Non compact toric resolutions for M-Theory

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Motivation

- We study local Calabi–Yau resolutions of orbifold singularities of the kind $\mathbb{C}_3/\mathbb{Z}_{2n+1}$. The aim is to consider M-Theory in these resolutions, in particular $\boxed{\text{degrees of freedom of } C_3}$.
- The main example presented here are

$$X = \widehat{\mathbb{C}^3/\mathbb{Z}_5(1,1,3)}, \text{ and } X = \widehat{\mathbb{C}^3/\mathbb{Z}_7(1,2,4)},$$

- An anti-holomorphic involution defines a real fixed locus $L = X^\sigma$.
- For that we search for the metric of the resolution, first with the **Guillemin prescription**, then by solving the **Monge-Ampére equation** in a numerical form. We study the metric asymptotic behavior.
- L^2 harmonic one-forms on L can be used to build L^2 harmonic two-forms in local G_2 quotients. We obtain numerical solutions for those forms and obtain their asymptotic behaviour.

Strategy

$$\begin{tikzcd} \begin{array}{c} X = \widehat{\mathbb{C}^3/\mathbb{Z}_{2n+1}} \\ \text{toric CY resolution} \end{array} \arrow[r] & \begin{array}{c} L = X^\sigma \\ \text{real fixed locus} \end{array} \arrow[d] \\ \begin{array}{c} Y = (X \times S^1)/(\sigma, -1) \\ \text{local } G_2 \text{ space} \end{array} & \begin{array}{c} \alpha \in H^1_{L^2}(L) \\ \text{harmonic one-forms} \end{array} \arrow[l] \end{tikzcd}$$

$$\omega = \alpha \wedge d\theta \in H^2_{L^2}(Y).$$

The orbifold $\mathbb{C}^3/\mathbb{Z}_5(1,1,3)$ and its resolution

The orbifold action is

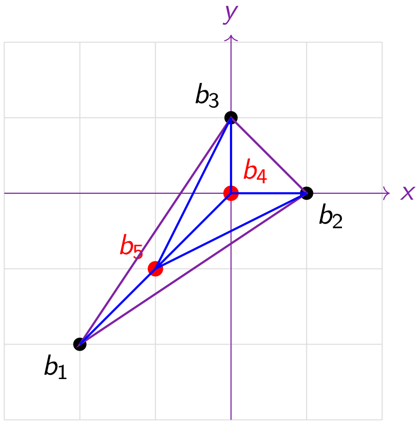
$$(z_1, z_2, z_3) \mapsto (\omega z_1, \omega z_2, \omega^3 z_3), \qquad \omega = e^{2\pi i/5}.$$

A convenient toric description of the resolved space uses the rays

$$\begin{aligned} b_1 &= (-2,-2,1), \quad b_2 = (1,0,1), \qquad b_3 = (0,1,1), \\ b_4 &= (0,0,1), \qquad b_5 = (-1,-1,1). \end{aligned}$$

All rays lie in the affine hyperplane $z = 1$, hence the resolution is toric Calabi–Yau.

Toric diagram of the resolution



The red points give the exceptional divisors of the crepant resolution.

Fan and divisors

One smooth triangulation is generated by the maximal cones

$$\sigma_1 = \langle b_1, b_2, b_5 \rangle, \quad \sigma_2 = \langle b_1, b_5, b_3 \rangle,$$
$$\sigma_3 = \langle b_5, b_2, b_4 \rangle, \quad \sigma_4 = \langle b_5, b_4, b_3 \rangle,$$
$$\sigma_5 = \langle b_4, b_2, b_3 \rangle.$$

