by Γ_1 & Γ_2

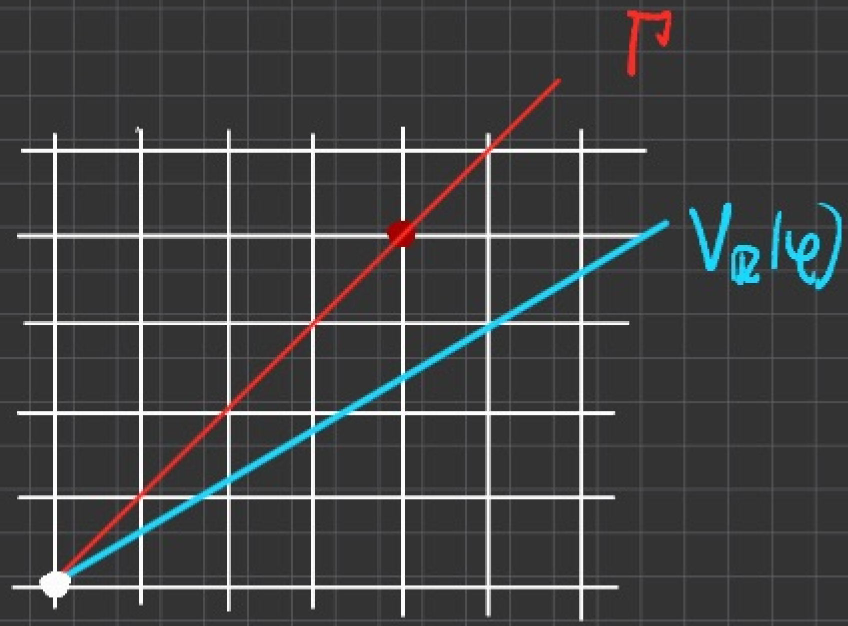
RARE

very difficult to find a CY which has rank 2 attractor points

Why?

A line which passes through the origin in general will <u>not</u> pass through another lattice point unless the slope is rational.

Not too hard to find φ st $V_{\mathbb{R}}(\varphi)$ coincides with Γ



For rank 2 attractors we have a plane and it is then <u>much harder</u> to find φ st it coincides with $V_{\mathbb{R}}(\varphi)$

Geometrically: at $\varphi = \varphi^*$

$\Gamma_1, \Gamma_2 \in \Lambda := V_{\mathbb{R}}(\varphi) \cap H^3(X, \mathbb{Z})$ is a lattice plane

<u>and</u> $\Lambda \otimes \mathbb{C} = H^{(3,0)} \oplus H^{(0,3)}$

Also: $\Lambda^{\perp} \subset H^3(X, \mathbb{Z})$ lattice orthogonal to Λ

(under the natural symplectic product on 3-forms)

satisfies $\Lambda^{\perp} \otimes \mathbb{C} = H^{(2,1)} \oplus H^{(1,2)}$

We obtain an isomorphism

$$\boxed{H^3(X, \mathbb{Q}) = \Lambda_{\mathbb{Q}} \oplus \Lambda^{\perp}_{\mathbb{Q}}}$$

Λ has finite index so extends to $\mathbb{Q}$

splitting of the Hodge structure $H^3(X, \mathbb{Q})$!

Hodge conjecture $\Rightarrow$ splitting has a geometrical origin

$\boxed{\text{VERY HARD}}$ $\rightsquigarrow$ open question

Problem: how do we find varieties with rank 2 attractor points?

However: the splitting becomes apparent in the arithmetic structure of X

$\therefore$ arithmetic strategy !

