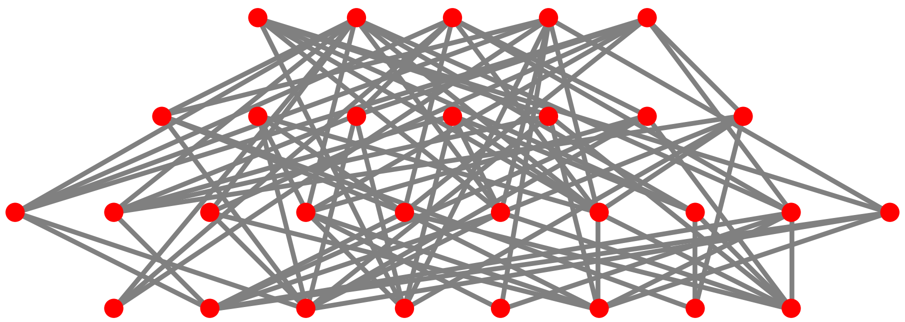
Causal Set Kinematics:
Reconstructing Spacetime from Randomly Embedded Posets



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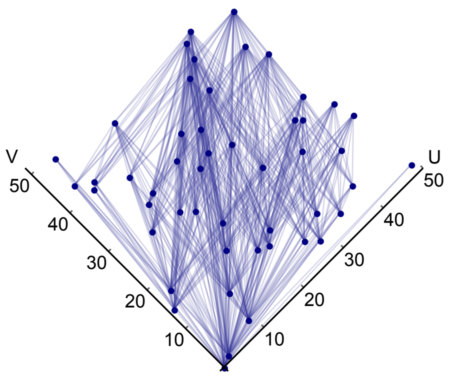
Non-Regular Spacetime Geometry,
ESI, March, 2023

Outline

- The Hawking-King-Macarthy-Malament Theorem
- The Causal Set Paradigm : locally finite posets replace spacetime
- Properties of the Continuum Approximation
    - Local Lorentz invariance
    - Non-locality
    - The Fundamental Conjecture



- Geometric Reconstruction: dimension, topology, curvature from Order
- Some thoughts on GH distance for 2d orders using the null distance function..

Hawking, King, McCarthy: 1976

Malament: 1977

<u>*Theorem:*</u>

If a chronological bijection exists between two future and past distinguishing spacetimes then they are conformally isometric .

Causal Bijection: $f : (M_1, \ll_1) \to (M_2, \ll_2), \quad f(x) \ll_2 f(y) \Leftrightarrow x \ll_1 y, \forall\, x, y \in M_1$

Causal Bijection: $f : (M_1, \prec_1) \to (M_2, \prec_2), \quad f(x) \prec_2 f(y) \Leftrightarrow x \prec_1 y, \forall\, x, y \in M_1$

Chronological Bijection ⇒ Causal Bijection *if they are future and past distinguishing*

—Kronheimer and Penrose, 1967

Conformal Isometry : $F : (M_1, g_1) \to (M_2, g_2), \quad F \circ g_1 = \Omega^2 g_2$

The existence of a chronological bijection implies that the dimension and more generally, the topology are the same (the latter iff s.c violating regions satisfy an additional condition).

-- Parrikar and Surya: 2011

Hawking, King, McCarthy & Malament.

Spacetime "=" Causal Structure Poset + Local Volume Element

$$(M, g) = (M, \prec) + \epsilon$$

Hinted at by many: Zeeman, Kronheimer, Penrose, Finkelstein, Myrheim, Hemion.

The Causal Set Hypothesis:

" Discretise" $(M, \prec) \rightarrow$ locally finite poset or causal set C

Order $\leftrightarrow$ Causal Order

Cardinality $\leftrightarrow$ Volume Element

Order + Number ~Spacetime

-Myrheim, 1978

-Bombelli, Lee, Meyer and Sorkin,1987

Acyclic:

$x \prec y \Rightarrow y \nprec x$

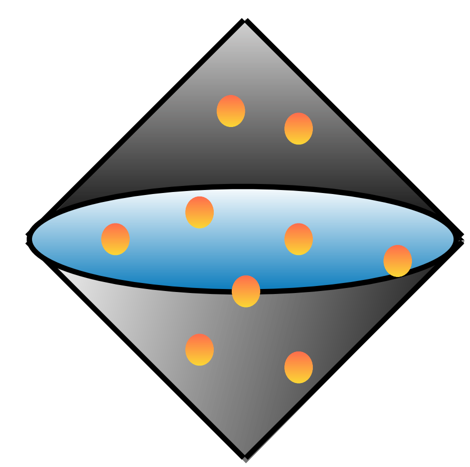
Transitive:

$x \prec y, y \prec z \Rightarrow x \prec z$

Locally Finite:

$|\mathrm{Fut}(x) \cap \mathrm{Past}(y)| < \infty$

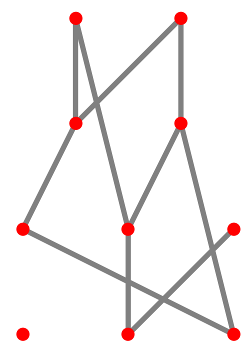
Finite number of spacetime atoms in a finite spacetime volume

