ML, CALABI YAU METRICS & BEYOND

WE WANT ~~[illegible]~~ RICCI FLAT METRICS ON CY (& MORE..)

WHY?

IN STRING COMPACTIFICATION!

- MASSES OF PARTICLES (NORMALIZED YUKAWAS - eg EVERYONE ALWAYS GIVES)

(BUT MUCH BEYOND THAT!)

- SPECTRUM OF MASSIVE STATES (IMPORTANT EG FOR SWAMPLAND CONJECTURES)

- MODULI STABILIZATION / FINDING VACUA

→ NOTE FOR SUCH GOALS WILL NEED METRIC AS FN OF MODULI..

⋮

WHY HARD?

- 6D (SAY) MANIFOLD WITH NO CONTINUOUS ISOMETRIES
- NON CONSTRUCTIVE EXISTENCE PROOF WIN FIELDS MEDAL (YAU)

→ APPROXIMATION

THAT SAID: WOULD LIKE TO SEE FEATURES WE CAN PROVE ANALYTICALLY

PIPE DREAM: ANALYTIC ANSWER...

NOTE ML CAN MEAN A LOT OF THINGS.. BUT IN BROAD BRUSHSTROKES:

IDEAL ML PROBLEM: HARD TO DO, EASY TO CHECK (WITH CONCRETE ANSWER

PLAN:

- REVIEW OLD APPROACH:

  MODULAR & MODERN ML TECHNIQUES CAN BE SEEN AS REPLACING ONE OR MORE OF THOSE MODULES...

- DISCUSS IMPACT OF ML & HOW IT HAS LEAD TO QUALITATIVE (AS WELL AS QUANTITATIVE) ADVANCES IN WHAT WE CAN DO.

GOAL : FIND RICCI FLAT METRIC ON CY_3

PLAN :

- MAKE AN ANSATZ FOR $g_{i\bar{j}}$ (METRIC)

→ MORE PRECISELY FOR K ON EACH PATCH IN AN OPEN COVER S.T.

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$$

→ ANSATZ DEPENDS ON SOME PARAMETERS h

→ DEVELOP MECHANISM TO ADJUST PARAMS TO APPROXIMATE RICCI FLAT g AS CLOSELY AS POSSIBLE

FOR INITIAL PASS :

ILLUSTRATE WITH EFS OF FORM $\left[\mathbb{P}^N \,/\, N+1\right] = X$

METRIC ANSATZES

OPEN COVER OF $\mathbb{P}^N$

HOMOG COORD : $\quad z^a \qquad a = 1, \dots, N+1$

PATCHES : $\quad z^c \neq 0$

AFFINE COORD : $\quad u^a{}_c = \dfrac{z^a}{z^c} \qquad a \neq c$

$$u^c{}_c \equiv 1$$

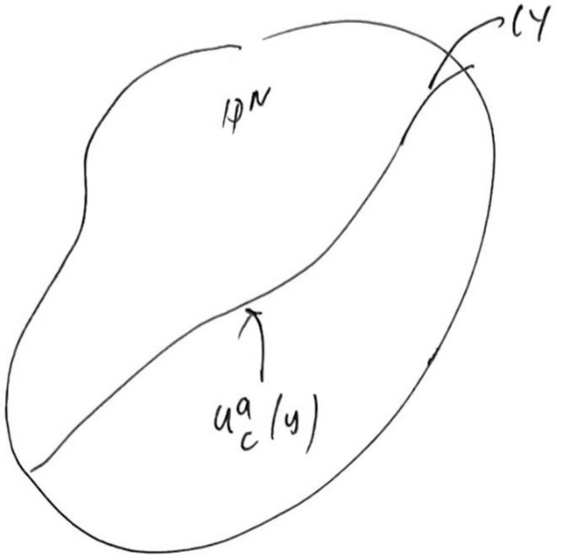
FUBINI STUDY METRIC

CONSTANT HERMITIAN MATRIX

$$k_c^{FS} := \ln \left(\frac{G_{a\bar{b}}\, z^a \bar{z}^{\bar{b}}}{|z^c|^2} \right) = \ln \left(G_{a\bar{b}}\, u^a_{\{c\}}\, \bar{u}_c^{\bar{b}} \right)$$

$$g^{FS}_{a\bar{b}} = \partial_a \partial_{\bar{b}} k_c^{FS}$$

$\rightarrow$ OBVIOUSLY CAN PULL BACK UNDER EMBEDDING MAP TO METRIC ON $[\mathbb{P}^N | N+1]$ BUT...