P. Candelas, M. Elmi
& D van Straten, 2021

P Candelas, XD, D van Straten April 2021

K Bönish + A Klemm + Scheidegger + Zagier 2022
K. Bönish + M Elmi + A.K. Kashani-Poor + A Klemm 2022

Candelas, XD, J McGovern, P Kuusela 2021 & Feb 2023

P Candelas, XD, P Kuusela 2024

4 Arithmetic of attractor varieties

$\boxed{\text{At } \varphi = \varphi_*}$

Recall $H^3(X_*, \mathbb{Z}) = \underbrace{V}_{(3,0)+(0,3)} \oplus \underbrace{V^\perp}_{(2,1)+(1,2)}$

$\mathcal{U} \rightsquigarrow$

$$R(T) = \det(\mathbb{1} - T\,\mathcal{U})$$

$$= \underbrace{(1 - p\alpha_p T + p^3 T^2)}_{H^{2,1} \oplus H^{1,2}} \underbrace{(1 - \beta_p T + p^3 T^2)}_{H^{3,0} \oplus H^{0,3}}$$

$\boxed{\text{factors over } \mathbb{Z} \ \forall p}$

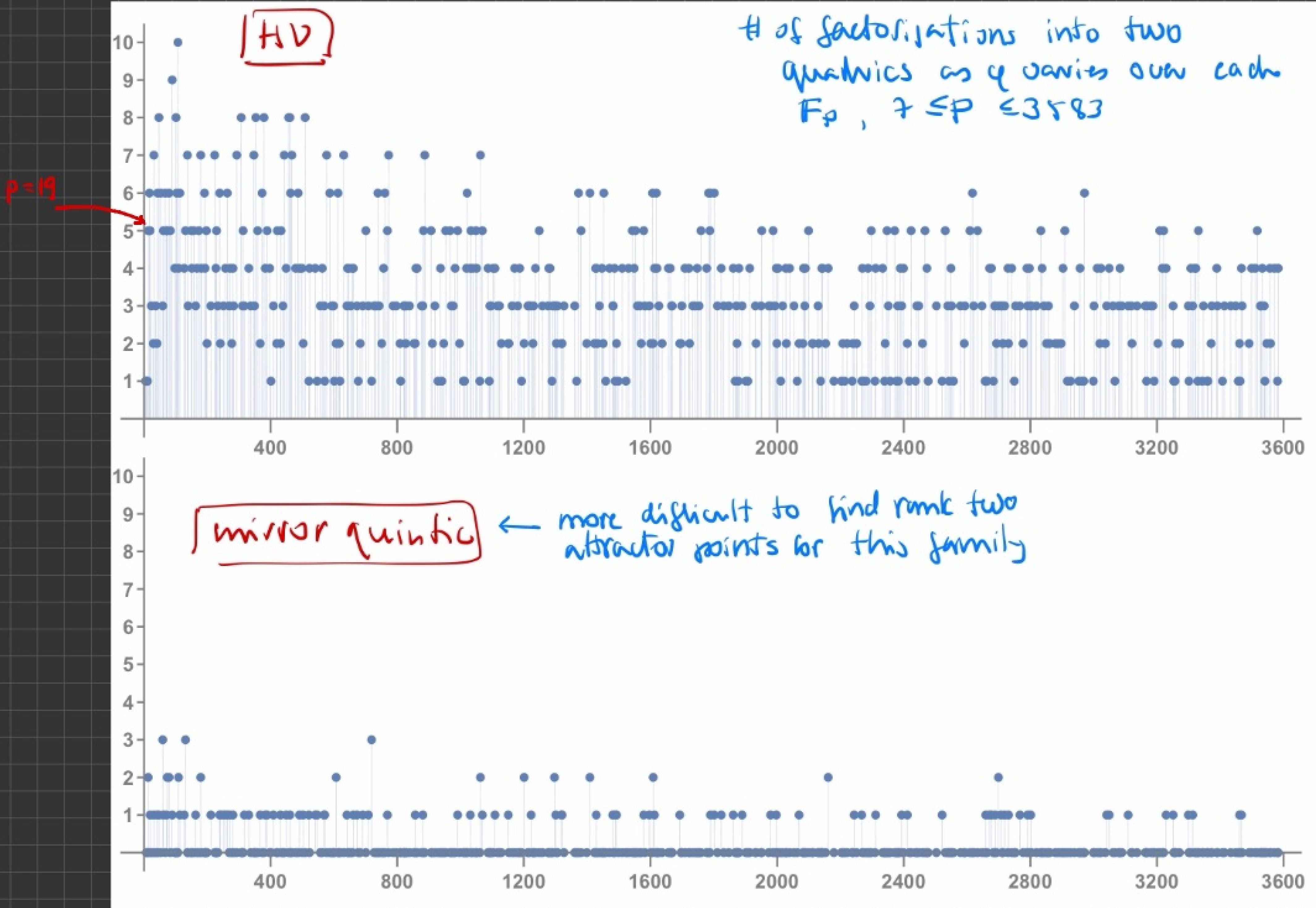
Moreover: expect α_p, β_p to be coeffs of modular forms
(Tate & Serre conjectures)