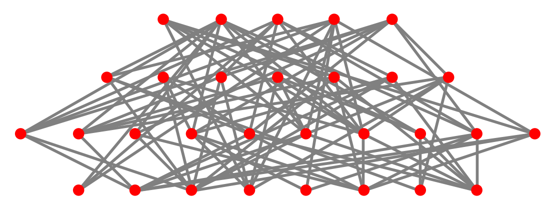


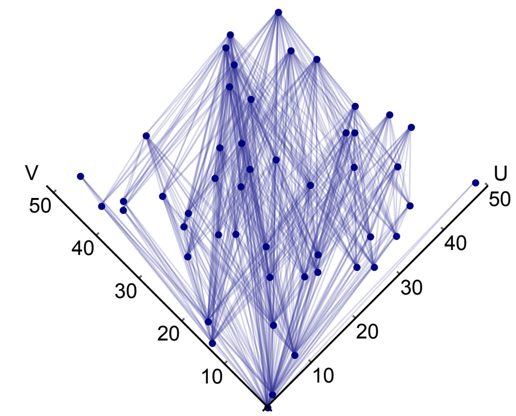
Causal Set Paradigm

-Bombelli, Lee, Meyer and Sorkin,1987

1. Causal Sets are the fine grained structure of spacetime

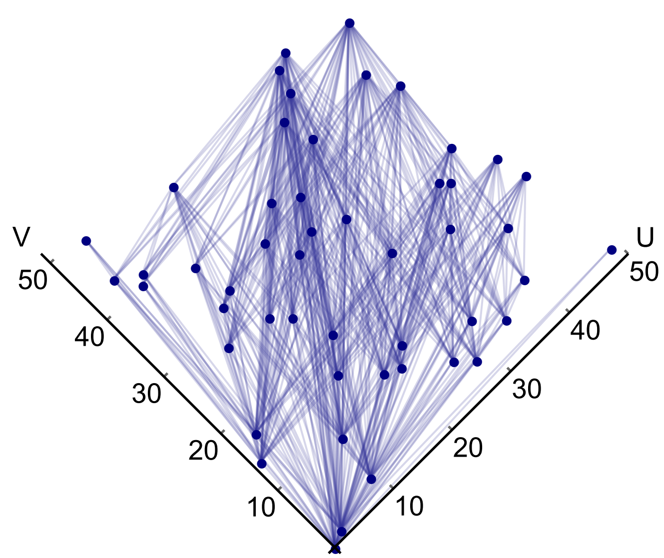






<u>*Acyclic:*</u> $x \prec y \Rightarrow y \nprec x$
<u>*Transitive:*</u>
$x \prec y, y \prec z \Rightarrow x \prec z$
<u>*Locally Finite:*</u>
$|\mathrm{Fut}(x) \cap \mathrm{Past}(y)| < \infty$

2. Continuum Approximation: $C \sim (M, g)$



Order + Number ~ Spacetime,
- ✳ Order ↔ Causal Structure
- ✳ Number ↔ Volume

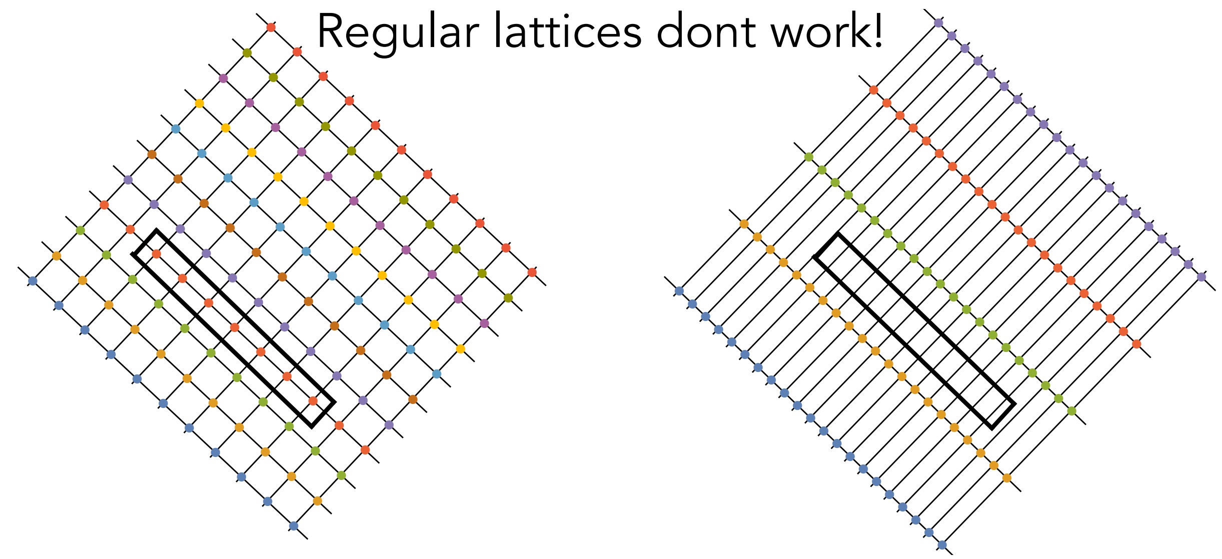
Poisson Sprinkling

$$P_V(n) = \frac{(\rho V)^n}{n!} e^{-\rho V}$$

$$\langle n \rangle = \rho V, \quad \Delta n = \sqrt{\rho V}$$

Poisson Point Process, Faithful Embedding, etc

Regular lattices dont work!

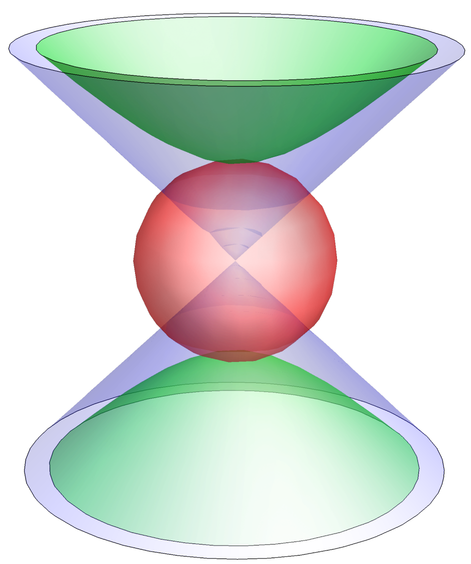


- $n \sim \rho V$ correspondence has to be diffeo invariant:

- Random discretisation via a Poisson sprinkling process:
    - $P_V(n) = \frac{(\rho V)^n}{n!} e^{-\rho V}$,
    - $\langle n \rangle = \rho V$: correspondence works in the mean.
    - $\Delta n = \sqrt{\rho V}$ — is Poisson optimal? — *Saravani and Aslanbeigi 2014*
    - Given a causal (distinguising) spacetime (M, g), extract an ensemble $\{C\}_\rho$
- We will say $C \sim_\rho (M, g)$ is a faithful embedding at density ρ if :
    - $C \hookrightarrow (M, g)$ is order preserving
    - n_V : number of points in spacetime volume V is a random variable
    - $P_V(n) = \frac{(\rho V)^n}{n!} e^{-\rho V}$
- Important feature of the Poisson sprinkling: Statistical independence of process in disjoint regions of (M, g)

