Infinite algebras and intertwining networks for Calogero models

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joint work with **F. Correa**, **L. Inzunza**, **I. Marquette**, **M. Plyushchay**

- Calogero invariants and their algebra
- A W_3 algebra and a Casimir operator
- A Casimir operator
- Horizontal intertwiners and algebraic integrability
- Vertical intertwiners and a network
- Conclusions

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Calogero invariants and their algebra

$$H \;=\; \tfrac12 \sum_i p_i^2 \;+\; \sum_{i<j} \frac{\hbar^2 g(g-1)}{(x_i-x_j)^2}\;, \qquad i,j = 1,2,\ldots,N\;, \qquad g \geq \tfrac12$$

$$P \;=\; \sum_{i=1}^N p_i \qquad \text{and} \qquad X \;=\; \frac{1}{N}\sum_{i=1}^N x_i$$

$$[x_i,p_j] = \mathrm{i}\hbar\,\delta_{ij} \qquad \Rightarrow \qquad [X,P] = \mathrm{i}\hbar$$

permutations $\quad s_{ij} = s_{ji}\;, \quad s_{ij}x_i = x_j s_{ij}\;, \quad s_{ij}p_i = p_j s_{ij}\;, \quad s_{ij}^2 = 1$

Dunkl operators $\quad \pi_i \;:=\; p_i + \mathrm{i}\sum_{j(\neq i)} \frac{\hbar\, g}{x_i-x_j} s_{ij} \qquad \Rightarrow \qquad [\pi_i,\pi_j] = 0$

Liouville charges $\quad I_k \;\equiv\; B_{0,k} \;=\; \mathrm{res}\Big(\sum_i \pi_i^k\Big) \qquad \Rightarrow \qquad [B_{0,k},B_{0,\ell}] \;=\; 0$

$$B_{0,1} = P\,, \qquad B_{0,2} = 2H\,, \qquad B_{0,3} = \sum_i p_i^3 + 3\sum_{i<j} \frac{\hbar^2 g(g-1)}{(x_i-x_j)^2}(p_i+p_j)$$

$$B_{1,1} = \tfrac{1}{2}\sum_i (x_i p_i + p_i x_i) =: D \quad \text{and} \quad B_{2,0} = \sum_i x_i^2 =: 2K$$

$$sl(2,\mathbb{R}): \qquad \tfrac{1}{\hbar}[D,H] = 2\mathrm{i}H\,, \qquad \tfrac{1}{\hbar}[D,K] = -2\mathrm{i}K\,, \qquad \tfrac{1}{\hbar}[K,H] = \mathrm{i}D$$

$$B_{k,\ell} := \mathrm{res}\Big(\sum_i \mathrm{weyl}(x_i^k \pi_i^\ell)\Big) \quad \text{with} \quad \mathrm{e}^{\alpha x + \beta\pi} = \sum_{k,\ell=0}^{\infty} \frac{\alpha^k}{k!}\frac{\beta^\ell}{\ell!}\,\mathrm{weyl}(x^k\pi^\ell)$$

$k+\ell =:$ level

$$B_{0,0} = N\,\mathbb{1}$$

$$B_{1,0} = NX \qquad B_{0,1} = P$$

$$B_{2,0} = 2K \qquad B_{1,1} = D \qquad B_{0,2} = 2H$$

$$B_{3,0} \qquad B_{2,1} \qquad B_{1,2} \qquad B_{0,3}$$

$$B_{4,0} \qquad B_{3,1} \qquad B_{2,2} \qquad B_{1,3} \qquad B_{0,4}$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$\frac{1}{\mathrm{i}\hbar}\big[x_i,\pi_j\big] \;=\; \begin{Bmatrix} 1+\hbar\, g \sum_{k(\neq i)} s_{ik} & \text{for} \quad i=j \\ -\hbar\, g\, s_{ij} & \text{for} \quad i\neq j \end{Bmatrix}$$ and others commute

$$\frac{1}{\mathrm{i}\hbar}\big[B_{k,\ell},B_{m,n}\big] \;=\; (kn-\ell m)\, B_{k+m-1,\ell+n-1} + \sum_{r=1}^{\infty} \hbar^{2r} c^{2r+1}_{k\ell mn}\, B_{k+m-1-2r,\ell+n-1-2r}$$

$$c^{2r+1}_{k\ell mn} = \sum_{s=0}^{2r+1} (-)^{r+s} \frac{(k)_{2r+1-s}(\ell)_s(m)_s(n)_{2r+1-s}}{2^{2r} s!\,(2r+1-s)!}\,, \qquad (x)_q = x(x-1)\cdots(x-q+1)$$

$$k+m=\ell+n=2r+1 \qquad \Rightarrow \qquad \hbar^{2r}\Big(c^{2r+1}_{k\ell mn}\, B_{0,0}+P_r(g(g-1))\Big)$$ deformation

dependence on g only in central term and symmetric under $g \leftrightarrow 1-g$

$$\frac{1}{\mathrm{i}\hbar}\big[B_{1,0},B_{m,n}\big] = n\, B_{m,n-1}\ , \quad \frac{1}{\mathrm{i}\hbar}\big[B_{0,1},B_{m,n}\big] = -m\, B_{m-1,n}$$

$$\frac{1}{\mathrm{i}\hbar}\big[B_{2,0},B_{m,n}\big] = 2n\, B_{m+1,n-1}\ , \quad \frac{1}{\mathrm{i}\hbar}\big[B_{1,1},B_{m,n}\big] = (n-m)\, B_{m,n}\ , \quad \frac{1}{\mathrm{i}\hbar}\big[B_{0,2},B_{m,n}\big] = -2m\, B_{m-1,n+1}$$

