Renormalization and C*-algebras

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Introduction

Perturbative construction of interacting renormalized QFT can be reduced to a construction of time-ordered products of composite local fields of a free theory (Stückelberg, Bogoliubov, Epstein-Glaser).

Basic object: S-matrix as a generating functional of time ordered functions,

$$S(F) = \sum_n \frac{i^n}{n!} T_n(F, \ldots, F)$$

with

$$T_n(F, \ldots, F) = \int dx_1 \ldots dx_n \, T_n(F(x_1), \ldots, F(x_n))$$

Epstein-Glaser axioms fix T_n up to renormalization freedom.

Main-Theorem of renormalization (Stora-Popineau):

Two different choices of time ordered functions are related by a renormalization group transformation Z which maps local composite fields to each other such that the corresponding S-matrices satisfy

$$\hat{S}(F) = S(Z(F))$$

The dynamical law is implemented by the Schwinger-Dyson equation which has for a scalar field ϕ the form

$$T_{n+1}(F(x_1), \ldots, F(x_n), (\Box + m^2)\phi(y))$$

$$= \sum T_n(F(x_1), \ldots, \frac{\delta F(x_k)}{\delta \phi(y)}, \ldots, F(x_n))$$

Question: Can one go beyond a construction via formal power series?

Problem treated by Constructive QFT with some success in low dimensions.

Other constructions exploit higher symmetries (conformal symmetry, integrability).

New ansatz: Use unitarity to construct an abstract C*-algebra generated by S-matrices.

Goal: Formulate all axioms of perturbation theory in terms of relations between S-matrices.

Causal factorization

Basic property of time ordered products

$$T_k(F(x_1),\ldots,F(x_k))T_n(F(y_1),\ldots,F(y_n))$$

$$= T_{k+n}(F(x_1),\ldots,F(x_k),F(y_1),\ldots,F(y_n))$$

if none of the x_i is in the past of one of the y_j.

This leads to the causal factorization relation for the S-matrix

$$S(F)S(G) = S(F+G)$$

if the support of F does not intersect the past of the support of G.

Epstein-Glaser renormalization:

This factorization yields an inductive construction of time ordered products up to coinciding points.

The extension to coinciding points corresponds to UV renormalization.

A complete solution can be obtained in terms of distribution theory.

Factorization not well defined if supports have common points.
Heuristically, split interaction $G = G_+ + G_-$,
G_+ in the future, G_- in the past of some Cauchy surface.
Supports of F and H separated by the Cauchy surface $\Longrightarrow$

$$S(F + G + H) = S(F + G_+)S(G_- + H)$$

$$= S(F + G_+)S(G_-)S(G_-)^{-1}S(G_+)^{-1}S(G_+)S(G_- + H)$$

$$= S(F + G)S(G)^{-1}S(G + H)$$

The resulting relation *Causal Factorization*

$$S(F + G + H) = S(F + G)S(G)^{-1}S(G + H)$$

is meaningful and can in fact be derived from the simpler factorization relation in terms of formal power series. In an abstract framework it has to be postulated as an axiom.

Dynamics

Implemented by a version of the Schwinger-Dyson equation formulated in terms of S-matrices.

Field equation (functional derivative of the action) replaced by finite difference:

$$\delta_\psi L(\phi) = \int L(\phi+\psi) - L(\phi) \ ,$$

L Lagrangian, ψ compactly supported field configuration

$$F^\psi(\phi) = F(\phi+\psi) \text{ shifted interaction}$$

We then require the relation *Dynamics*

$$S(F) = S(F^\psi + \delta_\psi L) \ .$$

In perturbation theory, for the free Lagrangian, this is equivalent to the Schwinger-Dyson equation.

Other Lagrangians: add interaction terms F, $\mathrm{supp}F$ compact, and construct the retarded or advanced relative S-matrices

$$S_F^{\mathrm{ret}}(G) = S(F)^{-1}S(F+G)\ ,\ S_F^{\mathrm{adv}}(G) = S(F+G)S(F)^{-1}\ .$$

They satisfy again the causal factorization and the dynamical relation, now for the new action.

S-matrices satisfying dynamical relation for general interaction: combination of retarded and advanced relative S-matrices (algebraic adiabatic limit).