Lorentz Invariance

— Bombelli, Henson and Sorkin,2006



$SO(3,1)$ is non-compact

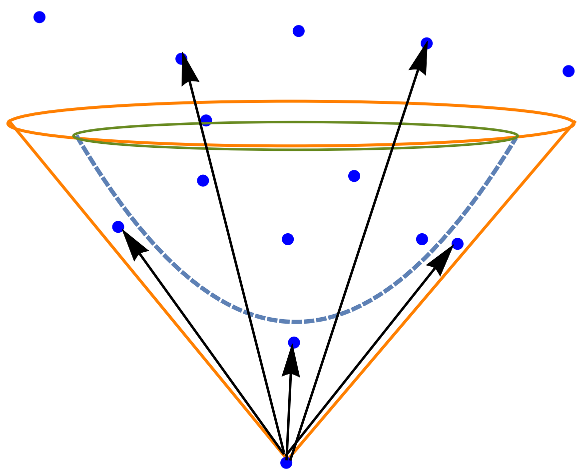
- *Ω: space of all Poisson sprinklings into (M, g)*
- *Poisson process gives a probability measure μ on Ω: (Ω, Σ, μ)*
- *μ is volume preserving and therefore Lorentz invariant*
- *Set of all f.d. timelike directions forms a unit hyperbola $H \subset \mathbb{M}^d$*
- *A good direction map $D : \Omega \to H$ should be equivariant:*

$$\begin{array}{ccc} \Omega & \xrightarrow{\Lambda} & \Omega \\ D \downarrow & & \downarrow D \\ H & \xrightarrow[\Lambda]{} & H \end{array}$$

<u>Theorem</u> :
There is no measurable map $D : \Omega \to H$ which is equivariant, i.e., $D \circ \Lambda = \Lambda \circ D$.

<u>Proof:</u>
If such a map existed, then $\mu_D = \mu \circ D^{-1}$ is a Lorentz invariant probability measure on H which is not possible since is H non-compact.

Non-Locality



Notation:

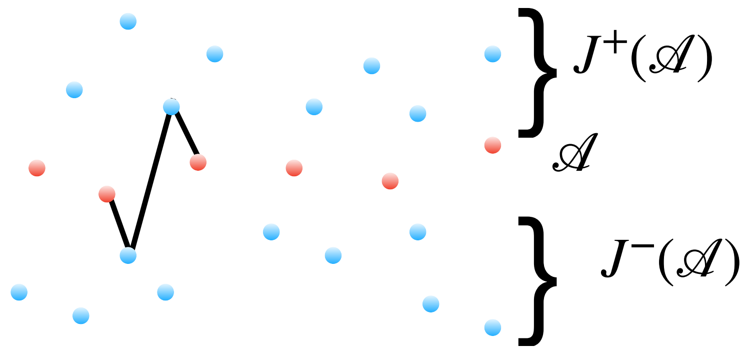
$\prec$, $\preceq$: as usual

$\prec_L$: Link/nearest neighbour/cover

#: spacelike related

- Let $C \sim_\rho (M, g)$ (causal, finite volume spacetime region)
- Define Links: $e \prec_L e'$, iff $e \prec e'$ & $\nexists\, e'', e \prec e'' \prec e'$
- $Prob(e \prec_L e') = P_{V(e,e')}(n=0) = \exp(-\rho V(e,e'))$: links lie all along the light cone
- Causal sets $C \sim_\rho (M, g)$ are graphs without a fixed valency : no tangent spaces!

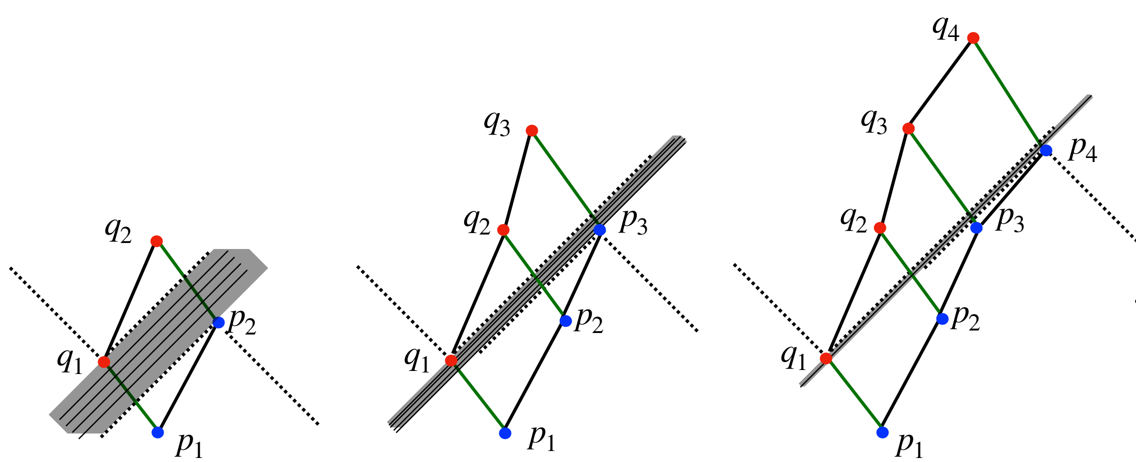
- Spacelike hypersurfaces ~ Antichains: $\mathscr{A} = \{e \,|\, \forall e, e', e \# e'\}$ are non-Cauchy:



$\exists\, e, e' \in C$, s.t. $e \prec \mathscr{A} \prec e'$, $e \prec_L e'$

- $\exists\, e, e' \in C$, st $\mathrm{Fut}(e) \cup \mathrm{Past}(e) = \mathrm{Fut}(e') \cup \mathrm{Past}(e')$: not itself future or past distinguishing

- Since Poisson is uniform wrt spacetime volume $V \Rightarrow$ sets of measure zero are not realised:



$x \to y$ almost surely will not be realised!

-- Bhattacharya & Surya, 2022

- An *almost* Lorentzian Length space: $(C, \prec, T)$, T is the length of the longest chain

The Fundamental Conjecture

If $C \sim_\rho (M,g)$ & $C \sim_\rho (M',g') \Rightarrow (M,g) \simeq_\rho (M',g')$

- $(M,g) \simeq_\rho (M',g')$: what does this mean?
- Closeness of Lorentzian spacetimes via GH convergence : which distance function to use?

— Bombelli, 2000, Bombelli and Noldus, 2004

—Burtscher and Allen, 2021,

--Kunzinger and Steinbauer, 2021

- Can we define ρ-closeness?

- Convergence in $\rho \to \infty$ limit : direct limits, nets, poset of posets : (M,g) is the reference spacetime.