$$g^{CY}_{i\bar{j}} = \frac{\partial u^a_c}{\partial y^i} \frac{\partial \bar{u}^{\bar{b}}_{\bar{c}}}{\partial \bar{y}^{\bar{j}}} g^{FS}_{a\bar{b}}$$

→ NOT RICCI FLAT

→ GENERALIZE TO MORE FLEXIBLE METRIC ANSATZ WITH PARAMETERS WE CAN VARY TO TRY TO GET CLOSE TO RICCI FLAT.

FIRST n EIGENSPACES OF LAPLACIAN ON $\mathbb{P}^n$ SPANNED BY

$$\frac{P_n^I(z)\, \bar{P}_n^{\bar{J}}(\bar{z})}{\left(G_{a\bar{b}}\, z^a \bar{z}^{\bar{b}}\right)^n}$$

DEGREE n POLYNOMIALS

CONSTANT HERMITIAN MATRIX

COULD WRITE K AS SOME EXPANSION IN THESE BUT....

ON CY:

$$P_n^I(z) \sim P_n^I(z) + \alpha_{(n-(N+1))} \underbrace{P_{N+1}}_{}$$

DEFINING

DEFINING EQN OF CY.

WORK INSTEAD WITH $P_n^A \in H^0(X, \mathcal{O}(n))$

BASIS

THEN...

POSSIBLE ANSATZ FOR K:

HERMITIAN MATRIX OF PARAMETERS

$$K_c = K_c^{FS} + \frac{1}{n} \ln \left(\frac{h_{A\bar{B}} P_n^A \bar{P}_n^{\bar{B}}}{(G_{a\bar{b}} z^a \bar{z}^{\bar{b}})^n} \right)$$

$$= \frac{1}{n} \ln \left(h_{A\bar{B}} \frac{P_n^A \bar{P}_n^{\bar{B}}}{|z^c|^{2n}} \right)$$

$$= \frac{1}{n} \ln \left(h_{A\bar{B}} P_n^A(u_c) \bar{P}_n^{\bar{B}}(\bar{u}_c) \right)$$

COULD TRY TO VARY $h_{A\bar{B}}$ TO OBTAIN RICCI FLAT.

DOES THIS WORK IN PRINCIPLE? & NICER WAY TO VIEW THIS K?

- OUR LINE BUNDLE SECTIONS DESCRIBE AN MAP

$$i : X \longrightarrow \mathbb{P}^{h^0(X,\mathcal{O}(n))-1}$$

VIA

$$i\,(\underbrace{z_1, \ldots, z_{N+1}}_{\text{HOMOG COORD OF POINT ON } X}) = \begin{array}{c} X_{A1} \quad X_{A2} \quad \ldots \\ \left(p_n^1,\ p_n^2,\ \ldots\right) \end{array}$$

- CONSISTENT WITH SCALING $z \to \lambda z \Rightarrow p_n \to \lambda^n p_n$
- NOTE p_n^A MUST NOT BE ABLE TO VANISH SIMULTANEOUSLY FOR GOOD MAP

- i IS AN EMBEDDING IF

$$p_n^A \in H^0(X, L)$$ FOR US $\Rightarrow n \geq 1$

VERY AMPLE

SO HOW DO WE VIEW OUR KÄHLER POTENTIAL ANSATZ?

$$K_c = \frac{1}{n} \ln \left(h_{A\bar{B}} \, P_n^A(u_c) \, \bar{P}_n^{\bar{B}}(\bar{u}_c) \right) \qquad (*)$$

IS RESTRICTION OF K_c^{FS} ON $\mathbb{P}^{h^0(X, \mathcal{O}(n))-1}$ TO X.