C*-algebras

One can consider the group generated by S-matrices with the relations

Causal Factorization and Dynamics for any Lagrangian.

This group generates a unique C*-algebra. This works for any region of spacetime and yields a Haag-Kastler net of local C*-algebras:

For any region $\mathcal{O}$ of spacetime one builds the C*-algebra $\mathfrak{A}(\mathcal{O})$ generated by $S(F)$ with $\mathrm{supp}F \subset \mathcal{O}$. The association $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ satisfies the conditions

- Inclusion: $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$
- Local commutativity: If $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{O}$ and $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$

  then the commutator $[A_1, A_2] \in \mathfrak{A}(\mathcal{O})$

  vanishes for all $A_1 \in \mathfrak{A}(\mathcal{O}_1), A_2 \in \mathfrak{A}(\mathcal{O}_2)$.
- Covariance If g is a symmetry of the spacetime then there exist isomorphisms $\alpha_{g,\mathcal{O}} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(g\mathcal{O})$ such that

  $$\alpha_{g,\mathcal{O}_2}\big|_{\mathfrak{A}(\mathcal{O}_1)} = \alpha_{g,\mathcal{O}_1} \text{ for } \mathcal{O}_1 \subset \mathcal{O}_2$$

  and

  $$\alpha_{g_1 g_2,\mathcal{O}} = \alpha_{g_1,g_2\mathcal{O}} \circ \alpha_{g_2,\mathcal{O}} \ .$$

Question: How far is this from a construction of the theory?

First results:

- If L_0 is the free Lagrangian, then the S-matrices of linear fields generate the Weyl algebra.
- In the Dereszinski-Meissner representation of the free massless field ϕ in 2 dimensions the S-matrices of $\cos\phi$ and $\sin\phi$ can be defined and yield the algebra of observables of the sine-Gordon model (Bahns-F-Rejzner)
- The Fock representation of the Weyl algebra can be extended to S-matrices of quadratic composite fields in 4 dimensions (Buchholz-F). This result goes beyond perturbation theory since it includes also changes of the spacetime metric and therefore of causal relations.

π irreducible representation of Weyl algebra on Hilbert space $\mathcal{H}$.

$\tilde{\pi}$ extension to the full algebra (if it exists).

$\alpha \neq \mathrm{id}$ acts trivially on the Weyl algebra $\Longrightarrow \tilde{\pi} \circ \alpha \not\simeq \tilde{\pi}$.

Postulate: All such automorphisms are of the form

$$\alpha : S(F) \mapsto S(Z(F))$$

Z renormalization group transformation, characterized by

$$Z(0) = 0,\ Z(F) \text{ local, } \mathrm{supp} Z(F) = \mathrm{supp} F$$

$$Z(F+G+H) = Z(F+G) - Z(G) + Z(G+H)$$
$$\text{if } \mathrm{supp} F \cap \mathrm{supp} H = \emptyset$$

$$Z(F^{\psi} + \delta L(\psi)) = Z(F)^{\psi} + \delta L(\psi)$$

$$Z(\int \phi f) = \int \phi f$$

Motivation: Main Theorem on renormalization (Stora-Popineau)

Let g be an invertible affine transformation on the field space which maps any local functional F to another local functional $g_* F$ such that the Lagrangian is invariant. Then

$$\alpha_g(S(F)) \mapsto S(g_* F)$$

is an automorphism. Let $U(g)$ be a unitary operator on $\mathcal{H}$ which implements α_g on the Weyl algebra. Then $\mathrm{Ad}U(g) \circ \alpha_g^{-1}$ acts trivially on the Weyl algebra

$$U(g)\alpha_g^{-1}(S(F))U(g)^{-1} = S(Z_g(F))$$

$g \mapsto Z_g$ fulfils the cocycle relation

$$Z_{gh} = Z_g g_* Z_h g_*^{-1}$$

Outlook

- Algebraic structures of formal perturbation theory induce a construction of quantum field theories in terms of C*-algebras.
- Problem of existence of QFT's reduced to search for suitable representations.
- New nonperturbative aspects on symmetries and renormalization.
- Formalism has to be generalized to Fermi fields and gauge theories.