-- *Bombelli and Meyer, 1989*

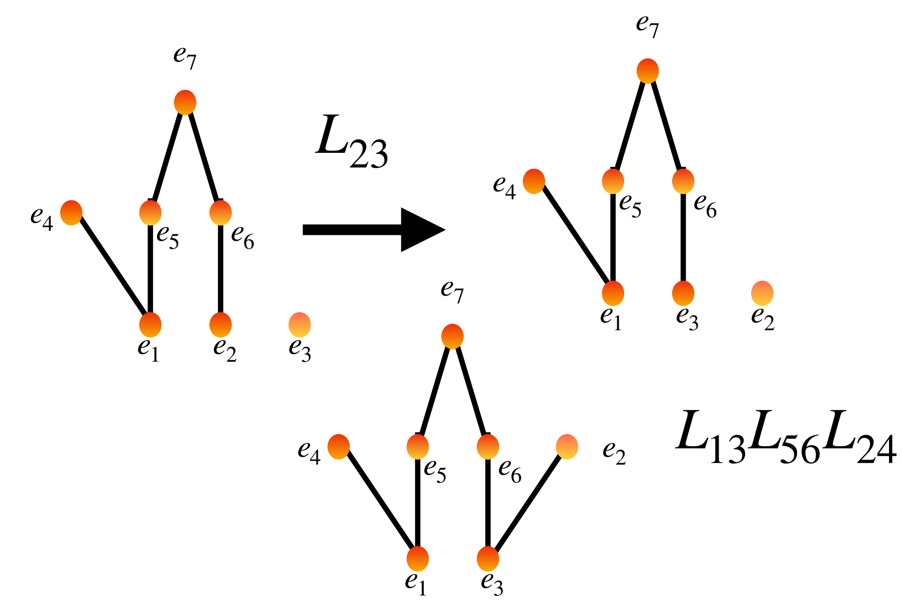
-- Minguzzi and Suhr , 2022

-- Muller, 2022

Conjecture: Causal sets contain all **physically** relevant information, i.e., upto the discreteness scale

Geometric Reconstruction from Random Order

- Labelling of a causal set $L : C \to \mathbb{N}$ is the analogue of diffeomorphisms
- Order invariants are label invariants (example: Number of linked pairs N_0)
- IF: Geometric/Topological observable $\mathcal{O} \leftrightarrow$ Order invariant $\mathcal{U}$
- $\mathcal{O}$ -Hauptvermutung:
    - $C \hookrightarrow_\rho (M, g)$ and $C \hookrightarrow_\rho (M', g')$
    - Then $(M, g) \simeq_{\mathcal{O}} (M', g')$ if $\langle \mathcal{U} \rangle = \rho^m \mathcal{O}$
- How should we identify the right order invariants $\mathcal{U}$?



Good Guess work — at least to start with!

- Poisson process $P_v(n) : \quad (M, g) \to \{C\}$
- C samples (M, g) uniformly at random at density ρ

Examples:

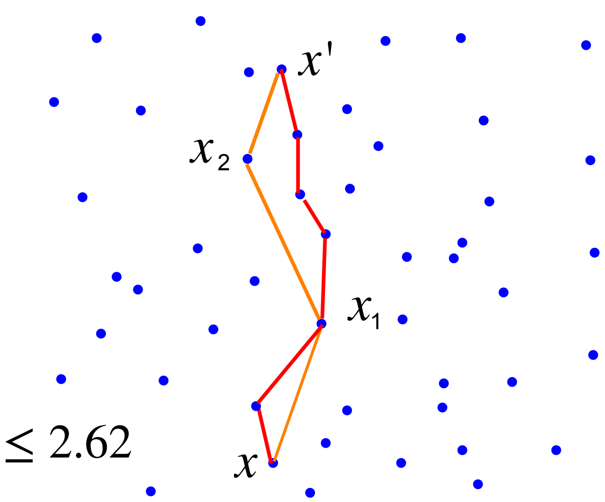
- *Myrheim-Meyer* Dimension Estimator : — *Myrheim, 1978, Meyer, 1987*

    - Let $(\mathbb{D}^d, \eta) \to_\rho \{C\}, \quad \langle n \rangle = \rho \mathrm{vol}(\mathbb{D}^d)$, $\mathbb{D}^d$ a flat causal diamond
    - Number of relations $e \prec e'$, $\langle R \rangle = \rho^2 \int_{\mathbb{D}^d} dx_1 \int_{J^+(x_1)\cap \mathbb{D}^d} dx_2 = \langle n \rangle^2 \frac{\Gamma(d+1)\Gamma(d/2)}{4\Gamma(3d/2)}$
    - Ordering fraction $\langle r \rangle = \frac{2\langle R \rangle}{\langle n \rangle^2} = f(d)$: **read off the spacetime dimension!**

- Mid-point scaling dimension: $2^d = V/V_-$, V_- is the largest smallest volume for the mid-point of V

- Time-like Distance: Maximise the length of a chain: — *Brightwell and Gregory,1987*

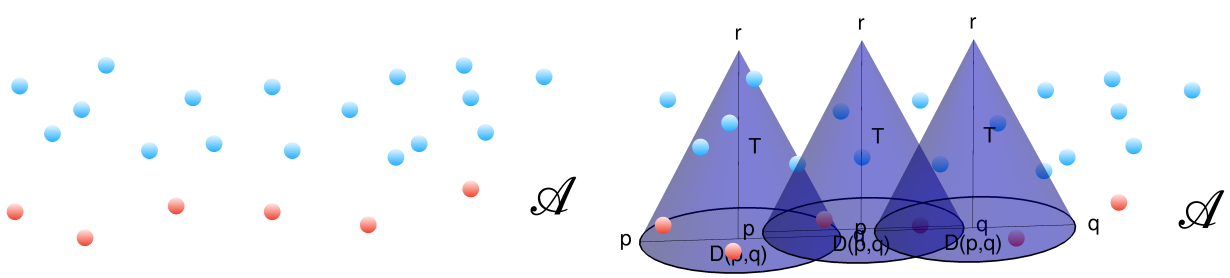
    - $e \prec e'$, a chain or total order $c = \{e, e_1, \ldots e_{k-1}, e'\}$, $l(c) = k$
    - $l(e, e') = \sup_{c \in C(e,e')} l(c)$ then $\langle l(e,e') \rangle = \zeta_d \tau(e,e') + o(\rho^{-m})$
    - $\lim_{\rho \to \infty} \frac{\langle l(e,e') \rangle}{(\rho V(e,e'))^{1/d}} = m_d,$
    - For $d = 2$, $m_2 = 2$, and for $d \geq 3$, $1.77 \leq \frac{2^{1-\frac{1}{d}}}{\Gamma(1+\frac{1}{d})} \leq m_d \leq \frac{2^{1-\frac{1}{d}} e\, (\Gamma(1+d))^{\frac{1}{d}}}{d} \leq 2.62$