Each ray b_i defines a toric divisor

$$D_i = \{Z_i = 0\}.$$

The compact exceptional divisors are

$$E_4 = D_4, \qquad E_5 = D_5.$$

Exceptional divisors and real projections

With the convention

$$b_4 = (0, 0, 1), \qquad b_5 = (-1, -1, 1),$$

the exceptional divisors are

$$\boxed{D_4 \simeq \mathbb{P}^2}, \qquad \boxed{D_5 \simeq \mathbb{F}_3}.$$

Under complex conjugation,

$$\sigma : [Z_1 : \cdots : Z_5] \mapsto [\overline{Z}_1 : \cdots : \overline{Z}_5],$$

their real loci are

$$\boxed{D_4(\mathbb{R}) \simeq \mathbb{RP}^2}, \qquad \boxed{D_5(\mathbb{R}) \simeq \mathbb{F}_3(\mathbb{R}) \simeq \text{Klein bottle}}.$$

Global fixed locus from Cox coordinates

The fixed locus is described as the real Cox quotient

$$L = X^{\sigma} = \frac{\mathbb{R}^5 \setminus Z(\Sigma)_{\mathbb{R}}}{(\mathbb{R}^*)^2}.$$

For the triangulation above, the Stanley–Reisner ideal is

$$SR = \langle Z_1 Z_4,\ Z_2 Z_3 Z_5 \rangle.$$

Thus

$$Z(\Sigma)_{\mathbb{R}} = \{Z_1 = Z_4 = 0\} \cup \{Z_2 = Z_3 = Z_5 = 0\}.$$

Hence

$$L = \frac{\mathbb{R}^5 \setminus (\{Z_1 = Z_4 = 0\} \cup \{Z_2 = Z_3 = Z_5 = 0\})}{(\mathbb{R}^*)^2}.$$

Scalings

The linear relations among rays give

$$b_1 + 2b_2 + 2b_3 - 5b_4 = 0,$$

$$b_2 + b_3 - 3b_4 + b_5 = 0.$$

Equivalently, the charge matrix is

$$Q = \begin{pmatrix} 1 & 2 & 2 & -5 & 0 \\ 0 & 1 & 1 & -3 & 1 \end{pmatrix}.$$

Thus

$$(Z_1, Z_2, Z_3, Z_4, Z_5) \sim (\lambda Z_1, \lambda^2 \mu Z_2, \lambda^2 \mu Z_3, \lambda^{-5} \mu^{-3} Z_4, \mu Z_5).$$

Searching L^2 integrable 1-forms in he Hosomichi–Page model

A reference example is the Calabi–Yau metric on

$$\mathcal{O}(-2,-2) \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

The metric is

$$ds^2 = \kappa^{-1}(\rho) d\rho^2 + \frac{1}{9}\kappa(\rho)\rho^2(\sigma_3 - \widetilde{\sigma}_3)^2 + \frac{1}{6}\rho^2(\sigma_1^2 + \sigma_2^2) + \frac{1}{6}\beta^2(\widetilde{\sigma}_1^2 + \widetilde{\sigma}_2^2),$$

where

$$\kappa(\rho) = \frac{1 + 9a^2/\rho^2 - b^6/\rho^6}{1 + 6a^2/\rho^2}, \qquad \beta^2 = \rho^2 + 6a^2.$$

Fixed locus in the Hosomichi–Page geometry

Complex conjugation fixes a real three-manifold. The induced metric is

$$ds^2\big|_\sigma = \kappa^{-1}(\rho)d\rho^2 + \frac{1}{6}\rho^2 d\phi^2 + \frac{1}{6}(\rho^2 + 6a^2)d\widetilde{\phi}^2.$$

Since ρ covers only half of the real fiber, introduce

$$\rho - \rho_0 = x^2, \qquad x \in \mathbb{R}.$$

Then

$$ds^2\big|_\sigma = 4\kappa^{-1}(\rho(x))x^2 dx^2 + \frac{1}{6}\rho(x)^2 d\phi^2 + \frac{1}{6}(\rho(x)^2 + 6a^2)d\widetilde{\phi}^2.$$

L^2 harmonic one-form in the reference model

Take

$$\alpha = f(x)dx.$$

Then $d\alpha = 0$. Harmonicity reduces to

$$\delta\alpha = 0.$$

For a diagonal metric,

$$\delta\alpha = -\frac{1}{\sqrt{g}}\partial_x\left(\sqrt{g}\, g^{xx} f(x)\right).$$

Hence

$$\boxed{f(x) = \frac{C}{\sqrt{g}\, g^{xx}}.}$$

The L^2 norm is

$$\|\alpha\|^2 = \int f(x)^2 g^{xx}\sqrt{g}\, dx\, d\phi\, d\widetilde{\phi}.$$

In the Hosomichi–Page case this integral converges at infinity.