$$\frac{1}{\mathrm{i}\hbar}\big[B_{k+1,1},B_{m+1,1}\big] = (k-m)\, B_{k+m+1,1}\ , \quad \frac{1}{\mathrm{i}\hbar}\big[B_{1,\ell+1},B_{1,n+1}\big] = (n-\ell)\, B_{1,\ell+n+1}$$

only N functionally indep't symmetric polynomials in N commuting variables π_i $\Rightarrow$
higher Liouville charges $B_{0,k>N}$ depend on N indep't ones $\{B_{0,1}, B_{0,2}, \ldots, B_{0,N}\}$

maximal superintegrability: $\exists\ N-1$ additional indep't (non-Liouville) charges

version 1, independent for $\ell = 1, \ldots, N-1$:

$$\widetilde{B}_{1,\ell} := B_{1,\ell} - t\, B_{0,\ell+1} \quad \Longrightarrow \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\widetilde{B}_{1,\ell} = \tfrac{1}{2\mathrm{i}\hbar}\big[\widetilde{B}_{1,\ell}, B_{0,2}\big] + \tfrac{\partial}{\partial t}\widetilde{B}_{1,\ell} = 0 \quad \checkmark$$

version 2:

$${}_kL_\ell := B_{0,k}B_{1,\ell-1} + B_{1,\ell-1}B_{0,k} - B_{0,\ell}B_{1,k-1} - B_{1,k-1}B_{0,\ell} \quad \Longrightarrow \quad \tfrac{\mathrm{d}}{\mathrm{d}t}({}_kL_\ell) = 0$$

independent example set for $k{=}2$: $F_\ell := {}_2L_\ell$ for $\ell{=}1, \ldots, N$ but $F_2{\equiv}0$

adjoint action of $sl(2,\mathbb{R})$ Casimir $\mathcal{C}_2$ generates the F_ℓ: $[\mathcal{C}_2, I_2]{=}0$

$$\mathcal{C}_2 = \tfrac{1}{2}(B_{2,0}B_{0,2} + B_{0,2}B_{2,0}) - B_{1,1}^2 \quad \Longrightarrow \quad \tfrac{1}{\mathrm{i}\hbar}\big[\mathcal{C}_2\, , I_\ell\big] = \ell\, F_\ell$$

additional charges F_ℓ not in involution but obey a polynomial algebra of order $2N-1$

A W_3 algebra

$$N{=}2: \quad B_{1,0}\,,\quad B_{0,1}\,;\quad B_{2,0}\,,\quad B_{1,1}\,,\quad B_{0,2}$$

$$B'_{2,0} := B_{2,0} - \tfrac{1}{2}B_{1,0}B_{1,0}\,,\ B'_{1,1} := B_{1,1} - \tfrac{1}{4}\{B_{1,0},B_{0,1}\}\,,\ B'_{0,2} := B_{0,2} - \tfrac{1}{2}B_{0,1}B_{0,1}$$

$$\{B'_{k,\ell}\} \Rightarrow sl(2,\mathbb{R}) \qquad \text{and} \qquad [B_{1,0},B'_{k,\ell}] = 0 = [B_{0,1},B'_{k,\ell}] \qquad \text{for} \quad k+\ell = 2$$

$$W_2 = W'_2 \oplus W_1 = sl(2,\mathbb{R})' \oplus \text{Heisenberg}$$

$$\mathcal{C}'_2 = 2\{K',H'\} - D'^2 = \tfrac{1}{2}\{B'_{2,0},B'_{0,2}\} - B'^2_{1,1} \overset{p_i \mapsto \frac{\hbar}{\mathrm{i}}\partial_i}{=} \hbar^2 g(g-1) \geq -\tfrac{1}{4}\hbar^2$$

$$N{=}3: \quad B_{1,0}\,,\quad B_{0,1}\,;\quad B_{2,0}\,,\quad B_{1,1}\,,\quad B_{0,2}\,;\quad B_{3,0}\,,\quad B_{2,1}\,,\quad B_{1,2}\,,\quad B_{0,3}$$

$$B_{3,0} = \sum_i x_i^3\,,\quad B_{2,1} = \sum_i \text{weyl}(x_i^2 p_i) = \tfrac{1}{2}\sum_i (x_i^2 p_i + p_i\,x_i^2) = \sum_i x_i p_i x_i$$

$$B_{1,2} = \sum_i \text{weyl}(x_i\,p_i^2) + \sum_{i<j} \frac{\hbar^2 g(g-1)}{(x_i-x_j)^2}(x_i+x_j)\,,\quad B_{0,3} = \sum_i p_i^3 + 3\sum_{i<j}\frac{\hbar^2 g(g-1)}{(x_i-x_j)^2}(p_i+p_j)$$

$\mathcal{U}(W_3)$: Weyl$\Big(A_1 A_2 \cdots A_q\Big) := \frac{1}{q!} \sum_{\sigma \in S_q} A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(q)}$ for $A_s \in \{B_{k,\ell}\}$

notation $B_{k,\ell} =: (k\ell)$ and Weyl$(B_{k,\ell} B_{m,n} \ldots B_{s,t}) =: (k\ell|mn|\ldots|st)$

