Twists & Localization: Local and Non-local models in QFT

Gandalf Lechner

joint with Ricardo Correa da Silva, Luca Giorgetti, Harald Grosse

FAU

Heisenberg-Programm

DFG Deutsche Forschungsgemeinschaft

Exactly solvable models – ESI Vienna – July 24, 2024

Mathematical Physics in Vienna in ~2009-10

Noncommutative Quantum Field Theory: Renormalization, Causality and Emergent Gravity

universität wien

Harald Grosse, Gandalf Lechner, Harold Steinacker, Fabien Vignes-Tourneret

PROJECT: Merge General Relativity with Quantum Physics through Noncommutative Geometry

- Quantum Field Theory on $\mathbb{R}^4$ suffers from infrared (IR) and

Minkowski Space Deformations and Causality Properties

There also exists a **general method** for deforming quantum field theories on noncommutative **Minkowski space**

. . . .

Some persons involved at Vienna University

H. Grosse, D. Klammer, G. Lechner, T. Ludwig, P. Schreivogl, H. Steinacker, F. Vignes-Tourneret

$$G^{\mu\nu} \sim \theta^{\mu\mu'}\theta^{\nu\nu'}g_{\mu'\nu'}$$
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- **Examples:** quantum plane $\mathbb{R}^4_\theta$, fuzzy spheres, ...
- effective (emergent) gravity, distinct from general relativity advantages upon quantization, consistent with vacuum energy

H. Steinacker: JHEP 2007, H. Grosse, H. Steinacker, M. Wohlgenannt: JHEP 2008, D. Klammer, H. Steinacker: JHEP 2008

PhD and Diploma Students, Projects, Coworkers:

PhD Students: *D. Klammer, T. Ludwig, P. Schreivogl*, Diploma Students: *T. Kaltenbrunner, A. Much, S. Urach*, Projects: *FWF P18657, FWF P20017, EU Marie Curie MEIF-CT-2007-041771*, Collaborators: *P. Aschieri, D. Blaschke, M. Buric, E. Kronberger, P. Presnajder, F. Lizzi, J. Madore, K.-G. Schlesinger, M. Schweda, R. Sedmik, Z. Wang, M. Wohlgenannt, R. Wulkenhaar, G. Zoupanos*

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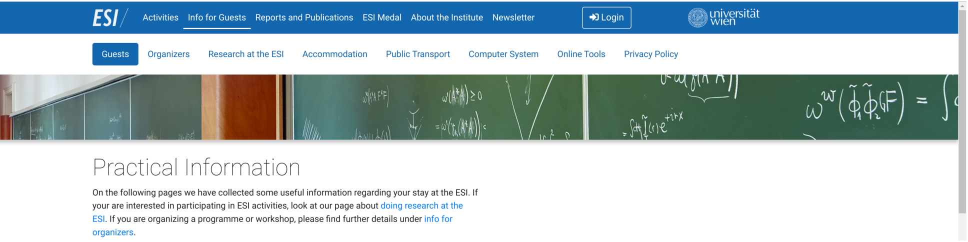

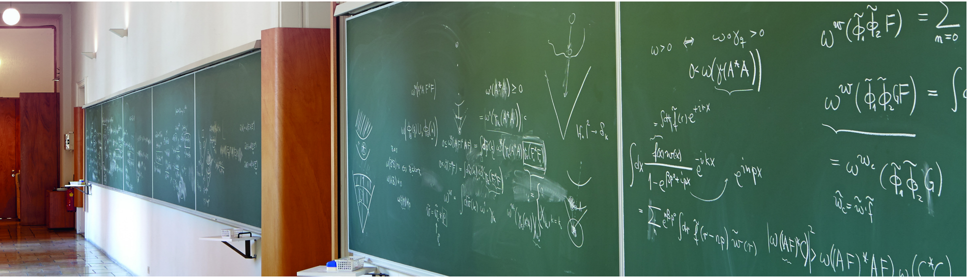

discussions about thermal NCQFT at ESI, documented on ESI website

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Many thanks, and Happy Birthday, Harald!

Overview of talk

Describe QFT models that are build from two ingredients:

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$$T : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}.$$

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So far, **no** conditions on T.

Aim: Build a Fock space and then quantum fields from T.