Canonical toric metric: Guillemin potential

For the resolved $\mathbb{C}^3/\mathbb{Z}_5(1,1,3)$, introduce symplectic coordinates

$$y = (y_1, y_2, y_3) \in M_{\mathbb{R}}.$$

The affine functions are

$$\ell_i(y) = \langle y, b_i \rangle - \lambda_i.$$

Guillemin's canonical symplectic potential is

$$\boxed{G_{\mathrm{can}}(y) = \frac{1}{2}\sum_{i=1}^{5} \ell_i(y) \log \ell_i(y).}$$

The associated toric Kähler metric is

$$\boxed{g_{ij} = \frac{\partial^2 G_{\mathrm{can}}}{\partial y_i \partial y_j}.}$$

This metric is toric and explicit, but not generally Ricci-flat.

Restricting the toric metric to the real locus

In action-angle coordinates, a toric Kähler metric has the form

$$ds^2 = g_{ij}(y) dy_i dy_j + g^{ij}(y) d\theta_i d\theta_j.$$

Complex conjugation fixes the angles,

$$\theta_i = 0 \quad \text{or} \quad \pi,$$

so

$$d\theta_i = 0.$$

Therefore the metric induced on the real fixed locus is

$$\boxed{ds_L^2 = g_{ij}(y) dy_i dy_j.}$$

This gives a first explicit metric on $L = X^\sigma$ from the toric resolution.

Towards a Ricci-flat metric

To search for a Ricci-flat metric, deform the symplectic potential:

$$G(y) = G_{\mathrm{can}}(y) + \varphi(y).$$

The Calabi–Yau condition is expressed as a real Monge–Ampère equation,

$$\boxed{\det\left(\frac{\partial^2(G_{\mathrm{can}} + \varphi)}{\partial y_i \partial y_j}\right) = Ce^{-2y_3}.}$$

Here the factor e^{-2y_3} follows from the toric Calabi–Yau condition: all rays lie in the hyperplane $z = 1$, so

$$m_{\mathrm{CY}} = (0, 0, 1).$$

Once G is found, we restrict again to

$$ds_L^2 = g_{ij}^{\mathrm{CY}}(y)dy_i dy_j.$$

Numerical construction of the Ricci-flat metric

We consider the toric resolution with rays

$$(-2,-2,1),\ (1,0,1),\ (0,1,1),\ (0,0,1),\ (-1,-1,1),$$

using the gauge

$$y_1 = xu_1, \qquad y_2 = xu_2, \qquad y_3 = x.$$

The Guillemin symplectic potential is

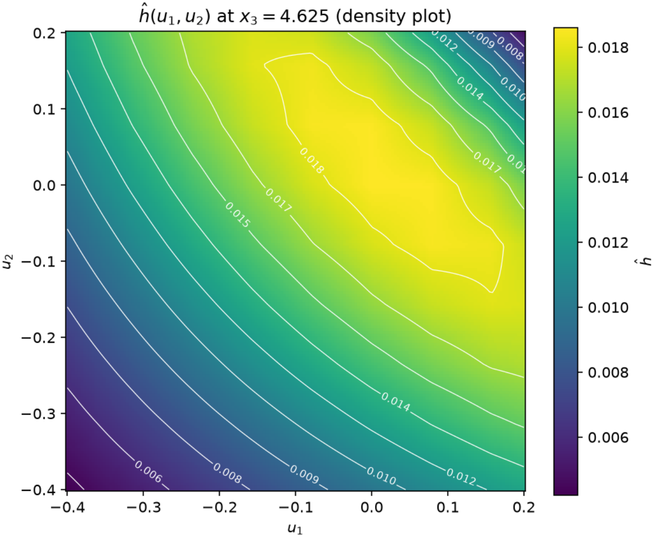
$$G_{\mathrm{Gul}} = \frac{1}{2}\sum_{i=1}^{5} \ell_i \log \ell_i,$$