$\frac{1}{i\hbar}[B_{k,\ell}, B_{m,n}]$	(30)	(21)	(12)	(03)
(30)	0	3(40)	6(31)	$9(22) - \frac{3}{2}\hbar^2(00)$ $+9\hbar^2 g(g-1)$
(21)	$-3(40)$	0	$3(22) + \frac{1}{2}\hbar^2(00)$ $-3\hbar^2 g(g-1)$	6(13)
(12)	$-6(31)$	$-3(22) - \frac{1}{2}\hbar^2(00)$ $+3\hbar^2 g(g-1)$	0	3(04)
(03)	$-9(22) + \frac{3}{2}\hbar^2(00)$ $-9\hbar^2 g(g-1)$	$-6(13)$	$-3(04)$	0

dependent observables:

$$(40) = \tfrac{4}{3}(30|10) + \tfrac{1}{2}(20|20) - (20|10|10) + \tfrac{1}{6}(10|10|10|10)$$
$$(31) = \tfrac{1}{3}(30|01) + (21|10) + \tfrac{1}{2}(20|11) - \tfrac{1}{2}(20|10|01) - \tfrac{1}{2}(11|10|10) + \tfrac{1}{6}(10|10|10|01)$$
$$(22) = \tfrac{2}{3}(21|01) + \tfrac{1}{6}(20|02) + \tfrac{2}{3}(12|10) + \tfrac{1}{3}(11|11) - \tfrac{1}{6}(20|01|01) - \tfrac{2}{3}(11|10|01) - \tfrac{1}{6}(10|10|02) + \tfrac{1}{6}(10|10|01|01)$$
$$(13) = (12|01) + \tfrac{1}{2}(11|02) + \tfrac{1}{3}(10|03) - \tfrac{1}{2}(11|01|01) - \tfrac{1}{2}(10|02|01) + \tfrac{1}{6}(10|01|01|01)$$
$$(04) = \tfrac{4}{3}(03|01) + \tfrac{1}{2}(02|02) - (02|01|01) + \tfrac{1}{6}(01|01|01|01)$$

center-of-mass decoupling: $W_3 = W_3' \oplus$ Heisenberg

$$B'_{2,0} \equiv (20)' = (20) - \tfrac{1}{3}(10|10)$$
$$B'_{1,1} \equiv (11)' = (11) - \tfrac{1}{3}(10|01)$$
$$B'_{0,2} \equiv (02)' = (02) - \tfrac{1}{3}(01|01)$$
$$B'_{3,0} \equiv (30)' = (30) - (20|10) + \tfrac{2}{9}(10|10|10)$$
$$B'_{2,1} \equiv (21)' = (21) - \tfrac{1}{3}(20|01) - \tfrac{2}{3}(11|10) + \tfrac{2}{9}(10|10|01)$$
$$B'_{1,2} \equiv (12)' = (12) - \tfrac{2}{3}(11|01) - \tfrac{1}{3}(10|02) + \tfrac{2}{9}(10|01|01)$$
$$B'_{0,3} \equiv (03)' = (03) - (02|01) + \tfrac{2}{9}(01|01|01)$$

nested Weyl ordering:

$$\begin{aligned}
\big(a\big|(b|c)\big) &= (a|b|c) + \tfrac{1}{12}\{[[a,b],c] + [[a,c],b]\} \\
\big(a\big|b\big|(c|d)\big) &= (a|b|c|d) + \tfrac{1}{12}\big\{\big(a\big|[[b,c],d]\big) + \big(a\big|[[b,d],c]\big) + \big([a,c]\big|[b,d]\big) + \ (a \leftrightarrow b)\big\} \\
\big(a\big|(b|c|d)\big) &= (a|b|c|d) + \tfrac{1}{12}\big\{\big(b\big|[[a,c],d]\big) + \big(b\big|[[a,d],c]\big) + \text{ cyclic in } (b,c,d)\big\} \\
\big((a|b)\big|(c|d)\big) &= (a|b|c|d) + \tfrac{1}{12}\big\{\big(a\big|[[b,c],d]\big) + \big(a\big|[[b,d],c]\big) + \big(b\big|[[a,c],d]\big) + \big(b\big|[[a,d],c]\big) + \big(\begin{smallmatrix} a\leftrightarrow c \\ b\leftrightarrow d\end{smallmatrix}\big)\big\} \\
&\quad + \tfrac{1}{4}\big\{\big([a,c]\big|[b,d]\big) + \big([a,d]\big|[b,c]\big)\big\}
\end{aligned}$$

$\frac{1}{i\hbar}[B'_{k,\ell}, B'_{m,n}]$	$(30)'$	$(21)'$	$(12)'$	$(03)'$
$(30)'$	0	$\frac{1}{2}(20\|20)'$	$(20\|11)'$	$-\frac{3}{2}(20\|02)'+3(11\|11)'$ $+\hbar^2[9g(g-1)-4]$
$(21)'$	$-\frac{1}{2}(20\|20)'$	0	$\frac{5}{6}(20\|02)'-\frac{1}{3}(11\|11)'$ $-\hbar^2[3g(g-1)-\frac{4}{3}]$	$(11\|02)'$
$(12)'$	$-(20\|11)'$	$-\frac{5}{6}(20\|02)'+\frac{1}{3}(11\|11)'$ $+\hbar^2[3g(g-1)-\frac{4}{3}]$	0	$\frac{1}{2}(02\|02)'$
$(03)'$	$\frac{3}{2}(20\|02)'-3(11\|11)'$ $-\hbar^2[9g(g-1)-4]$	$-(11\|02)'$	$-\frac{1}{2}(02\|02)'$	0

in $sl(2,\mathbb{R})'$ covariant notation:

$$(20)' =: \sqrt{8}\, J_{-1}\,, \qquad (11)' =: 2\, J_0\,, \qquad (02)' =: \sqrt{8}\, J_{+1}$$

$$(30)' =: 2\, K_{-3/2}\,, \quad (21)' =: \frac{2}{\sqrt{3}} K_{-1/2}\,, \quad (12)' =: \frac{2}{\sqrt{3}} K_{+1/2}\,, \quad (03)' =: 2\, K_{+3/2}$$