Twisted Fock spaces

- Define $T_k := 1^{\otimes(k-1)} \otimes T \otimes 1^{\otimes(n-k-1)}$ on $\mathcal{H}^{\otimes n}$
- Quantum symmetrizers ($\rightsquigarrow$ quantum groups)

$$P_T^{(n)} := \sum_{\pi \in S_n} \varphi_{T,n}(\pi), \qquad P_T^{(3)} = 1 + T_1 + T_2 + T_1T_2 + T_2T_1 + T_1T_2T_1$$

$$P_T^{(n+1)} = (1 \otimes P_T^{(n)})(1 + T_1 + T_1T_2 + \ldots + T_1 \cdots T_n).$$

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Examples

- $T = 0$: $\mathcal{F}_0(\mathcal{H})$ = Boltzmann Fock space over $\mathcal{H}$
- $T = F$ **(tensor flip)**: $\mathcal{F}_F(\mathcal{H})$ = Bose Fock space over $\mathcal{H}$
- $T = -F$: $\mathcal{F}_{-F}(\mathcal{H})$ = Fermi Fock space over $\mathcal{H}$
- $T = qF$: $\mathcal{F}_{qF}(\mathcal{H})$ = q-deformed Fock space over $\mathcal{H}$
- many more examples ...

Positivity

- Natural inner product on "n-particle space" $\mathcal{H}^{\otimes n}/\ker P_T^{(n)}$

$$\langle[\Psi],[\Phi]\rangle_{T,n} = \langle\Psi, P_T^{(n)}\Phi\rangle_{\mathcal{H}^{\otimes n}}.$$

needs to be **positive** $\rightsquigarrow$ $P_T^{(n)} \geq 0$.

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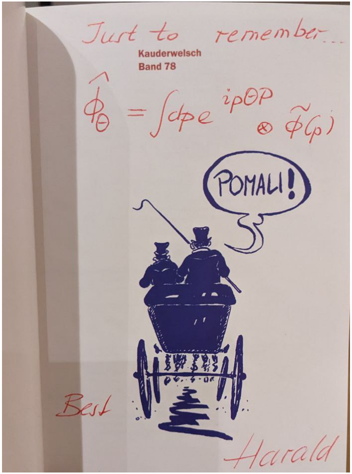
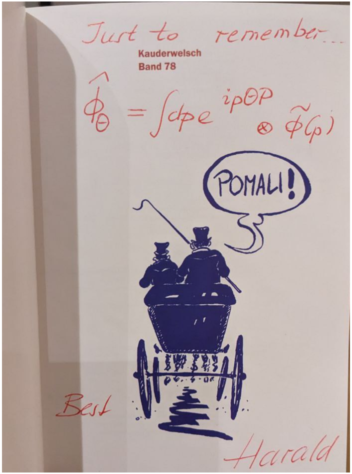
- Twists are selfadjoint, $T = T^*$, because $P_T^{(2)} = 1 + T \geq 0$.
- Simple examples: $T = 0 \Rightarrow P_T^{(n)} = 1$, and $T = \pm F \Rightarrow P_T^{(n)}$ =(anti-)symmetrization
- More examples ... ?

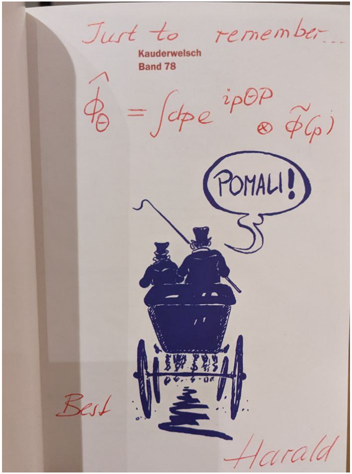

REISE KNOW-HOW
Kauderwelsch
Wienerisch
das andere Deutsch


Just to remember...
Kauderwelsch
Band 78
$\hat{\phi}_\Theta = \int dp\, e^{ip\Theta P} \otimes \tilde{\phi}(p)$
POMALI!
Best
Harald



$$\mathcal{H} = L^2(\mathbb{R}^d, dp), \qquad (Tf)(p,q) = e^{ip\cdot\theta q} \cdot f(q,p),$$

with θ antisymmetric real matrix, is a unitary twist.



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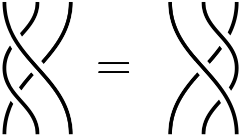
Theorem ([Jørgensen/Schmitt/Werner; Bożejko/Speicher])

Let $T = T^* \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$.