where the affine functions ℓ_i depend on the two Kähler parameters t_1, t_2.
We deform the canonical metric through

$$G = G_{\mathrm{Gul}} + \phi(u_1, u_2, x),$$

and solve numerically the real Monge–Ampère equation

Numeric Ricci-flat metric on $\widehat{\mathbb{C}^3/\mathbb{Z}_5(1,1,3)}$



Crepant Resolutions and Asymptotic Decay

Let

$$\pi : X \longrightarrow \mathbb{C}^3/\Gamma, \qquad \Gamma \subset SU(3),$$

be a crepant resolution of an orbifold singularity.
Joyce proved the existence of ALE Ricci-flat Kähler metrics on crepant resolutions of

$$\mathbb{C}^m/G, \qquad G \subset SU(m).$$

Away from the exceptional set, the metric is asymptotic to the flat cone metric:

$$g = g_{\mathrm{cone}} + h.$$

More generally, Conlon–Hein show that for asymptotically conical Calabi–Yau metrics one has

$$|\nabla^k h| = O(r^{-\lambda-k}),$$

for some positive decay rate λ, determined by the geometry at infinity.
For crepant resolutions of $\mathbb{C}^3/\mathbb{Z}_n$, one expects

$$g - g_{\mathrm{cone}} = O(r^{-6}), \qquad \nabla^k(g - g_{\mathrm{cone}}) = O(r^{-6-k}).$$

Numeric Ricci-flat metric on $\mathbb{C}^3/\widehat{\mathbb{Z}_5(1,1,3)}$

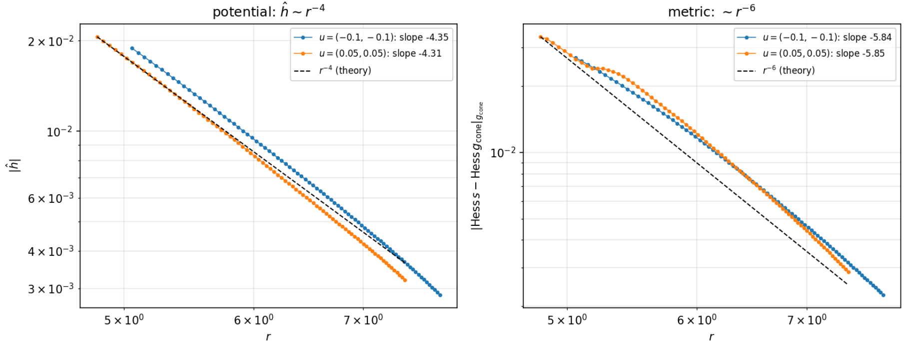


Figure: Numerical solution of the Monge–Ampère equation describing the deformation of the Guillemin metric. Dependence on the potential and the component g_{rr} in the radial coordinate.

Numerical harmonic one-form

After obtaining the numerical Ricci-flat metric

$$g_{ij} = g_{ij}^{\mathrm{Gul}} + \partial_i \partial_j \phi,$$

we construct a harmonic one-form by solving

$$\Delta_g \Phi = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g}\, g^{ij} \partial_j \Phi \right) = 0,$$

including the full inverse metric g^{ij}.
The harmonic one-form is then

$$\alpha = d\Phi.$$

The numerical implementation computes

$$g^{ij}, \qquad \sqrt{\det g}, \qquad \alpha_i = \partial_i \Phi,$$

and evaluates $\|\alpha\|^2 = \int g^{ij} \partial_i \Phi \partial_j \Phi \sqrt{g}\, d^3 q$.