Arithmetic strategy:

make tables of $R(T,\varphi)$ for many p & φ and look for persistent factorisations of R into two quadrics

factorisations occuring when φ_* is a root of a polynomial with integer coeffs

(P. Candelas, XD, A Thorne, D v Straten 2018
P. Candelas, XD, D v Straten 04/2021)


HV
# of factorisations into two quadrics as ψ varies over each $\mathbb{F}_p$, $7 \le p \le 3583$
p=19
mirror quintic
← more difficult to find rank two attractor points for this family
400
800
1200
1600
2000
2400
2800
3200
3600

We find that for the <u>HV manifold</u> there is always a factorisation when (P Candelas, XD, M Elmi & D van Straten)

- $7\varphi+1=0$: $\varphi=-1/7$ $(\forall p\neq 7)$

and

- $\varphi^2-66\varphi+1=0$: $\varphi_\pm = 33\pm 8\sqrt{17}$

(exists in $\mathbb{F}_p$ when 17 is a square modulo p)

For $p=19$ (say)

$$\varphi=-\frac{1}{7}\equiv 8 \,, \qquad \varphi_\pm \equiv 4,5$$

$(17\equiv 6^2)$

Remark: X_{φ_*} is <u>smooth</u>

	$p = 19$			
	φ	smooth/sing.	singularity	$R(T)$
conifold	1	singular	1	$(1 - pT)(1 - 20T + p^3T^2)$
	2	smooth		$1 + 4pT + 2pT^2 + 4p^4T^3 + p^6T^4$
	3	smooth		$1 - 8T + 242pT^2 - 8p^3T^3 + p^6T^4$
φ_+	4	smooth		$(1 + 4pT + p^3T^2)(1 - 60T + p^3T^2)$
φ_-	5	smooth		$(1 + 4pT + p^3T^2)(1 - 60T + p^3T^2)$
	6	smooth		$1 + 8T - 318pT^2 + 8p^3T^3 + p^6T^4$
	7	smooth		$1 - 44T - 238pT^2 - 44p^3T^3 + p^6T^4$
$\varphi = -1/7$	8	smooth		$(1 - 2pT + p^3T^2)(1 - 80T + p^3T^2)$
	9	smooth		$(1 + 4pT + p^3T^2)(1 - 160T + p^3T^2)$
	10	smooth		$1 + 12T + 562pT^2 + 12p^3T^3 + p^6T^4$
	11	smooth		$(1 + 4pT + p^3T^2)(1 - 140T + p^3T^2)$
	12	smooth		$1 + 12T + 82pT^2 + 12p^3T^3 + p^6T^4$
	13	smooth		$1 + 178T + 1082pT^2 + 178p^3T^3 + p^6T^4$
	14	smooth		$1 + 12T - 158pT^2 + 12p^3T^3 + p^6T^4$
	15	smooth		$1 + 42T - 2p^2T^2 + 42p^3T^3 + p^6T^4$
conifold	16	singular	$\frac{1}{25}$	$(1 - pT)(1 + 76T + p^3T^2)$
conifold	17	singular	$\frac{1}{9}$	$(1 - pT)(1 - 20T + p^3T^2)$
	18	smooth		$1 - 54T + 322pT^2 - 54p^3T^3 + p^6T^4$