spin-1 and spin-$\frac{3}{2}$ representations of $sl(2,\mathbb{R})'$:

$$\frac{1}{i\hbar}[J_i, J_k] = f_{ik}^{\ \ \ell} J_\ell \qquad \text{and} \qquad \frac{1}{i\hbar}[J_i, K_\alpha] = f_{i\alpha}^{\ \ \beta} K_\beta$$

antisymmetric coupling of two spin–$\frac{3}{2}$ representations:

$$[K, K] \sim J\, J + \text{central} : \quad \left[\frac{\mathbf{3}}{\mathbf{2}} \otimes \frac{\mathbf{3}}{\mathbf{2}}\right]_{\mathrm{A}} = \mathbf{2} \oplus \mathbf{0} = \left[\mathbf{1} \otimes \mathbf{1}\right]_{\mathrm{S}}$$

singlet **0** = $sl(2,\mathbb{R})'$ Casimir:

$$\mathcal{C}_2' = (20|02)' - (11|11)' = 8\,(J_{+1}|J_{-1}) - 4\,(J_0|J_0)$$

$\frac{1}{\mathrm{i}\hbar}[K_\alpha, K_\beta]$:

	$K_{3/2}$	$K_{1/2}$	$K_{-1/2}$	$K_{-3/2}$
$K_{3/2}$	0	$-\sqrt{3}\,(J_{+1}\vert J_{+1})$	$-\sqrt{6}\,(J_{+1}\vert J_0)$	$-(J_{+1}\vert J_{-1}) - (J_0\vert J_0)$ $-\frac{1}{2}\mathcal{C}'_2 - \hbar^2\,C$
$K_{1/2}$	$\sqrt{3}\,(J_{+1}\vert J_{+1})$	0	$-(J_{+1}\vert J_{-1}) - (J_0\vert J_0)$ $+\frac{1}{2}\mathcal{C}'_2 + \hbar^2\,C$	$-\sqrt{6}\,(J_0\vert J_{-1})$
$K_{-1/2}$	$\sqrt{6}\,(J_{+1}\vert J_0)$	$(J_{+1}\vert J_{-1}) + (J_0\vert J_0)$ $-\frac{1}{2}\mathcal{C}'_2 - \hbar^2\,C$	0	$-\sqrt{3}\,(J_{-1}\vert J_{-1})$
$K_{-3/2}$	$(J_{+1}\vert J_{-1}) + (J_0\vert J_0)$ $+\frac{1}{2}\mathcal{C}'_2 + \hbar^2\,C$	$\sqrt{6}\,(J_0\vert J_{-1})$	$\sqrt{3}\,(J_{-1}\vert J_{-1})$	0

with central term $C = \frac{9}{4}\,g(g-1) - 1$

nontrivial nonlinear commutator:

$$\frac{1}{\mathrm{i}\hbar}[K_\alpha, K_\beta] = f_{\alpha\beta}^{i\,k}\,(J_i|J_k) + \epsilon_{\alpha\beta}\Big(\frac{1}{2}\,\mathcal{C}'_2 + \hbar^2[\tfrac{9}{4}g(g-1)-1]\Big)$$

A Casimir operator

N=3: 9 generators but dim(phase space)=6 $\Rightarrow$ expect three Casimir operators

classical ansatz: $\mathcal{C}_6^{\text{class}} = \alpha\, T'^{6}_{66} + \beta\, T'^{5}_{66} + \gamma\, T'^{4}_{66}$ with $\alpha, \beta, \gamma \in \mathbb{R}$

$$T'^{6}_{66} = (20|20|20|02|02|02)' - 3\,(20|20|11|11|02|02)' + 3\,(20|11|11|11|11|02)' - (11|11|11|11|11|11)'$$

$$\begin{aligned} T'^{5}_{66} = {} & (30|30|02|02|02)' - 6\,(30|21|11|02|02)' + 6\,(30|20|12|02|02)' - 6\,(30|20|11|03|02)' \\ & + 4\,(30|11|11|11|03)' - 3\,(21|21|20|02|02)' + 12\,(21|21|11|11|02)' + 6\,(21|20|20|03|02)' \\ & - 6\,(21|20|12|11|02)' - 12\,(21|12|11|11|11)' + (20|20|20|03|03)' - 3\,(20|20|12|12|02)' \\ & - 6\,(20|20|12|11|03)' + 12\,(20|12|12|11|11)' \end{aligned}$$

$$T'^{4}_{66} = (30|30|03|03)' - 6\,(30|21|12|03)' + 4\,(30|12|12|12)' + 4\,(21|21|21|03)' - 3\,(21|21|12|12)'$$

$$[T'^{s}_{66}, (20)'] = [T'^{s}_{66}, (11)'] = [T'^{s}_{66}, (02)'] = 0 \quad \text{for} \quad s = 6, 5, 4 \quad \checkmark$$

$$[\mathcal{C}_6^{\text{class}}, (30)'] = [\mathcal{C}_6^{\text{class}}, (21)'] = [\mathcal{C}_6^{\text{class}}, (12)'] = [\mathcal{C}_6^{\text{class}}, (03)'] \stackrel{!}{=} 0$$

$$[T'^{6}_{66}, (30)'] \longrightarrow T'^{6}_{85}\,, \quad [T'^{5}_{66}, (30)'] \stackrel{\hbar=0}{\longrightarrow} T'^{6}_{85} \;\&\; T'^{5}_{85}\,, \quad [T'^{4}_{66}, (30)'] \stackrel{\hbar=0}{\longrightarrow} T'^{5}_{85}$$

classical solution: $\mathcal{C}_6^{\text{class}} = 6\,T'^6_{66} + 9\,T'^5_{66} - 54\,T'^4_{66}$

turn on $\hbar$: $[T'^5_{66}, (30)'] \longrightarrow T'^6_{85}\ \&\ T'^5_{85}\ \&\ \hbar^2 T'^4_{63}\ \&\ \hbar^4 T'^2_{41}$