Numerical Pipeline

$$\boxed{\textbf{Toric Resolution}} \Longrightarrow \boxed{\textbf{Guillemin Metric}} \Longrightarrow \boxed{\begin{array}{c}\textbf{Monge–Ampère}\\ \det(g) = F\end{array}} \Longrightarrow \boxed{\begin{array}{c}g_{\text{def}}\\ = g_{\text{Gul}} + \partial_i\partial_j\phi\end{array}}$$

$$\boxed{\textbf{Asymptotic Analysis}} \Longrightarrow \boxed{\begin{array}{c}\Delta_g\Phi = 0\\ \alpha = d\Phi\end{array}} \Longrightarrow \boxed{\begin{array}{c}d\alpha = 0\\ \delta\alpha = 0\end{array}} \Longrightarrow \boxed{\begin{array}{c}\Delta_H\alpha = 0\\ \|\alpha\|_{L^2} < \infty\end{array}}$$

Search for L^2 harmonic one-forms

Numerical search for L^2 harmonic one-forms

The Ricci-flat metric is obtained by solving numerically the Monge–Ampère equation

$$g_{ij} = g_{ij}^{\text{Guillemin}} + \partial_i \partial_j \phi,$$

using the Guillemin symplectic potential as the background metric.

The harmonic one-form

$$\alpha = \alpha_{u_1}\, du_1 + \alpha_{u_2}\, du_2 + \alpha_x\, dx$$

is obtained from the Hodge system $d\alpha = 0, \qquad \delta\alpha = 0$, with asymptotically decaying boundary conditions ensuring L^2 integrability.

We found not definitive existence results.

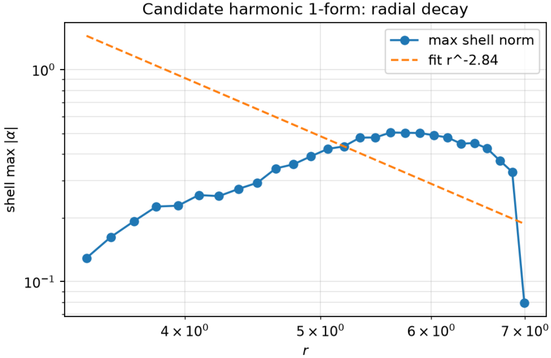


Figure: Numerical profiles of the three components of the candidate L^2 harmonic one-form on the real locus.

Numerical search for L^2 harmonic two-forms

Using the numerical Ricci-flat metric, we consider the general two-form

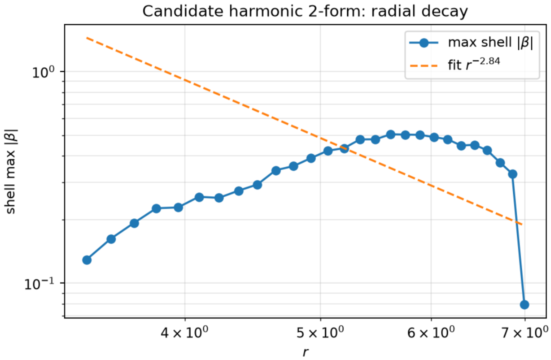
$$\beta = B_{12}\, du_1 \wedge du_2 + B_{1x}\, du_1 \wedge dx + B_{2x}\, du_2 \wedge dx.$$

The numerical solver imposes the Hodge equations $d\beta = 0, \qquad \delta\beta = 0$, together with asymptotically conical boundary conditions.

The L^2 norm is evaluated as

$$\|\beta\|^2_{L^2} = \int \beta \wedge *\beta.$$

We found not definitive existence results.



Numerical profiles of the candidate L^2 harmonic two-form on the real locus. The solver simultaneously verifies closure, co-closure and asymptotic decay.

The resolution of $\mathbb{C}^3/\mathbb{Z}_7(1,2,4)$: toric data

The $(P \mid Q)$ matrix.
Rays $D_1{=}(2,0,1)$, $D_2{=}(-1,2,1)$, $D_3{=}(0,-1,1)$ (non-compact) and $E_1{=}(0,0,1)$, $E_2{=}(0,1,1)$, $E_4{=}(1,0,1)$ (*compact* exceptional divisors).