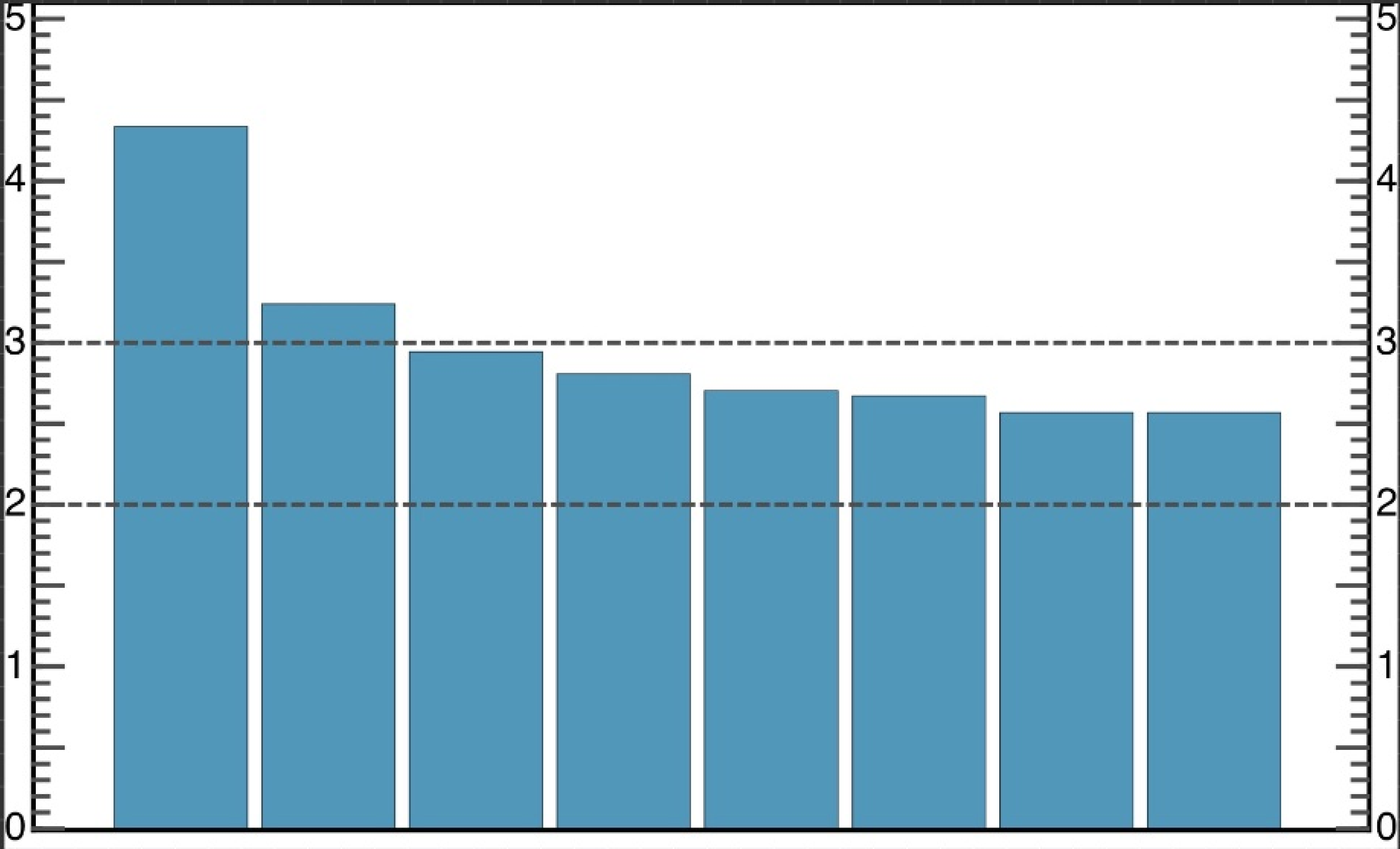
Table 1: *The R-factors for $\varphi \in \mathbb{F}_{19}$. Note the factorisations into two quadrics for the five values $\varphi = 4, 5, 8, 9, 11$.*