$[T'^4_{66}, (30)'] \longrightarrow T'^5_{85}\ \&\ \hbar^2 T'^3_{63}\ \&\ \hbar^4 T'^2_{41}$

quantum ansatz : $\mathcal{C}_6^{\text{quant}} = \mathcal{C}_6^{\text{class}} + \hbar^2(\delta\,T'^4_{44} + \epsilon\,T'^3_{44}) + \hbar^4 \zeta\,T'^2_{22}$ with $\delta, \epsilon, \zeta \in \mathbb{R}$

$$T'^4_{44} = (20|20|02|02)' - 2\,(20|11|11|02)' + (11|11|11|11)'$$

$$T'^3_{44} = (30|12|02)' - (30|11|03)' - (21|21|02)' + (21|20|03)' + (21|12|11)' - (20|12|12)'$$

$$T'^2_{22} = (20|02)' - (11|11)'$$

$$\hbar^2[T'^4_{44}, (30)'] \longrightarrow \hbar^2 T'^4_{63}$$

$$\hbar^2[T'^3_{44}, (30)'] \longrightarrow \hbar^2 T'^4_{63}\ \&\ \hbar^2 T'^3_{63}\ \&\ \hbar^4 T'^2_{41}$$

$$\hbar^4[T'^2_{22}, (30)'] \longrightarrow \hbar^4 T'^2_{41}$$

highly overdetermined system! quantum solution:

$$(\delta, \epsilon, \zeta) = (207 - 108\lambda\,,\ 648 - 324\lambda\,,\ 709 - 1656\lambda + 486\lambda^2)\ , \qquad \lambda \equiv g(g-1)$$

the lowest quantum W_3' Casimir in one formula:

$$\mathcal{C}_6^{\rm quant} =$$

$$6\{(20|20|20|02|02|02)' - 3\,(20|20|11|11|02|02)' + 3\,(20|11|11|11|11|02)' - (11|11|11|11|11|11)'\}$$

$$+ 9\,\{(30|30|02|02|02)' - 6\,(30|21|11|02|02)' + 6\,(30|20|12|02|02)' - 6\,(30|20|11|03|02)'$$
$$+ 4\,(30|11|11|11|03)' - 3\,(21|21|20|02|02)' + 12\,(21|21|11|11|02)' + 6\,(21|20|20|03|02)'$$
$$- 6\,(21|20|12|11|02)' - 12\,(21|12|11|11|11)' + (20|20|20|03|03)' - 3\,(20|20|12|12|02)'$$
$$- 6\,(20|20|12|11|03)' + 12\,(20|12|12|11|11)'\}$$

$$- 54\,\{(30|30|03|03)' - 6\,(30|21|12|03)' + 4\,(30|12|12|12)' + 4\,(21|21|21|03)' - 3\,(21|21|12|12)'\}$$

$$+ 9(23-12\lambda)\hbar^2\{(20|20|02|02)' - 2\,(20|11|11|02)' + (11|11|11|11)'\}$$

$$+ 324(2-\lambda)\hbar^2\{(30|12|02)' - (30|11|03)' - (21|21|02)' + (21|20|03)' + (21|12|11)' - (20|12|12)'\}$$

$$+ (709-1656\lambda+486\lambda^2)\hbar^4\{(20|02)' - (11|11)'\}$$

its value in the Calogero realization:

$$p_i \mapsto \frac{\hbar}{\mathrm{i}}\partial_i \qquad \Rightarrow \qquad \mathcal{C}_6^{\rm quant} \mapsto (144 + 216\,\lambda - 1215\,\lambda^2)\,\hbar^6$$

putting back the center-of-mass degree of freedom (10) and (01) $\Rightarrow$

massive Weyl re-ordering required $\Rightarrow$

lowest quantum W_3 Casimir:

$$\mathcal{C}_6^{\text{quant}} = 3T_{66}^9 - 3T_{66}^8 + 9T_{66}^7 - 3T_{66}^6 + 9T_{66}^5 - 54T_{66}^4 - \tfrac{9}{2}\hbar^2 T_{44}^6 + 27\hbar^2 T_{44}^5 - \tfrac{9}{2}\hbar^2 T_{44}^4 + 54\hbar^2 T_{44}^3 - \tfrac{27}{8}\hbar^4 T_{22}^3 + \tfrac{81}{8}\hbar^4 T_{22}^2$$

$$
\begin{aligned}
T^9_{66} = {} & (20|20|20|01|01|01|01|01|01) - 6(20|20|11|10|01|01|01|01|01) + 3(20|20|10|10|02|01|01|01|01) \\
& + 12(20|11|11|10|10|01|01|01|01) - 12(20|11|10|10|10|02|01|01|01) + 3(20|10|10|10|10|02|02|01|01) \\
& - 8(11|11|11|10|10|10|01|01|01) + 12(11|11|10|10|10|10|02|01|01) - 6(11|10|10|10|10|10|02|02|01) \\
& + (10|10|10|10|10|10|02|02|02)
\end{aligned}
$$