There are three linear relations

$$D_1 + E_1 = 2E_4, \quad D_2 + E_4 = 2E_2, \quad D_3 + E_2 = 2E_1.$$

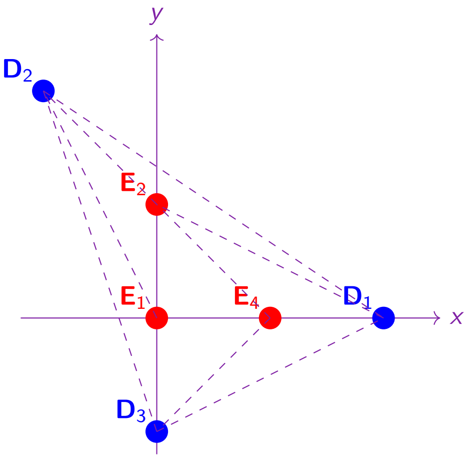
They pin down the unique unimodular triangulation (7 triangles): the crepant resolution is unique.

The $(\mathbb{C}^*)^3$ action defining the toric variety is given by:

$$G = (\lambda_1, \lambda_2, \lambda_2^2\lambda_3, \lambda_1\lambda_2^{-4}\lambda_3^{-2}, \lambda_3, \lambda_1^{-2}\lambda_2). \tag{1}$$

The exclusion set is given by

$$Z(\Sigma) = ((z_3, y_2) = (0,0), (z_2, y_3) = (0,0) \\ , (z_1, y_1) = (0,0), (z_1, z_2, z_3) = (0,0,0)). \tag{2}$$



Triangulation of the junior simplex; dashed: the 3 collinearities. Red = compact, blue = non-compact.

Numeric Ricci-flat metric on $\mathbb{C}^3/\widehat{\mathbb{Z}_7(1,2,4)}$

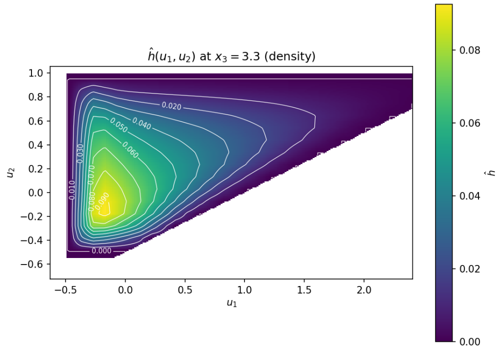


Figure: Numerical solution of the Monge–Ampère equation describing the deformation of the Guillemin metric. The numerical potential $\phi(u_1, u_2, x)$ generates the Ricci-flat metric $g_{ij} = g_{ij}^{\mathrm{Gul}} + \partial_i \partial_j \phi$, which provides the geometric input for the computation of harmonic L^2 one-forms.

Numeric Ricci-flat metric on $\mathbb{C}^3/\widehat{\mathbb{Z}_7(1,2,4)}$

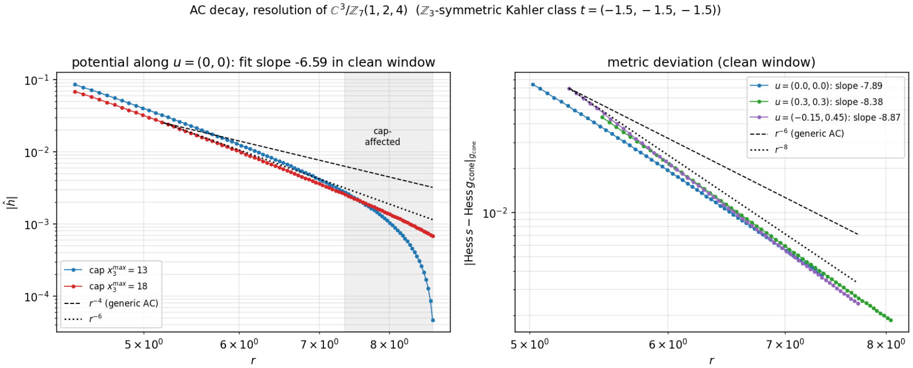


Figure: Numerical solution of the Monge–Ampère equation describing the deformation of the Guillemin metric. Dependence on the potential and the component g_{rr} in the radial coordinate. .