(METRIC IS $\underline{g = i^* g_{FS}}$: JUST PULL BACK OF FS METRIC)

SO THATS WHAT METRIC IS. CAN THIS ANSATZ ALLOW US TO GET RICCI FLAT IN PRINCIPLE? :

THEOREM (TIAN) : LET $\{P_n^A\}$ BE A BASIS FOR $H^0(X, \mathcal{L}^n)$ FOR SOME AMPLE LINE BUNDLE $\mathcal{L}$. THEN THE SPACE OF ALL ALGEBRAIC KÄHLER POTENTIALS OF THE FORM $(*)$ WHERE $n \in \mathbb{Z}$ IS DENSE IN SPACE OF KÄHLER POTENTIALS

IN OTHER WORDS

→ GO TO HIGH ENOUGH DEGREE

→ GET ENOUGH PARAMETERS IN h TO TUNE

→ CAN GET ARBITRARILY CLOSE TO ANY KÄHLER g (INCL. $g_{\text{RICCI FLAT}}$)

NUMBERS OF PARAMETERS :

- TO GIVE A ROUGH IDEA :

$$N_{p_n^A} = h^0\left(X, \mathcal{O}(n)\right) = \begin{cases} \binom{n+N}{n} & \text{IF } n \leq N \\[2ex] \binom{n+N}{n} - \underbrace{\binom{n-1}{n-(N+1)}}_{N_{n-(N+1)} \text{ POLYNOMIALS}} & \text{IF } n > N \end{cases}$$

- GOES AS n^{N-1}

- FOR QUINIC FOR EXAMPLE

$$\frac{5}{6} n \left(5+n^2\right)$$

n	5	6	7	8	9
$\# p_n^A$	125	205	315	460	645
$\# h_{A\bar{B}}$	15625	42025	99225	211600	416025

$\rightarrow$ AND THOSE ARE INDEX RANGE OF $h_{A\bar{B}}$ IN OUR ANSATZ ...

THATS A LOT OF PARAMETERS...

IN OLDER WORK:

SYMMETRIES

- TUNE DEFINING RELN TO ADMIT DISCRETE SYMMETRIES
- RESTRICT TO SPACE OF INVARIANT POLYNOMIALS (UNDER INDUCE ACTION ON COM ASSUMING TRIVIAL EQUIV STRUCTURE.)

E.G: $[\mathbb{P}^4|5]$

TAKE $P_5^{DEFINING} = z_1^5 + \dots z_5^5$ ('FERMAT' QUINTIC)

SYMMETRIES:

- PERMUTATIONS ON z_a
- $z_a \to w_a z_a$
  5TH ROOT OF UNITY
- COMPLEX CONJUGATION

ORDER OF THAT DISCRETE GROUP: $2(N+1)^N (N+1)!$ → 150,000 FOR QUINTIC
(HEADRICK NOTES)

→ BRINGS # PARAMETERS DOWN TO $\mathcal{O}(10)$ & $n=9$

→ AVOIDING NEED TO DO THIS WILL BE ONE IMPROVEMENT WITH ML!

DERIVATIVES OF THE ALGEBRAIC K

- LOCALLY, ON CY HYPERSURFACE, CAN USE ANY SUBSET OF N-1 OF OUR N AFFINE COORDS u_c^a AS COORDINATES

$$z^i = u^i_c \qquad i \neq c, \delta$$

→ u^δ_c IS THEN FN OF z^i OBTAINED BY SOLVING

$$p_{N+1}^{\text{DEFINING}} = 0$$

- WE NEED DERIVATIVES OF FNs $f(u^a_c(z^i))$ W.R.T CY COORD. (NAIVELY HORRIBLE):

$$\partial_{z^i} f(u^a_c(z^i)) = \frac{\partial f}{\partial u^a_c} \frac{\partial u^a_c}{\partial z^i}$$

$$= \frac{\partial f}{\partial u^i_c} + \frac{\partial f}{\partial u^\delta_c} \frac{\partial u^\delta_c}{\partial u^i_c}$$

...