$$
\begin{aligned}
T^8_{66} = {} & (30|30|01|01|01|01|01|01) - 6(30|21|10|01|01|01|01|01) - 6(30|20|11|01|01|01|01|01) \\
& + 6(30|20|10|02|01|01|01|01) + 6(30|12|10|10|01|01|01|01) + 12(30|11|11|10|01|01|01|01) \\
& - 18(30|11|10|10|02|01|01|01) - 2(30|10|10|10|03|01|01|01) + 6(30|10|10|10|02|02|01|01) \\
& + 9(21|21|10|10|01|01|01|01) + 6(21|20|20|01|01|01|01|01) - 6(21|20|11|10|01|01|01|01) \\
& - 6(21|20|10|10|02|01|01|01) - 18(21|12|10|10|10|01|01|01) - 12(21|11|11|10|10|01|01|01) \\
& + 30(21|11|10|10|10|02|01|01) + 6(21|10|10|10|10|03|01|01) - 12(21|10|10|10|10|02|02|01) \\
& + 6(20|20|20|02|01|01|01|01) - 12(20|20|12|10|01|01|01|01) - 6(20|20|11|11|01|01|01|01) \\
& - 24(20|20|11|10|02|01|01|01) + 6(20|20|10|10|03|01|01|01) + 12(20|20|10|10|02|02|01|01) \\
& + 30(20|12|11|10|10|01|01|01) - 6(20|12|10|10|10|02|01|01) + 24(20|11|11|11|10|01|01|01) \\
& + 12(20|11|11|10|10|02|01|01) - 18(20|11|10|10|10|03|01|01) - 24(20|11|10|10|10|02|02|01) \\
& + 6(20|10|10|10|10|03|02|01) + 6(20|10|10|10|10|02|02|02) + 9(12|12|10|10|10|10|01|01) \\
& - 12(12|11|11|10|10|10|01|01) - 6(12|11|10|10|10|10|02|01) - 6(12|10|10|10|10|10|03|01) \\
& + 6(12|10|10|10|10|10|02|02) - 24(11|11|11|11|10|10|01|01) + 24(11|11|11|10|10|10|02|01) \\
& + 12(11|11|10|10|10|10|03|01) - 6(11|11|10|10|10|10|02|02) - 6(11|10|10|10|10|10|03|02) \\
& + (10|10|10|10|10|10|03|03)
\end{aligned}
$$

$$T^7_{66} \;=$$

$$3(30|30|02|01|01|01|01) - 6(30|21|11|01|01|01|01) - 12(30|21|10|02|01|01|01) - 2(30|20|12|01|01|01|01)$$

$$- 10(30|20|11|02|01|01|01) + 2(30|20|10|03|01|01|01) + 10(30|20|10|02|02|01|01) + 16(30|12|11|10|01|01|01)$$

$$+ 4(30|12|10|10|02|01|01) + 8(30|11|11|11|01|01|01) - 4(30|11|11|10|02|01|01) - 10(30|11|10|10|03|01|01)$$

$$- 6(30|11|10|10|02|02|01) + 2(30|10|10|10|03|02|01) + 2(30|10|10|10|02|02|02) + 5(21|21|20|01|01|01|01)$$

$$+ 8(21|21|11|10|01|01|01) + 14(21|21|10|10|02|01|01) + 10(21|20|20|02|01|01|01) - 14(21|20|12|10|01|01|01)$$

$$- 4(21|20|11|11|01|01|01) - 2(21|20|11|10|02|01|01) + 4(21|20|10|10|03|01|01) - 14(21|20|10|10|02|02|01)$$

$$- 26(21|12|11|10|10|01|01) - 14(21|12|10|10|10|02|01) - 16(21|11|11|11|10|01|01) + 32(21|11|11|10|10|02|01$$

$$+ 16(21|11|10|10|10|03|01) - 6(21|11|10|10|10|02|02) - 2(21|10|10|10|10|03|02) + 2(20|20|20|03|01|01|01)$$

$$+ 2(20|20|20|02|02|01|01) - 6(20|20|12|11|01|01|01) - 14(20|20|12|10|02|01|01) - 4(20|20|11|11|02|01|01)$$

$$- 6(20|20|11|10|03|01|01) - 4(20|20|11|10|02|02|01) + 10(20|20|10|10|03|02|01) + 2(20|20|10|10|02|02|02)$$

$$+ 14(20|12|12|10|10|01|01) + 32(20|12|11|11|10|01|01) - 2(20|12|11|10|10|02|01) - 12(20|12|10|10|10|03|01)$$

$$+ 10(20|12|10|10|10|02|02) + 2(20|11|11|11|11|01|01) + 8(20|11|11|11|10|02|01) - 4(20|11|11|10|10|03|01)$$

$$- 4(20|11|11|10|10|02|02) - 10(20|11|10|10|10|03|02) + 3(20|10|10|10|10|03|03) + 8(12|12|11|10|10|10|01)$$

$$+ 5(12|12|10|10|10|10|02) - 16(12|11|11|11|10|10|01) - 4(12|11|11|10|10|10|02) - 6(12|11|10|10|10|10|03)$$

$$- 4(11|11|11|11|11|10|01) + 8(11|11|11|10|10|10|03) + 2(11|11|11|11|10|10|02)$$