L^2 cohomology of the real locus: $\dim \mathcal{H}^1_{L^2} = \dim \mathcal{H}^2_{L^2} = 3$

Setting: the real locus L with the induced CY metric is a complete, smooth, asymptotically conical 3-manifold.

- **Explicit modes.** $\alpha_i = d\varphi_i$, with φ_i the harmonic *potential* of the compact wall E_i against its parity partner $D_{a(i)}$ $(0 < \varphi_i < 1)$; $\beta_i = \star\alpha_i$.
- **Harmonic and L^2.** Closed + co-closed + L^2 = harmonic on a complete manifold;

$$\|\alpha_i\|^2_{L^2} = q_i \;< \infty, \qquad |\alpha_i| = |\beta_i| \sim r^{-3} \;\; \text{(dipolar)}.$$

- **Counting:** The compact real locus is topologically $L_{\mathrm{comp}} \simeq T^3$, which implies $b_1(L_{\mathrm{comp}}) = 3$. This predicts three topological candidates for L^2 harmonic one-forms.
- **Numerical study:** The decay exponents are indicial roots of the link problem: unchanged by the AC decay of the true (Monge–Ampère) CY metric — only the coefficients q_i, p_i shift.

Application to local G_2 geometry

The local Joyce quotient is

$$Y = \frac{X \times S^1}{\sigma \times (-1)}.$$

Where X in our case is $X = \widehat{\mathbb{C}^3/\mathbb{Z}_5(1,1,3)}, \widehat{\mathbb{C}^3/\mathbb{Z}_7(1,3,4)}, \widehat{\mathbb{C}^3/\mathbb{Z}_{2n+1}}$. Singularities are localized along

$$L \times \{0, \pi\}.$$

A normalizable harmonic one-form on L gives a harmonic two-form on Y:

$$\boxed{\omega = \alpha \wedge d\theta.}$$

In M-theory,

$$C_3 = \sum_I A_I \wedge \omega_I + \cdots .$$

L^2 normalizability gives finite gauge kinetic terms in the lower-dimensional theory.

Summary

- We described the resolved $\mathbb{C}^3/\mathbb{Z}_5(1,1,3)$ and $\mathbb{C}^3/\mathbb{Z}_7(1,1,5)$ geometries torically.
- For $\mathbb{C}^3/\mathbb{Z}_5(1,1,3)$ resolution the exceptional divisors are $D_4 \simeq \mathbb{P}^2$, $D_5 \simeq \mathbb{F}_3$. Their real projections are $\mathbb{RP}^2$, Klein bottle .
- The global fixed locus is the real Cox quotient $L = X^\sigma$. Guillemin's metric provides an explicit toric metric on L.
- We solved numerically the Monge–Ampère equation for $\mathbb{C}^3/\widehat{\mathbb{Z}_5(1,1,3)}$ and study $H^1_{L^2}(L)$ and $H^2_{L^2}(L)$, not finding numerical solutions.
- Then we study $\mathbb{C}^3/\widehat{\mathbb{Z}_7(1,1,7)}$ to find harmonic forms on the corresponding L, that will allow to study dimensional reduction.

Next steps

- Continue with the resolutions of $\mathbb{C}^3/\mathbb{Z}_{2n+1}(1, 1, n)$ and the study of the harmonic forms.
- Construct local Joyce quotients associated with toric Calabi–Yau resolutions.
- Investigate localized L^2 harmonic forms on the resulting G_2 geometries.
- Derive the four-dimensional effective theory from the M-theory three-form C_3.
- Explore gauge sectors, localized modes and flux compactifications.

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Thank you!

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