Corollary to the Chebotarëv-theorem

Over $\mathbb{F}_p$, an irreducible polynomial $f(x)$ with integer coefficients has on average one root (over $\mathbb{F}_p$)

$\Rightarrow$ if $G(x)$ has k irreducible factors it has on average k roots

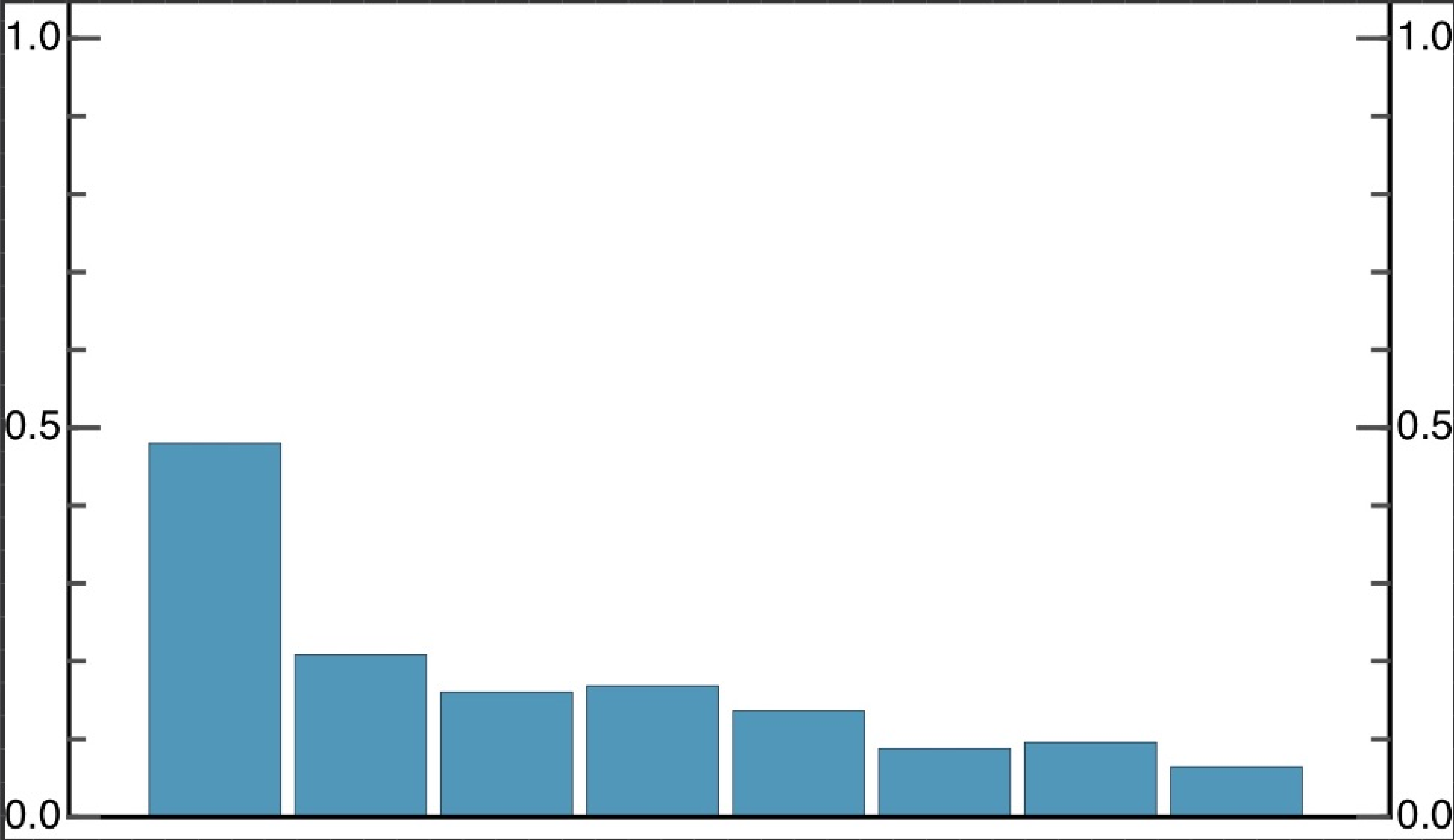
For Verrill's manifold



$\underbrace{\hspace{2em}}_{125}$

→ 1000 primes

For the mirror quintic



There is more information in the tables:

there are modular forms

$$R(T) = (1 - p\,\alpha_p T + p^3 T^2)(1 - \beta_p T + p^3 T^2)$$

Serre's conjecture (generalizing Taniyama-Weil)

"motives" of length two are modular

algebraically defined part of cohomology

[proof: Dieulefait, Khare & Wintenberger, Kisin]

For $\varphi = -1/7$

α_p & β_p are Fourier coefficients of a modular form for $\Gamma_0(14)$

LMFDB

$f_2 = \sum_n \alpha_n q^n$ weight 2 14.2.a.a

$f_4 = \sum_n \beta_n q^n$ weight 4 14.4.a.a

$q = e^{2\pi i \tau}$ $\tau \in \mathbb{H}$

Similarly, for $\varphi_{\pm}$

α_p & β_p $\longrightarrow$ coeffs of modular forms for $\Gamma_1(34) \subset \Gamma_0(34)$

$$SL(2,\mathbb{Z}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod 34$$

$f_2 \longrightarrow$ 34.2.b.a

$f_4 \longrightarrow$ 34.4.b.a

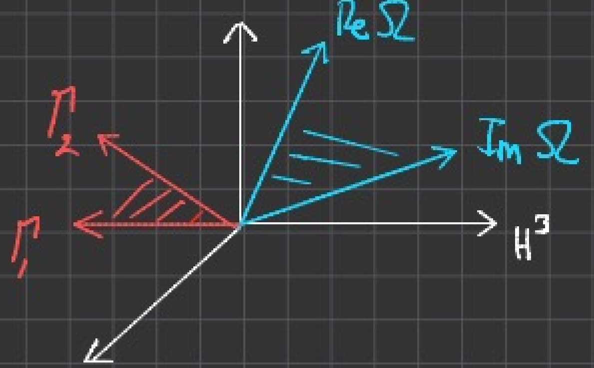
Associated with these modular forms are L-functions

Mellin transform of f

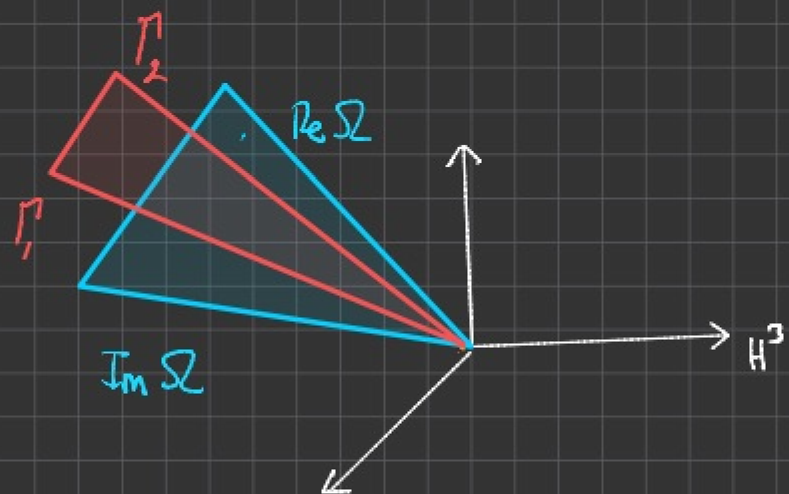
$$L_w(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty dy\, y^{s-1} f_w(iy) \qquad w = 2,4$$

At φ_* the periods of Ω are given by simple rational multiples of $L_4(1)$ and $L_4(2)$