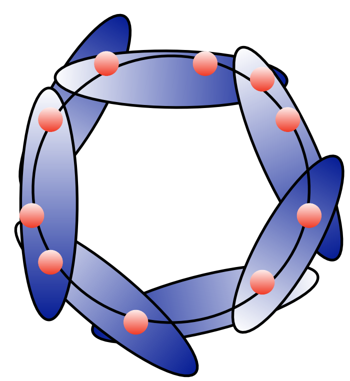


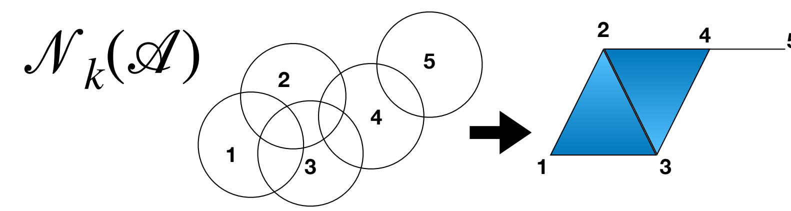
- Topological Invariants : Homology from thickened anti chains — *Major, Rideout and Surya, 2006,2007,2009*

  $C \hookrightarrow (M = \Sigma \times I, g)$, Σ : compact

  - Maximal antichain $\mathscr{A}$: has only the discrete topology
  - $\mathscr{T}_k(\mathscr{A}) = \{e \,|\, |\mathrm{Past}(e)| \leq k\}$
  - Future most elements: $\mathscr{M}_k \subset \mathscr{T}_k(\mathscr{A})$
  - $e \in \mathscr{M}_k, \mathscr{B}_k(e) = \mathrm{Past}(e) \cap \mathscr{A}$
  - $\{\mathscr{B}_k\}$ covers $\mathscr{A}$. Construct nerve simplicial complex $\mathscr{N}_k(\mathscr{A})$
  - For large enough ρ, k, using the De Rham-Weil Theorem $\mathscr{N}_k(\mathscr{A})$ is homological to Σ

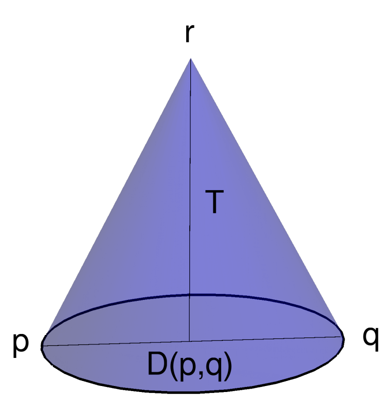


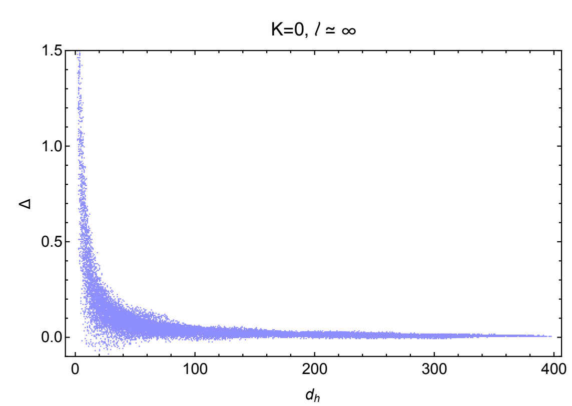


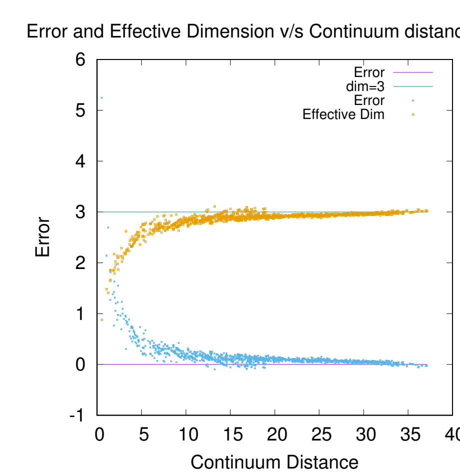


- Spatial Distance Function $V(r, \Sigma) = \zeta_d \left(\dfrac{D(p,q)}{2} \right)^d$

-Eichhorn, Mizera & Surya, 2017
-Eichhorn, Surya & Versteegen, 2018, 2019







- The Discrete Einstein-Hilbert or Benincasa-Dowker-Glaser Action(s)

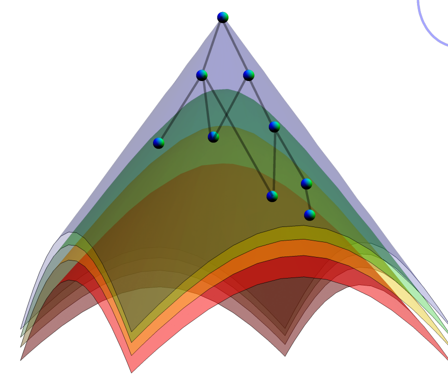
$$\frac{1}{\hbar}S^{(d)}_{BDG}(C) = \mu\left(n + \sum_{j=0}^{j_{max}} \lambda_j N_j\right),\ N_i = \#\text{ of i-element intervals}$$

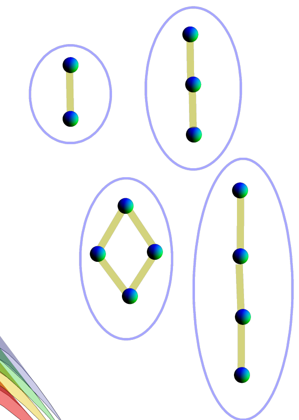
$$S^{(4)}_{BDG} = \frac{4}{\sqrt{6}}\left(n - N_0 + 9N_1 - 16N_2 + 8N_3\right)$$

$$\lim_{\rho_c \to \infty} \hbar \frac{l_c^2}{l_p^2}\langle S_{BDG}\rangle = S_{EH} + \text{bdry terms}$$

$$\text{In } \mathbb{M}^d:\ \lim_{N\to\infty} \frac{1}{\hbar}\left\langle S_{BDG}\right\rangle = \frac{1}{l_p^{d-2}}\text{vol}(\mathcal{J}^{(d-2)}),$$

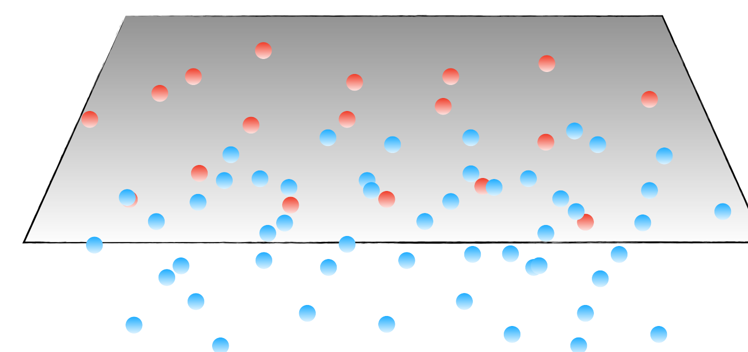
— *Benincasa & Dowker, 2010,*
— *Dowker & Glaser, 2012,*
— *Glaser, 2014*