- BUT $p^{\text{DEFINING}} = 0$ EVERYWHERE ON CY:

$$0 = \frac{\partial p^{\text{DEFINING}}}{\partial z^i} = \frac{\partial p^{\text{DEFINING}}}{\partial u^i_c} + \frac{\partial p^{\text{DEFINING}}}{\partial u^\delta_c} \frac{\partial u^\delta_c}{\partial u^i_c}$$

SO

$$\frac{\partial u^d_c}{\partial u^i_c} = -\frac{\partial p^{\text{DEFINING}}}{\partial u^i_c} \Big/ \frac{\partial p^{\text{DEFINING}}}{\partial u^d_c}$$

$\Rightarrow$

$$\frac{\partial f}{\partial z^i} = \frac{\partial f}{\partial u^i_c} - \left(\frac{\partial p^{\text{DEFINING}}}{\partial u^i_c} \Big/ \frac{\partial p^{\text{DEFINING}}}{\partial u^d_c} \right) \frac{\partial f}{\partial u^d_c}$$

$\rightarrow$ CY DERIVS IN TERMS OF AMBIENT SPACE ONES WITHOUT HAVING TO SOLVE FOR u^d_c AS A FN OF u^i_c

NUMERICAL INTEGRATION OVER THE CY

- CONSIDER OPEN SET $U \subset X$
- DEFINE $\mathbb{1}_U(x) = \begin{cases} 1 & x \in U \\ 0 & x \notin U \end{cases}$
- THEN $\int_X \mathbb{1}_U \, \mu = \mathrm{Vol}_\mu (U)$ ← VOLUME WRT THIS MEASURE

- IF WE HAVE SAMPLE OF N_p POINTS $\{q_i \in X\}$ UNIFORMLY DISTRIBUTED ACCORDING TO μ:

EXPECTED NUM POINTS IN $U \subset X$: $\sum_{i=1}^{N_p} \mathbb{1}_U(q_i) = N_p \frac{\mathrm{Vol}_\mu U}{\mathrm{Vol}_\mu X}$

SO APPROXIMATION TO INTEGRAL IS

$$\int_X \mathbb{1}_U \, \mu \simeq \frac{\mathrm{Vol}_\mu(X)}{N_p} \sum_{i=1}^{N_p} \mathbb{1}_U(q_i)$$

- FOR A MORE GENERAL, ARBITRARY FN:

$$\int_X f \, \mu \simeq \underbrace{\frac{\mathrm{Vol}_\mu(X)}{N_p}}_{} \sum_{i=1}^{N_p} f(q_i)$$

← VOLUME ASSOCIATED TO EACH POINT

STATISTICAL ERROR $\sim O\left(\frac{1}{\sqrt{N_p}}\right)$ (SEE "NUMERICAL RECIPES IN C." PRESS ET AL for example)

IN PRACTICE: PRODUCING SAMPLE DISTRIBUTED ACCORDING TO μ NOT SO EASY!

SO:

- PRODUCE POINTS UNIFORMLY DISTRIBUTED ACCORDING TO SOME OTHER MEASURE μ'
- ACCOUNT FOR FACT THAT POINTS NOT DISTRIBUTED ACCORDING TO μ BY USING 'MASS FNS'.