1. *If $\|T\| \leq \frac{1}{2}$ or $T \geq 0$, then T is a strict twist.*
2. *If T satisfies the Yang-Baxter Equation and $\|T\| \leq 1$, then T is a twist. Strict for $\|T\| < 1$.*

Reminder: Yang-Baxter Equation (YBE)



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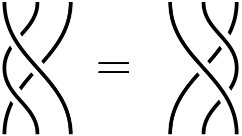

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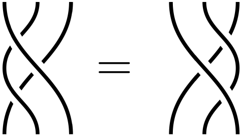

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From now on: $\mathcal{H}$ Hilbert space, T arbitrary twist.

Field operators

For every one-particle vector $\xi \in \mathcal{H}$, have

$$a_T^*(\xi) : \mathcal{F}_T(\mathcal{H}) \to \mathcal{F}_T(\mathcal{H}), \qquad [\Psi] \mapsto [\xi \otimes \Psi] \qquad \text{creation operator}$$

$$a_T(\xi) := T\text{-adjoint of } a_T^*(\xi) \qquad \text{annihilation operator}$$

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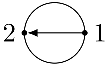
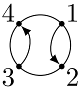
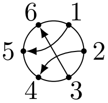
- The "n-point functions"

$$\langle \Omega, \phi_T(\xi_1) \cdots \phi_T(\xi_n) \Omega \rangle$$

can be expressed in diagrammatical form:



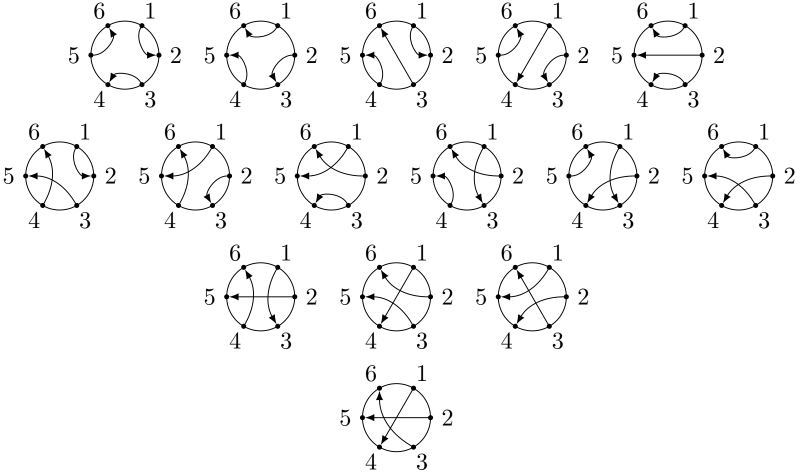




$\langle \xi_1, \xi_2 \rangle$ $\qquad$ $\langle \xi_1, \xi_2 \rangle \cdot \langle \xi_3, \xi_4 \rangle$ $\qquad$ $\langle \xi_3 \otimes T(\xi_2 \otimes \xi_1), T(\xi_4 \otimes \xi_5) \otimes \xi_6 \rangle$

Example: Six-point function

$$\langle \Omega, \phi_T(\xi_1) \cdots \phi_T(\xi_6) \Omega \rangle =$$



Field algebras and Locality

Note that $\xi \mapsto \phi_T(\xi)$ is only *real* linear.
Given a real linear subspace $H \subset \mathcal{H}$, consider algebra generated by fields,

$$\mathcal{L}_T(H) := \{\text{Polynomials in } \phi_T(h),\ h \in H\}''.$$

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Minimal requirement: $\mathcal{L}_T(H)$ should comply with the Reeh-Schlieder Theorem: Want

- $\mathcal{L}_T(H)\Omega$ is a dense subspace (Ω cyclic), and
- $\mathcal{L}_T(H)$ contains no vacuum annihilators (Ω separating)

Standard subspaces

We need to understand when Ω is cyclic and separating for $\mathcal{L}_T(H)$.

- For Ω cyclic ("large" algebra), need $H + iH \subset \mathcal{H}$ dense.
- For Ω separating ("small" algebra), need $H \cap iH = \{0\}$.
  (Otherwise $2a_T(h) = \phi_T(h) + i\phi_T(ih) \in \mathcal{L}_T(H)$)

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- H being standard **does imply** that Ω is cyclic for $\mathcal{L}_T(H)$, but **does not imply** that it is separating. Counterexample e.g. $T = 1$.