$$
\begin{aligned}
T^6_{66} \;=\;& 8(30|30|03|01|01|01) + 21(30|30|02|02|01|01) - 24(30|21|12|01|01|01) - 84(30|21|11|02|01|01) \\
& - 24(30|21|10|03|01|01) - 42(30|21|10|02|02|01) - 24(30|20|12|02|01|01) - 30(30|20|11|03|01|01) \\
& - 12(30|20|11|02|02|01) + 54(30|20|10|03|02|01) + 12(30|20|10|02|02|02) + 48(30|12|12|10|01|01) \\
& + 96(30|12|11|11|01|01) + 24(30|12|11|10|02|01) - 24(30|12|10|10|03|01) + 30(30|12|10|10|02|02) \\
& + 12(30|11|11|11|02|01) - 36(30|11|11|10|03|01) - 12(30|11|11|10|02|02) - 30(30|11|10|10|03|02) \\
& + 8(30|10|10|10|03|03) + 16(21|21|21|01|01|01) + 66(21|21|20|02|01|01) - 24(21|21|12|10|01|01) \\
& - 12(21|21|11|11|01|01) + 144(21|21|11|10|02|01) + 48(21|21|10|10|03|01) - 9(21|21|10|10|02|02) \\
& + 30(21|20|20|03|01|01) + 12(21|20|20|02|02|01) - 54(21|20|12|11|01|01) - 138(21|20|12|10|02|01) \\
& + 12(21|20|11|11|02|01) + 24(21|20|11|10|03|01) - 36(21|20|11|10|02|02) - 24(21|20|10|10|03|02) \\
& - 24(21|12|12|10|10|01) - 132(21|12|11|11|10|01) - 54(21|12|11|10|10|02) - 24(21|12|10|10|10|03) \\
& - 24(21|11|11|11|11|01) + 36(21|11|11|11|10|02) + 96(21|11|11|10|10|03) + 12(20|20|20|03|02|01) \\
& - 2(20|20|20|02|02|02) - 9(20|20|12|12|01|01) - 36(20|20|12|11|02|01) - 42(20|20|12|10|03|01) \\
& + 12(20|20|12|10|02|02) - 12(20|20|11|11|03|01) + 6(20|20|11|11|02|02) - 12(20|20|11|10|03|02) \\
& + 21(20|20|10|10|03|03) + 144(20|12|12|11|10|01) + 66(20|12|12|10|10|02) + 36(20|12|11|11|11|01) \\
& + 12(20|12|11|11|10|02) - 84(20|12|11|10|10|03) - 6(20|11|11|11|11|02) + 12(20|11|11|11|10|03) \\
& + 16(12|12|12|10|10|10) - 12(12|12|11|11|10|10) - 24(12|11|11|11|11|10) + 2(11|11|11|11|11|11)
\end{aligned}
$$

$$
\begin{aligned}
T^5_{66} \;=\;& 12(30|30|03|02|01) + (30|30|02|02|02) - 36(30|21|12|02|01) - 24(30|21|11|03|01) \\
& - 6(30|21|11|02|02) - 12(30|21|10|03|02) - 12(30|20|12|03|01) + 6(30|20|12|02|02) \\
& - 6(30|20|11|03|02) + 12(30|20|10|03|03) + 48(30|12|12|11|01) + 24(30|12|12|10|02) \\
& - 24(30|12|11|10|03) + 4(30|11|11|11|03) + 24(21|21|21|02|01) + 24(21|21|20|03|01) \\
& - 3(21|21|20|02|02) - 24(21|21|12|11|01) - 12(21|21|12|10|02) + 48(21|21|11|10|03) \\
& + 12(21|21|11|11|02) + 6(21|20|20|03|02) - 12(21|20|12|12|01) - 36(21|20|12|10|03) \\
& - 6(21|20|12|11|02) - 24(21|12|12|11|10) - 12(21|12|11|11|11) + (20|20|20|03|03) \\
& - 3(20|20|12|12|02) - 6(20|20|12|11|03) + 24(20|12|12|12|10) + 12(20|12|12|11|11)
\end{aligned}
$$

$$T^4_{66} = (30|30|03|03) - 6(30|21|12|03) + 4(30|12|12|12) + 4(21|21|21|03) - 3(21|21|12|12)$$

$$T^6_{44} = (20|20|01|01|01|01) - 4(20|11|10|01|01|01) + 2(20|10|10|02|01|01) + 4(11|11|10|10|01|01) - 4(11|10|10|10|02|01) + (10|10|10|10|02|02)$$

$$T^5_{44} = (20|20|02|01|01) - (20|11|11|01|01) - 2(20|11|10|02|01) + (20|10|10|02|02) + 2(11|11|11|10|01) - (11|11|10|10|02)$$

$$\begin{aligned} T^4_{44} = {} & 4(17-6\lambda)(30|12|01|01) - 4(17-6\lambda)(30|11|02|01) - 4(17-6\lambda)(30|10|03|01) \\ & + 4(17-6\lambda)(30|10|02|02) - 4(17-6\lambda)(21|21|01|01) + 4(17-6\lambda)(21|20|02|01) \\ & + 4(17-6\lambda)(21|12|10|01) + 8(17-6\lambda)(21|11|11|01) - 12(17-6\lambda)(21|11|10|02) \\ & + 4(17-6\lambda)(21|10|10|03) + 4(17-6\lambda)(20|20|03|01) - (59-24\lambda)(20|20|02|02) \\ & - 12(17-6\lambda)(20|12|11|01) + 4(17-6\lambda)(20|12|10|02) + 2(59-24\lambda)(20|11|11|02) \\ & - 4(17-6\lambda)(20|11|10|03) - 4(17-6\lambda)(12|12|10|10) + 8(17-6\lambda)(12|11|11|10) \\ & - (59-24\lambda)(11|11|11|11) \end{aligned}$$

$$T^3_{44} = (17-6\lambda)(30|12|02) - (17-6\lambda)(30|11|03) - (17-6\lambda)(21|21|02) + (17-6\lambda)(21|20|03) + (17-6\lambda)(21|12|11) - (17-6\lambda)(20|12|12)$$

$$T^3_{22} = \Big(177 - 16\lambda(13-3\lambda)\Big)(20|01|01) - 2\Big(177 - 16\lambda(13-3\lambda)\Big)(11|10|01) + \Big(177 - 16\lambda(13-3\lambda)\Big)(10|10|02)$$

$$T^2_{22} = \Big(177 - 16\lambda(13-3\lambda)\Big)(20|02) - \Big(177 - 16\lambda(13-3\lambda)\Big)(11|11)$$