[$L_w(s, f_w)$ critical values at $s = 1, 2, \dots, w-1$
Deligne: critical values of L related to periods of modular CYs]



$\varphi \to \varphi_*$



For example

$$\Pi\left(-\frac{1}{7}\right) = i\,\frac{L_4(1)}{4\pi}\begin{pmatrix} 8K \\ -30K \\ 0 \\ 5 \end{pmatrix} + \frac{7}{2}\,\frac{L_4(2)}{\pi^2}\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

The two integral vectors define a lattice plane

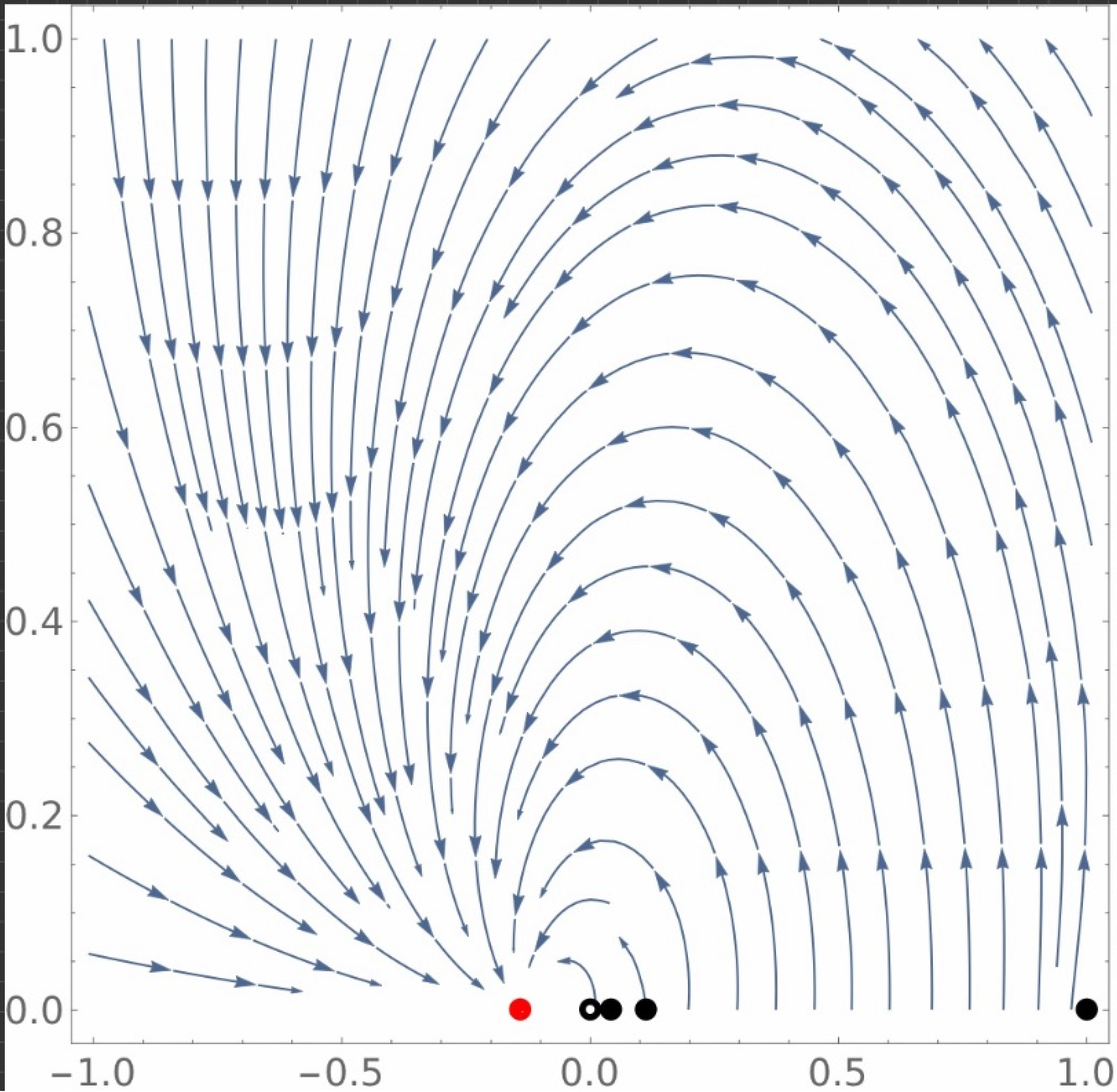
$\Rightarrow$ a two dimensional family of charge vectors.