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Theorem ([Correa da Silva / L 22])

$H \subset \mathcal{H}$ standard subspace, T compatible twist. Then Ω is separating for $\mathcal{L}_T(H)$ if and only if two conditions are satisfied:

1. *T solves the* **YBE***.*
2. *T is* **crossing symmetric w.r.t.** *H: for all $v_1, v_2, w_1, w_2 \in \mathcal{H}$,*

$$\langle v_1 \otimes v_2, T^*(w_1 \otimes w_2)\rangle = \langle v_2 \otimes J_H \Delta_H^{-1/2} w_2, T(J_H \Delta_H^{1/2} v_1 \otimes w_1)\rangle.$$

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- Crossing symmetry is a generalization of a property of scattering amplitudes in QFT ($\rightsquigarrow$ analytic S-matrix, conformal bootstrap ...)
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▶ In situation of theorem, have $\mathcal{L}_T(H)' = \mathcal{R}_T(H')$ (left-right duality).

- In general, crossing symmetry is a subtle property involving analytic continuation of

$$t \longmapsto \langle v_1 \otimes \Delta_H^{it} v_2, T(\Delta_H^{it} w_1 \otimes w_2)\rangle,$$

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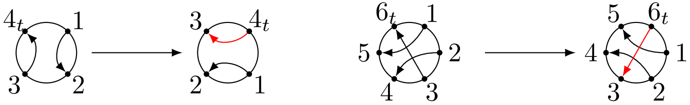

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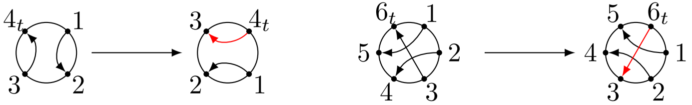
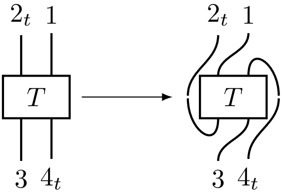


$\langle 2_t \otimes 1, T(3 \otimes 4_t)\rangle =$



$= \langle 1 \otimes \bar{4}_t, T(\bar{2}_t \otimes 3)\rangle$

- Leads to KMS (Gibbs type) property and then to separating vacuum

An NCQFT example, for simplicity in $d = 1 + 1$

Take $\mathcal{H} = L^2(\mathbb{R}, \frac{dp_1}{\sqrt{p_1^2+m^2}})$ and $\hat{p} = (p_0, p_1)$ with $p_0 = \sqrt{p_1^2 + m^2}$, and twist

$$(Tf)(p_1, q_1) = e^{i\hat{p}\cdot\theta\hat{q}} \cdot f(q_1, p_1).$$

This twist satisfies the YBE and is crossing symmetric w.r.t. the standard subspace H (a Hardy space) with

$$(J_H f)(p_1) = \overline{f(p_1)}, \qquad (\Delta_H^{it} f)(p_1) = f(\Lambda_{-2\pi t} p_1) \text{ Lorentz boost}$$

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- It turns out that this is an interacting model that is solvable in the sense that its S-matrix (factorizing, elastic) can be computed:

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- The localization given by H is localization in a spacelike wedge. We expect no better localization in this model.
- In a chiral situation, this is proven [GL/Scotford 22].

More models

- For *unitary* T, many models are known. In case T is compatible with an irreducible scalar positive energy rep. of the 2d Poincaré group, all twists are classified [Correa da Silva, Giorgetti, GL 24], namely
$$(Tf)(\eta_1, \eta_2) = s(\eta_2 - \eta_1) f(\eta_2, \eta_1),$$
with a bounded analytic function s on the strip $\mathbb{R} \times i(0, \pi)$ with
$$s(\eta + i\pi) = \overline{s(\eta)} = s(-\eta).$$
The corresponding models exist as local QFTs under a regularity condition on s, and can be solved: Their S-matrix is factorizing and asymptotically complete, with two-particle S-matrix s.

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- The models with $\|T\| < 1$ are expected to be more non-local (they are close to the free group factor at $T = 0$)
- Approach not restricted to Minkowski space. → Models on deSitter space, the real line, the circle (CFT), higher dimensions ..