—*Buck, Dowker, Jubb, Surya, 2016*
-- *Machet and Wang, 2020*

- Gibbons-Hawking-York Boundary Term for Spatial Σ
  - Consider $C \hookrightarrow_\rho (M = \Sigma \times I, g)$, Σ compact
  - Let Σ_f be future spatial boundary of $(M = \Sigma \times I, g)$
  - Future most antichain $\mathcal{A}_+ \subset C$
  - Let $\mathcal{A}_1 = \{e \,|\, |\text{Fut}(e) \cap C| = 1\}$
  - $S^{(d)}_{CBT}[\mathcal{A}_+] \equiv \dfrac{a_d}{\Gamma\left(\frac{2}{d}\right)}\left(d \times |\mathcal{A}_1| - |\mathcal{A}_+|\right)$
  - $\lim_{\rho\to\infty}\left(\dfrac{l_p}{l}\right)^{d-2}\left\langle \text{S}^{(d)}_{CBT}\right\rangle = \dfrac{1}{l_p^{d-2}}\int_\Sigma d^{d-1}x\,\sqrt{h}\,K = S_{GHY}(\Sigma, M^-),$



$\mathcal{A}_+$

—*Buck, Dowker, Jubb, Surya, 2016*

Many other examples in the literature..

Uniformly sampled discreteness contains a lot of information!

Typical posets in Ω_n

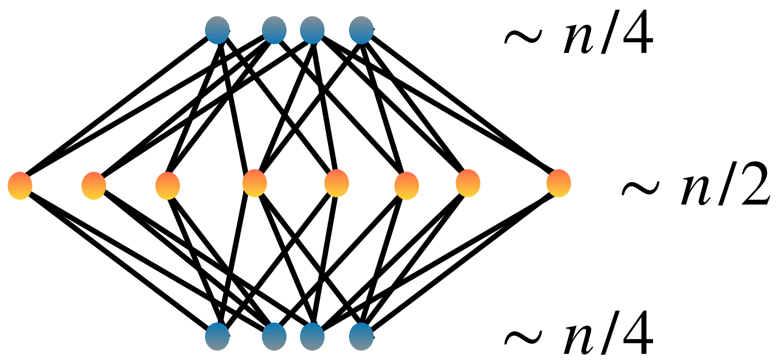
- Kleitman and Rothschild, Trans AMS, 1975

Ω_n : sample space of all n-element causal sets

$|\Omega_n| \sim 2^{\frac{n^2}{4}+\frac{3n}{2}+o(n)}$

Typical causal sets are Kleitmann-Rothschild (KR):

- 3 layers: $\mathbb{L}_k, \quad k = 1,2,3, \quad |\mathbb{L}_{1,3}| \sim \frac{n}{4}, |\mathbb{L}_2| \sim \frac{n}{2}$
- elements of $\mathbb{L}_k$ form an **antichain**
- $\forall\, e \in \mathbb{L}_1, \exists \sim \frac{n}{4}$ no. of $e' \in \mathbb{L}_2$ such that $e \prec_* e'$,
- $\forall\, e \in \mathbb{L}_3, \exists \sim \frac{n}{4}$ no. of $e' \in \mathbb{L}_2$ such that $e' \prec_* e$
- $\forall\, e \in \mathbb{L}_1, e' \in \mathbb{L}_3, \quad e' \prec e$