DEFINE $m(x) = \dfrac{\mu(x)}{\mu'(x)}$

THEN

$$\int_X f\mu = \int_X f\frac{\mu}{\mu'}\mu' \simeq \frac{\mathrm{Vol}_{\mu'}(X)}{N_p}\sum_{i=1}^{N_p} f(q_i)\, m(q_i)$$

NOW

$$\int_X \mu = \mathrm{Vol}_\mu(X) = \int_X \frac{\mu}{\mu'}\mu' \approx \frac{\mathrm{Vol}_{\mu'}(X)}{N_p}\sum_{i=1}^{N_p} m(q_i)$$

$$\therefore \frac{\mathrm{Vol}_{\mu'}(X)}{N_p} \simeq \frac{\mathrm{Vol}_\mu(X)}{\sum_{i=1}^{N_p} m(q_i)}$$

$$\therefore \int_X f \mu \simeq \frac{\mathrm{Vol}_\mu(X)}{\sum_{i=1}^{N_p} m(q_i)} \sum_{i=1}^{N_p} f(q_i)\, m(q_i)$$

NO MENTION OF μ' OUTSIDE OF m

HOW DO YOU GET A SAMPLE OF POINTS WRT SOME MEASURE μ' ON THE CY?

E.G. ADAPTED TO OUR CASE: HOMOGENEOUS SAMPLING IN PROJ. SPACE

- UNIFORMLY SAMPLE TWO POINTS $z_{(1)}^a$ & $z_{(2)}^a$ ON $\mathbb{P}^N$

- GENERATE UNIFORMLY DISTRIBUTED POINTS ON S^{2N+1} BY, E.G., USING GAUSSIAN DIST. (SPHERICALLY SYMMETRIC)

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{1}{\sigma^2}\sum |z^a|^2}$$

& DIVIDING BY NORM.

- QUOTIENT OUT BY $U(1)$

$$z^i \to e^{i\theta} z^i$$

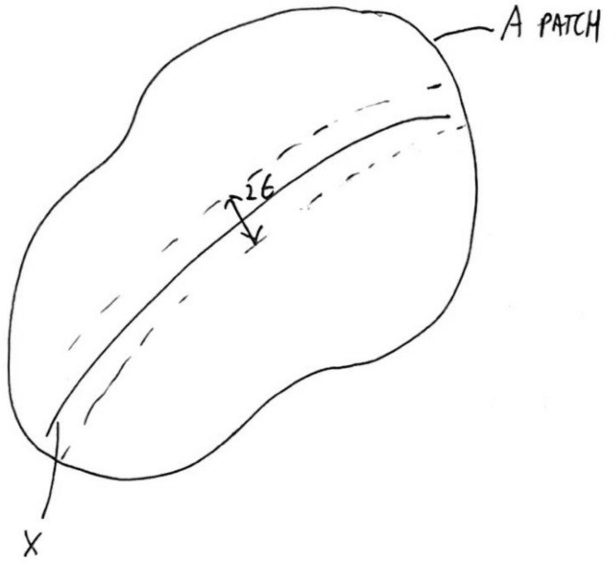
- COMPUTE $p^{\text{DEFINING}}(z_{(1)} + t\, z_{(2)})$ & SOLVE FOR t

- GIVES $(N+1)$ ROOTS ON CY

- RESULTING POINTS ARE DISTRIBUTED WRT TO KNOWN
F.S. MEASURE
→ CAN COMPUTE MASS FNS

[SEE HEP-TH/0612075 FOR A PROOF USING ORIGINAL RESULTS BY SHIFFMAN & ZELDITCH]

ANOTHER EXAMPLE : REJECTION SAMPLING (HEADRICK)



↳ TAKE $\left|\frac{z^a}{z^c}\right|^2 \leq 1 \quad \forall a \neq c$

as SET OF PATCHES

- CHOOSE POINTS RANDOMLY ON PATCHES (ACCORDING TO COORD MEASURE)
- THROW AWAY IF $|p| \geq \epsilon$
- PROJECT REMAINING POINTS ORTHOGONALLY WRT. FS METRIC.

ISSUES
- WASTEFUL (& SO RELATIVELY SLOW C.F. PREVIOUS EG)
- HARD TO KNOW MEASURE μ'

BUT...

- GETS JOB DONE!

- IN <u>SOME</u> RESPECTS GETTING μ' RIGHT DOESN'T MATTER <u>SO</u> MUCH!