$$K = \begin{cases} 1 & \text{Verrill}/G_1 \quad h^{21}=1 \quad h^{11}=9 \\ 2 & \text{Verrill}/G_1\times G_2 \quad h^{21}=1 \quad h^{11}=5 \end{cases}$$

$$P = \left(\sum_{i=1}^{5} X_i\right)\left(\sum_{i=1}^{5}\frac{a_i}{X_i}\right) - \varphi \qquad (X_1,\dots,X_5)\in\mathbb{P}^4,\ \prod X_i \neq 0$$

$$G_1 : X_i \longmapsto X_{i+1} \quad \mathbb{Z}_5 \qquad G_2 : X_i \longmapsto \frac{a_i}{X_i} \quad \mathbb{Z}_2$$

$\varphi = -\frac{1}{7}$



Area of the horizon of the BH

$\boxed{\varphi = -1/7}$

Charges $Q_{k\ell} = k(4,-15,-5,0) + \ell(0,0,2,1)$

(so a two parameter family of BHs)

let $\quad V_* = \dfrac{7}{\pi} \dfrac{L_4(2)}{L_4(1)}$ $\quad L_4 \to$ L-function associated to f_4

$L_4(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty dy\, y^{s-1} f_4(iy)$

Then

$$\boxed{A_{\ell k}(\varphi_*) = 4\pi |Z_\gamma(\varphi_*)|^2 = 14\pi \left\{ k^2 V_* + \left(\ell - \frac{5k}{2}\right)^2 \frac{1}{V_*} \right\}}$$

↳ BH entropy

What is the precise meaning of this?

Area of the horizon of the BH

$$\boxed{\varphi_\pm = 33 \pm 8\sqrt{17}}$$

φ_+ $\qquad Q_{kl} = k(4,-9,7,4) + l(4,-30,-30,-5)$

φ_- $\qquad Q_{kl} = k(-2,0,0,5) + l(0,3,1,0)$

$$v_* = \frac{17}{2}\left(\frac{9-\sqrt{17}}{2}\right)\frac{\lambda_4(2)}{\pi\,\lambda_4(1)} \qquad (\lambda_4 = \mathrm{Re}\, L_4)$$

$$A_{kl}(\varphi_\pm) = 34\pi\left(\frac{k^2}{v_*} + l^2 v_*\right)$$

Outlook

- arithmetic strategy to study the Hodge structure
  - Hope to understand HS better eg using the formulas for $N_q(\varphi)$ explicitly in terms of the periods
  - attractor mechanism ⇝ still open questions
    - the meaning of the formula for the area of the BH solns.
    - what is the property of a given family of CYs that allows for rank 2 attractor points in its moduli space
    - rigorous proof?

      ⇝ find the geometric reason for the splitting of the Hodge structure

      ⇝ Dommigan 2026 proof for HV $\varphi = -1/7$
    - other splittings / degenerations?

- IIB [X] flux vacua

Conjecture: (Kachru, Nally, Yang) A CY variety X_φ with CS φ which gives a supersymmetric flux vaccum is modular in the sense that it is associated to a modular form of $w=2$

(..... Candelas, XD, McGovern, Kuusela 2023)

▶ Modularity of CY varieties?

1 parameter families $\leftrightarrow$ paramodular forms of $\Gamma(N) \subseteq Sp(4)^\vee$

$\leadsto$ at certain values of the parameter this might "degenerate" into known modular forms of $\Gamma_0(N) \subseteq SL(2,\mathbb{Z})$

THANKS!