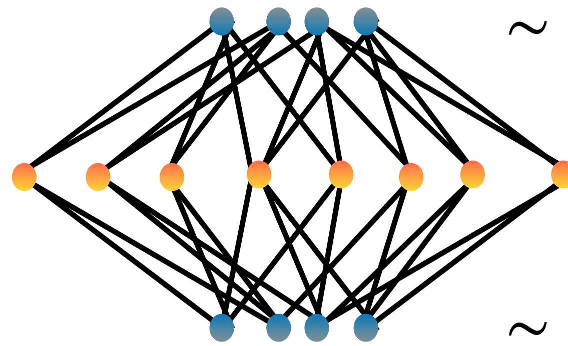


$|\Omega_{KR}| \sim 2^{\frac{n^2}{4}+\frac{3n}{2}+o(n)}$

Onset of asymptotic regime $n \sim 100$

- J. Henson, D. Rideout, R. Sorkin and S.Surya, JEM, 2015

A KR poset is not continuum-like

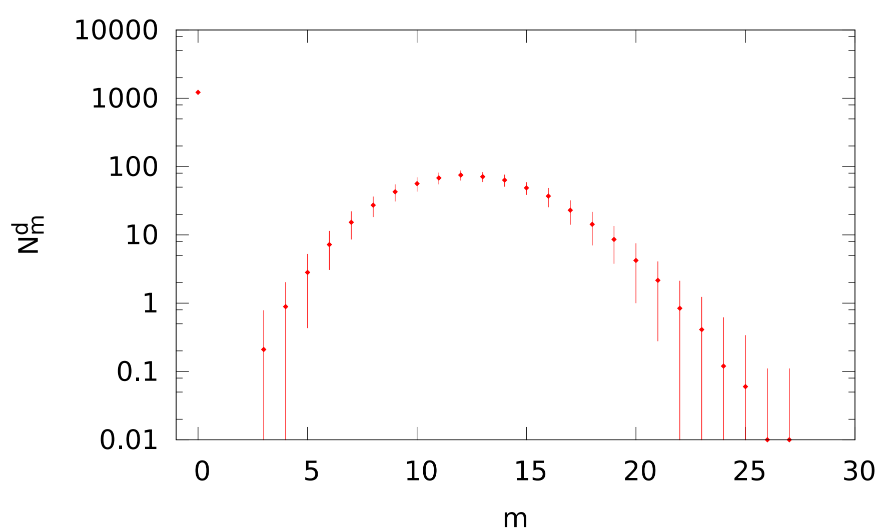


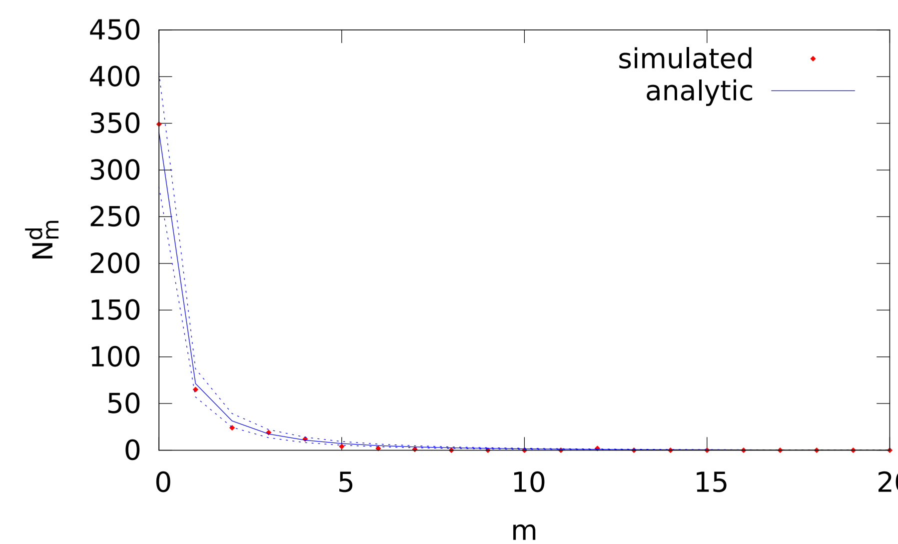
$$\langle R \rangle \sim \frac{3}{16} n^2$$

- Does not arise from a typical Poisson sprinkling into any continuum (M, g)
  - Myrheim-Myer Continuum Dimension is fractional :

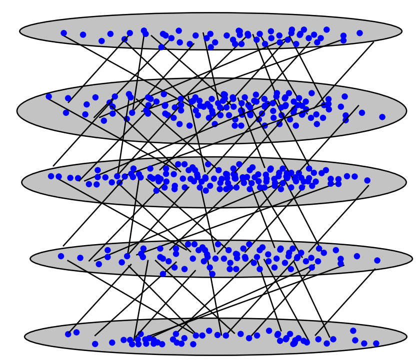
$$\frac{\langle R \rangle}{n^2} = \frac{\Gamma(d+1)\Gamma(d/2)}{4\Gamma(3d/2)} \Rightarrow \frac{\Gamma(d_{KR}+1)\Gamma(d_{KR}/2)}{4\Gamma(3d_{KR}/2)} = \frac{3}{16} \Rightarrow d_{KR} \sim 2.5$$

  - Maximal time-like distance $H_{KR} = 3$
  - Interval Abundances are not like the continuum:





The layered hierarchy



-D. Dhar, JMP, 1978
- Promel, Steger, Taraz 2001

- K-layered poset: $C = \mathbb{L}_1 \sqcup \mathbb{L}_2 \ldots \mathbb{L}_K : e \prec e', e \in \mathbb{L}_k, e' \in \mathbb{L}_{k'} \Rightarrow k < k'$

- $|\Omega_n^{(K)}| \sim 2^{c(d)n^2 + o(n^2)}, \quad c(d) \leq 1/4, \quad d = \text{ordering fraction},$

- Dominant hierarchy: $|\Omega_n^{(3)}| > |\Omega_n^{(2)}| > |\Omega_n^{(4)}| > |\Omega_n^{(5)}| \ldots$

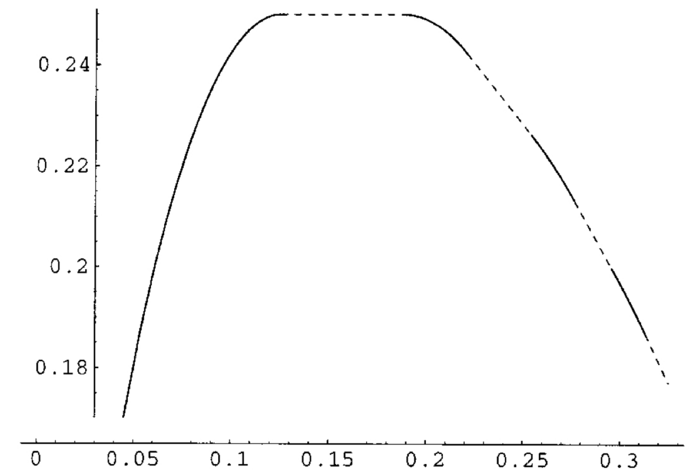


FIG. 5. c(d) in the range [0.05, 0.32].

While order Invariants are defined for all causal sets, they only carry geometric significance for continuum like causal sets

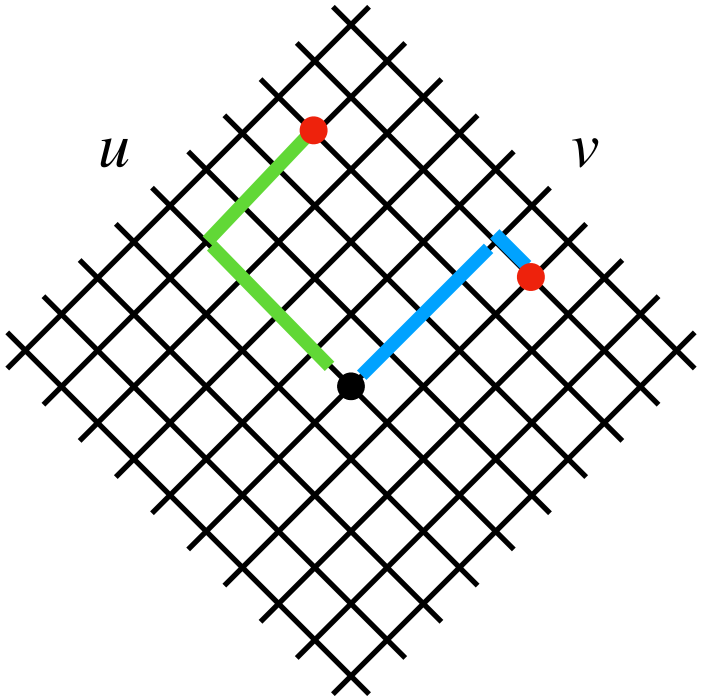
Some Thoughts on null distance and GH distance..

- Sormani-Vega Null distance function on causal sets
    - Time functions from maximal anti-chain $\mathscr{A}$
        i. strip-off maximal sets in $\mathrm{Past}(\mathscr{A})$ and minimal sets in $\mathrm{Fut}(\mathscr{A})$
        ii. OR obtain from Future/Past volume to $\mathscr{A}$
- How can one obtain a GH distance function on the space of causal sets?