  ↳ WE WILL SEE WHY WHEN WE USE THIS IN TUNING PARAMETERS

TUNING THE PARAMETERS

TWO COMMONLY EMPLOYED TECHNIQUES:

- DONALDSON ALGORITHM
  - 'PRINCIPLED' APPROACH
  - ITTERATION OF AN OPERATOR TO A FIXED POINT.

- MINIMIZATION OF AN APPROPRIATE FUNCTIONAL (HEADRICK & NASSAR)
  - PRACTICALLY MORE POWERFUL.
  - WORKS BETTER, EVEN IN PRINCIPLE, AWAY FROM IDEAL SITUATIONS

PREAMBLE

X IS AN $d=N-1$ FOLD

- DEFINE

HOL. 3 FORM

$$\mu = (-i)^d \, \Omega \wedge \bar{\Omega}$$

i FACTOR SO REAL

- GIVEN A KÄHLER FORM J DEFINE

$$V_J = \frac{J^d}{d!\,\mu}$$

(QUOTIENT OF TOP FORMS: FN)

DIRECT COMPUTATION $\longrightarrow$ $$V_J = \frac{\det g_{i\bar{j}}}{|\Omega_{1\cdots d}|^2}$$

NOTE

$$V_J = \text{CONST.} \qquad (\text{CALL IT } c)$$

$$\Rightarrow \frac{1}{d!} J^d = c\, \Omega \wedge \bar{\Omega}\, (-i)^d \qquad (\text{MONGE AMPERE EQN})$$

RICCI TENSOR:

$$R_{i\bar{j}}(J) = -\partial_i \partial_{\bar{j}} \ln V_J$$

Note $$\ln V_J = \ln \det g_{i\bar{j}} + \underbrace{\ln |\Omega_{1\cdots d}|^2}_{hol + hol} \rightarrow 0 \text{ ON DERIVS.}$$

DONALDSON'S ALGORITHM

DEFINE A 'T-OPERATOR'

$$T_n(h)^{A\bar{B}} := \frac{N_{p_n}}{\int_X \mu} \int_X \frac{p_n^A \bar{p}_n^{\bar{B}}}{h_{C\bar{D}}\, p_n^C \bar{p}_n^{\bar{D}}} \mu$$

THEOREM (DONALDSON)

FOR ANY INITIAL $h_{A\bar{B}}$ & FIXED n, THE SEQUENCE

$$h_{m+1} = \left(T_n(h_m)\right)^{-1}$$

CONVERGES TO A UNIQUE 'BALANCED' VALUE AS $m \to \infty$ & THE ASSOCIATED METRIC ON X IS CALLED THE BALANCED METRIC

(IN PRACTICE CONVERGES IN $\leq O(10)$ ITTERATIONS)

WHO CARES?

THEOREM (DONALDSON)

AS $n \to \infty$ THE BALANCED METRIC ON X CONVERGES TO THE UNIQUE RICCI-FLAT METRIC FOR A GIVEN KÄHLER CLASS & COMPLEX STRUCTURE

MORE GENERALLY ONE CAN CONSIDER NON CY CASES & BALANCED METRIC CONVERGES TO A KÄHLER METRIC ASSOCIATED TO THE KÄHLER CLASS $C_1(\mathcal{L})$ WHERE $P_n \in H^0(X, \mathcal{L})$. THIS WILL BE A CONSTANT SCALAR CURVATURE METRIC WITH CURVATURE DETERMINED BY $C_1(X)$

WHY WILL WE NOT TALK ABOUT THIS MUCH?

- ALTHOUGH AS $n \to \infty$ THE BALANCED METRIC $\to$ RICCI FLAT, FOR FIXED, FINITE, n THE BALANCED METRIC IS NOT THE CLOSEST YOU CAN GET TO RICCI FLAT! (HEADRICK & NASSAR)

- LETS GO ON TO METHODS THAT ARE ACTUALLY USED IN MODERN APPLICATIONS, & BY WHAT METRICS I MEAN 'CLOSEST TO RICCI FLAT' ETC...

