The BV formalism for finite spectral triple: towards the quantum case

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Non-commutative geometry meets topological recursion

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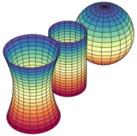
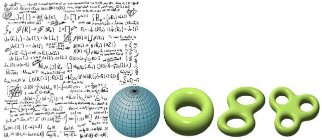
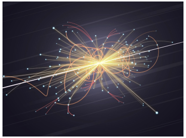
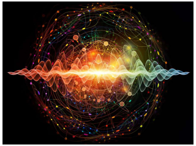
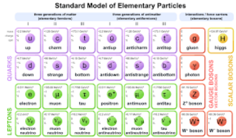










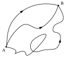
The BV formalism: a cohomological approach to gauge theory

Def. A gauge theory $(X_0, S_0, \mathcal{G})$ consists of a configuration space X_0, an action functional $S_0 : X_0 \to \mathbb{R}$, and a group $\mathcal{G}$ acting on X_0 such that $S_0(g \cdot \varphi)) = S_0(\varphi)$, $\forall \varphi \in X_0, \forall g \in \mathcal{G}$.


Gravity


Quantum interactions

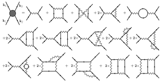
Context: quantization of a theory (X_0, S_0) via a perturbative approach to path integral


path integral

$$Z := \int_{X_0} e^{\frac{i}{\hbar} S_0} [d\mu]$$

partition function

$$\underset{\hbar \to 0}{\sim} \sum_{x_0 \in \{\text{crit. pts } S_0\}} e^{\frac{i}{\hbar} S_0(x_0)} \, |\det S_0''(x_0)|^{-\frac{1}{2}} \, e^{\frac{\pi i}{4} sign(S_0''(x_0))} (2\pi\hbar)^{\frac{\dim X_0}{2}} \sum_{\Gamma} \frac{\hbar^{-\chi(\Gamma)}}{|Aut(\Gamma)|} \Phi_{\Gamma}.$$


Γ Feynman diagrams

Problem: for gauge theories, the critical points appear in *orbits*

Solution: to add extra variables, called ghost fields

$$\langle g \rangle := \int_{X_0} g e^{\frac{i}{\hbar} S_0} [d\mu]$$

$$\langle \mathcal{O} \rangle := \int_{\mathcal{L} \subset \mathcal{F}_{BV}} \mathcal{O} e^{\frac{i}{\hbar} S_q|_{\mathcal{L}}} \sqrt{\mu|_{\mathcal{L}}}$$

- $\mathcal{L}$ Lagrangian $\subset \mathcal{F}_{BV}$ ghost sector
- $S_q \in \mathcal{C}^\infty(\mathcal{F}_{BV})[[-i\hbar]]$, sol. quant. master eq.
- $\mathcal{O} \in \mathcal{C}^\infty(\mathcal{F}_{BV})[[\hbar]]$, quantum BV cocycle

The classical/quantum BV construction

(X_0, S_0) (initial gauge theory) $\xrightarrow{+\text{ gh./anti-gh.}}$ $(\tilde{X}, \tilde{S}/S_q)$ (BV-extended th.) $\xrightarrow{+\text{ aux. flds}}$ $(X_t, S_t/S_{q,t})$ (total th) $\xrightarrow{\text{gauge-fixing}}$ $(X_t, S_t/S_{q,t}|_{\mathcal{L}})$ (gauge-fixed th.)

$\downarrow$

$\mathcal{C}^\bullet_{BV}(\tilde{X}, d_{\tilde{S}/S_q})$ (BV complex) $\cong$ $\mathcal{C}^\bullet_{BV}(X_t, d_{S_t/S_{q,t}})$ (total complex), $\mathcal{C}^\bullet_{BRST}(X_t, d_{S_t/S_{q,t}})|_{\mathcal{L}}$ (BRST complex)

The BV formalism for spectral triples: towards the quantum case

spectral triple → gauge theory

$(\mathcal{A}, \mathcal{H}, D)$ → $(X_0, S_0, \mathcal{G})$

- $\mathcal{A}$ = unital *-alg., $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$
- $\mathcal{H}$ = Hilbert space
- $D : \mathcal{H} \to \mathcal{H}$ = self-adj. op.

- $X_0 = \{\varphi = \sum_j a_j[D, b_j] : \varphi^* = \varphi\}$ $\rightsquigarrow$ conf. sp = inner fluctuations
- $S_0[D+\varphi] = Tr(f(D+\varphi))$ $\rightsquigarrow$ action func. = spectral action
- $\mathcal{G} = \mathcal{U}(\mathcal{A})$ $\rightsquigarrow$ gauge group = unitary elements in $\mathcal{A}$

noncommutative geometry

$(M_n(\mathbb{C}), \mathbb{C}^n, D, f)$ initial spectral triple → ? BV spectral triple → ? total spectral triple → ? gauge-fixing fermion

$\mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\widetilde{S}}) \cong$? BV complex $\cong$??

$\mathcal{C}^\bullet_{BV}(X_t, d_{S_t}) \cong$? BV tot complex $\cong$??

$\mathcal{C}^\bullet_{BRST}(X_t, d_{S_t})|_\Psi \cong$? BRST complex $\cong$??

(X_0, S_0) initial gauge theory → $(\widetilde{X}, \widetilde{S})$ extended theory → (X_t, S_t) total theory → $(X_t, S_t)|_\Psi$ gauge-fixed theory

BV construction

The BV formalism for spectral triples: towards the quantum case

spectral triple		gauge theory
$(\mathcal{A}, \mathcal{H}, D)$	⇨	$(X_0, S_0, \mathcal{G})$

- $\mathcal{A}$ = unital *-alg., $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$
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- $X_0 = \{\varphi = \sum_j a_j[D, b_j] : \varphi^* = \varphi\} \rightsquigarrow$ conf. sp = inner fluctuations
- $S_0[D + \varphi] = Tr(f(D + \varphi)) \rightsquigarrow$ action func. = spectral action
- $\mathcal{G} = \mathcal{U}(\mathcal{A}) \rightsquigarrow$ gauge group = unitary elements in $\mathcal{A}$

noncommutative geometry

$(M_n(\mathbb{C}), \mathbb{C}^n, D, f)$ → $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$ → $(\mathcal{A}_t, \mathcal{H}_t, D_t, J_t)$ → $\Psi \in \mathcal{F}^{-1}_{\mathcal{O}_{X_0}}(\mathcal{H}_{BV,f} \oplus \mathcal{H}_{aux,f})$

initial spectral triple — BV spectral triple — total spectral triple — gauge-fixing fermion

$\mathcal{C}^\bullet_{BV}(\tilde{X}, d_{\tilde{S}}) \cong \mathcal{C}^\bullet_H(\mathcal{B}, \mathcal{M})$ — BV/ Hochschild complex

$\mathcal{C}^\bullet_{BV}(X_t, d_{S_t}) \cong \mathcal{C}^\bullet_H(\mathcal{B}_t, \mathcal{M})$ — BV tot / Hochschild complex

$\mathcal{C}^\bullet_{BRST}(X_t, d_{S_t})|_\Psi \cong \mathcal{C}^\bullet_H(\mathcal{B}_\Psi, \mathcal{M})$ — BRST/ Hochschild complex

(X_0, S_0) → $(\tilde{X}, \tilde{S})$ → (X_t, S_t) → $(X_t, S_t)|_\Psi$

initial gauge th. — extended theory — total theory — gauge-fixed theory

BV construction