GH-like distance on the space of 2d orders

-Work in progress with Alan Daniel Santosh

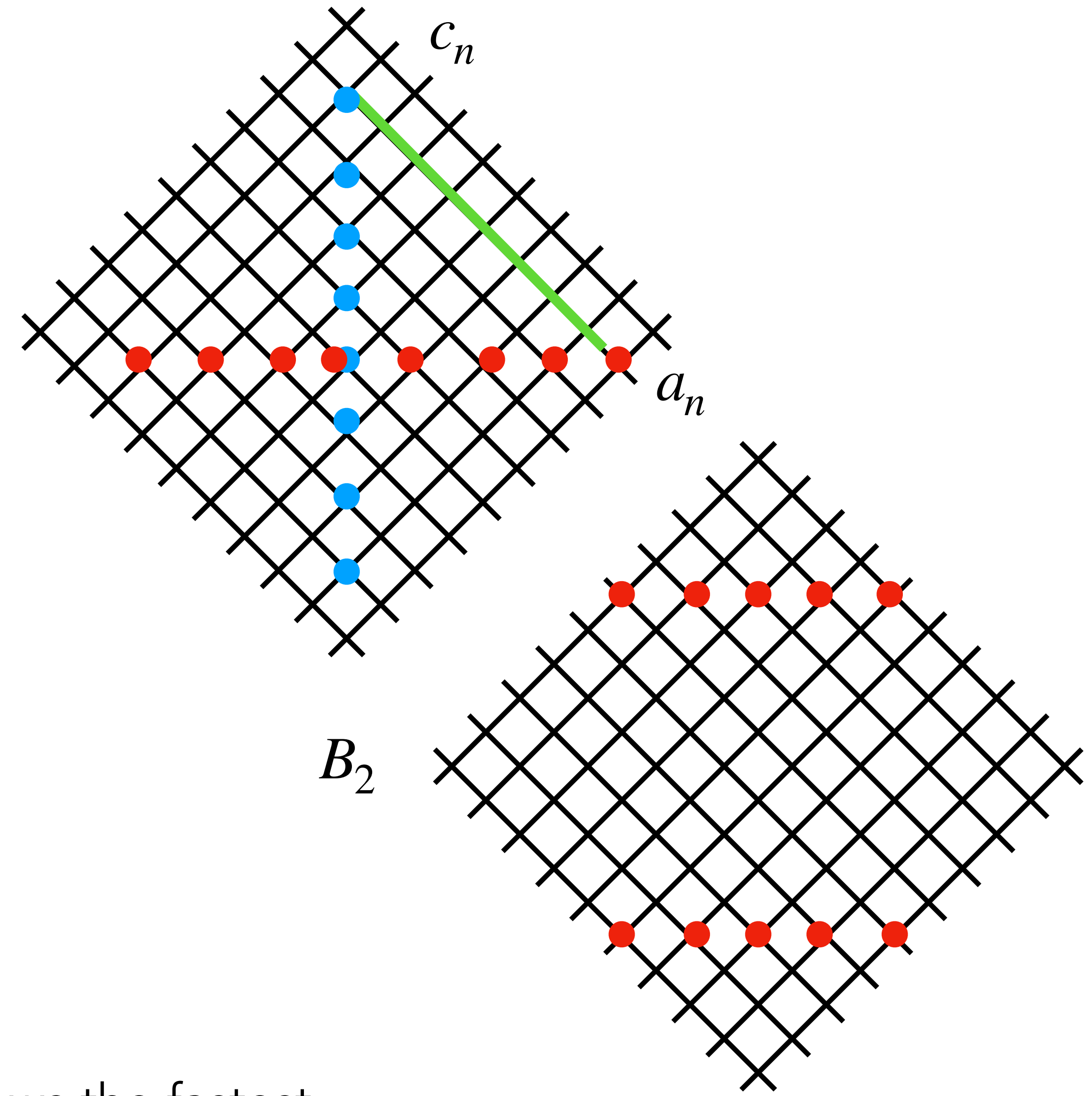
- $S = (1,2,\ldots n)$, $U = (u_1, \ldots u_n)$, $V = (v_1, \ldots v_n)$, $u_i \in S, v_i \in S$
- 2d order $C = U \cap V : e_i = (u_i, v_i) \prec e_j = (u_j, v_j) \Leftrightarrow u_i < u_j, v_i < v_j$
- Examples:
    - $u_1 < u_2 \ldots < u_n, v_1 < v_2 \ldots < v_n \Rightarrow C$ is a chain
    - $u_1 < u_2 \ldots < u_n, v_n < v_{n-1} \ldots < v_1 \Rightarrow C$ is an antichain
    - U, V randomly sampled : random 2d order $\sim (\mathbb{D}^2, \eta)$
- Every 2d order can be embedded *as a 2d order* into the light cone lattice $\mathscr{L}$
- The null distance function on $\mathscr{L}$: $d_N(a,b) = \frac{1}{2}(|u_b - u_a| + |v_b - v_a|)$
- $A, B \subseteq \mathscr{L}, \quad d_H(A,B) = \sup_{a \in A} \inf_{b \in B} d_N(a,b)$
- Let $c_1, c_2 \in \Omega_{2d}, \quad \mathscr{E}_i : c_i \hookrightarrow \mathscr{L}, \; d_{GH}(c_1, c_2) \equiv \inf_{\mathscr{E}_i} d_H^{\leftrightarrow}(\mathscr{E}_1(c_1), \mathscr{E}_2(c_2))$



$t = \frac{1}{2}(v+u), \quad x = \frac{1}{2}(v-u)$

Preliminary calculations...

- $d_{GH}(a_n, a_{n+1}) = 1, \quad d_{GH}(c_n, c_{n+1}) = 1$
- $d_{GH}(a_n, c_n) = m, \quad n = 2m \text{ or } n = 2m + 1$
- $d_{GH}(B_2, c_n) = \frac{n}{4}, \quad d_{GH}(B_2, a_n) = \frac{n}{4} + \frac{1}{2}$
- $d_{GH}(KR, c_n) \leq \frac{n}{4}, \quad d_{GH}(KR, a_n) = \frac{n}{6}$
- $d_{GH}(L_4, c_n) \leq \frac{n}{8}, \quad d_{GH}(L_4, a_n) \leq \frac{3n}{8}$



- Distance between Antichain a_n and Chain c_n grows the fastest
- Distance between the K-layer poset -- does it get closer to c_n than a_n as K increases?

Conjecture: For any $\epsilon > 0$ there exists n such that any two different realisations of random 2d orders P_n, P'_n are such that $d_{GH}(P_n, P'_n) \leq \epsilon$

Thank you!