MINIMIZATION APPROACH

THE IDEA: FORM A FUNCTIONAL THAT IS ONLY MINIMIZED BY THE RICCI FLAT METRIC

→ MINIMIZE IT AS A FN OF THE PARAMETERS h!

MONGE AMPERE FNL

$$M_{MA} := \int_X \mu \left(v - \langle v \rangle \right)^2$$

WHERE
$$\langle v \rangle := \frac{\int_X \mu v}{\int_X \mu}$$

- POSITIVE SEMI DEF WITH GLOBAL MIN @

$$v = \langle v \rangle$$
← A CONSTANT

FROM DEF OF v
$$\Rightarrow \frac{J^d}{d!} = \langle v \rangle \, \Omega \wedge \bar{\Omega} \, (-i)^d$$

→ MONGE AMPERE EQN!

- NO OTHER CRITICAL POINTS (CAN BE SHOWN BY DIRECT COMP.)

- NO DERIVATIVES OF J! ... μ FIXED & DETERMINED

→ PRETTY SIMPLE

NOTE THAT IF WE IGNORE THE MASS FNS THEN WE ARE JUST COMPUTING

$$\int_X \mu' \left(v - \langle v \rangle\right)^2$$

BOTH JUST SOME FIXED MEASURE...

INSTEAD OF

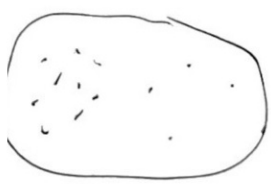
$$\int_X \mu \left(v - \langle v \rangle\right)^2$$

- HOWEVER $\Omega \wedge \bar{\Omega} \propto J \wedge J \wedge J$

→ SO μ IS REALLY CY VOL. FORM

... CAN EXPECT BETTER APPROX IN SOME CASE.

Eg PTS EVEN WRT μ' MEASURE DISPLAYED IN μ MEASURE!



(COULD BE LESS EFFICIENT)

~~- NO OTHER CRITICAL POINTS (CAN BE SHOWN BY DIRECT COMP.)~~

~~- NO DERIVATIVES OF J! ∴ μ FIXED & DETERMINED → PRETTY SIMPLE!~~

RICCI FUNC

$$H_R := -\frac{1}{2}\int_X \mu \underbrace{R}_{\text{RICCI SCALAR}}$$

$$= \int_X \mu\, g^{ij}\partial_i\partial_j \ln V$$

REWRITE TO MAKE PROPERTIES CLEARER:

$$H_R = \int_X \frac{J^n}{n!\,V}\, g^{ij}\partial_i\partial_j \ln V$$

$$= \int_X \frac{\sqrt{g}}{n!\,V}\, g^{ij}\partial_i\partial_j \ln V\, d^6x$$

COMES OUT AS $\det g_{ij} = \sqrt{g}$

IBP ↓

$$= -\int_X \partial_i\left(\frac{\sqrt{g}\, g^{ij}}{n!}\frac{1}{V}\right)\partial_j \ln V\, d^6x$$

$$\left\ulcorner \partial_i \left(\sqrt{g}\, g^{i\bar{j}}\right) = \sqrt{g}\, \Gamma^{\bar{j}}_{i\bar{k}}\, g^{i\bar{k}} = 0 \right\lrcorner$$

$$= -\int_X \frac{\sqrt{g}\, g^{i\bar{j}}}{n!}\, \partial_i \frac{1}{V}\, \partial_{\bar{j}} \ln V \, d^6x$$

$$= \int_X \mu\, g^{i\bar{j}}\, \partial_i \ln V\, \partial_{\bar{j}} \ln V$$

NOW WE CAN SEE

- +VE SEMI DEF. WITH GLOBAL MIN
  WHEN $V = \text{CONST}$ (RICCI FLAT!)

- NO OTHER CRITICAL PTS (DIRECT COMPUTATION)

BUT... NEEDS MORE DERIVS THAN HMA..