Horizontal intertwiners and algebraic integrability

Heckman (1991) constructed intertwiners $g \leftrightarrow g{+}1$ via Dunkl operators:

$$M(g)\,I_k(g) \;=\; I_k(g{+}1)\,M(g) \qquad \text{for} \qquad M(g) \;=\; \text{res}\Big(\prod_{i<j}(\pi_i{-}\pi_j)(g)\Big)$$

$$M(g)^*\,I_k(g{+}1) \;=\; I_k(g)\,M(g)^* \qquad \text{for} \qquad M(g)^* \;=\; \text{res}\Big(\prod_{i<j}(\pi_i{-}\pi_j)(-g)\Big)$$

$$M(1{-}g)\,I_k(g) \;=\; I_k(g{-}1)\,M(1{-}g) \qquad \Leftarrow \qquad M(g)^* \;=\; M(-g)$$

immediate consequence:

$$\big[M(g)^*M(g)\;,\;I_k(g)\big] \;=\; 0 \qquad \text{and} \qquad \big[M(g)\,M(g)^*\;,\;I_k(g{+}1)\big] \;=\; 0$$

new conserved charge? no, because it is a polynomial in the Liouville charges:

$$M(g)^*M(g) \;=\; M(-g)\,M(-g)^* \;=:\; \mathcal{R}\big(I(g)\big)$$

coefficients of polynomial $\mathcal{R}(I)$ do not depend on g $\quad\rightarrow\quad$ evaluate for $g{=}0$:

$$\mathcal{R}\big(I(0)\big) = M(0)^* M(0) = \prod_{i<j} (p_i - p_j)^2$$

$$= \begin{vmatrix} 1 & p_1 & \dots & p_1^{n-1} \\ 1 & p_2 & \dots & p_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & p_n & \dots & p_n^{n-1} \end{vmatrix}^2 = \begin{vmatrix} 1 & 1 & \dots & 1 \\ p_1 & p_2 & \dots & p_n \\ \vdots & \vdots & & \vdots \\ p_1^{n-1} & p_2^{n-1} & \dots & p_n^{n-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & p_1 & \dots & p_1^{n-1} \\ 1 & p_2 & \dots & p_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & p_n & \dots & p_n^{n-1} \end{vmatrix}$$

$$= \left| \Big(\sum_k p_k^{i+j-2} \Big)_{ij} \right| = \det\big(I_{i+j-2}(0)\big)_{ij}$$

hence:

$$\mathcal{R}\big(I(g)\big) = \det\big(I_{i+j-2}(g)\big)_{ij}$$

$$n=2: \quad \mathcal{R}(I) = -I_1^2 + 2I_2$$

$$n=3: \quad \mathcal{R}(I) = -I_1^2 I_4 + 2I_1 I_2 I_3 - I_2^3 + 3I_2 I_4 - 3I_3^2$$

$$= \tfrac{1}{6}\big(-I_1^6 + 9I_1^4 I_2 - 8I_1^3 I_3 - 21 I_1^2 I_2^2 + 36 I_1 I_2 I_3 + 3I_2^3 - 18 I_3^2\big)$$

so far, $g \in \mathbb{R}$ generic; but $g \in \mathbb{N}$ admits intertwiner with free theory ($g{=}1$):

$$\mathbb{M}(g) \;=\; M(g{-}1)M(g{-}2)\cdots M(2)M(1) \qquad \Rightarrow \qquad \mathbb{M}(g)I_k(1) \;=\; I_k(g)\mathbb{M}(g)$$

$$\mathbb{M}(g)^* \;=\; M(-1)M(-2)\cdots M(2{-}g)M(1{-}g)$$ is conjugate intertwiner

$$\Rightarrow \qquad \mathbb{M}(g)\,\mathbb{M}(g)^* \;=\; \big(\mathcal{R}(g)\big)^{g-1} \qquad \text{and} \qquad \mathbb{M}(g)^*\mathbb{M}(g) \;=\; \big(\mathcal{R}(g{+}1)\big)^{g-1}$$

Darboux dressing of some free $G(1)$ with $[G(1), I_k(1)] = 0$ for some k:

$$G(g) \;=\; \mathbb{M}(g)\,G(1)\,\mathbb{M}(g)^* \qquad \Rightarrow \qquad \big[G(g)\;,\;I_k(g)\big] \;=\; 0$$

consistent with involution of Liouville charges:

$$\mathbb{M}(g)\,I_k(1)\,\mathbb{M}(g)^* \;=\; \big(\mathcal{R}(g)\big)^{g-1} I_k(g)$$

large choice of 'naked' $G(1)$: any polynomial in $\{p_i\}$ with constant coefficients
identical particles $\rightarrow$ observables totally (anti)symmetric under s_{ij}
totally symmetric $\rightarrow$ Liouville integrals; totally antisymmetric $\rightarrow$ simplest is

$$G(1) \;=\; M(0) \;=\; \prod_{i<j}(p_i - p_j)$$

Darboux dressing:

$$\begin{aligned} Q(g) &= \mathbb{M}(g)\, M(0)\, \mathbb{M}(g)^* \\ &= M(g-1)M(g-2)\cdots M(1)M(0)M(-1)\cdots M(2-g)M(1-g) \end{aligned}$$

builds a chain relating $I_k(g) = I_k(1-g)$ back to $I_k(g)$:

$$Q(g)\, I_k(1-g) \;=\; I_k(g)\, Q(g) \qquad \Rightarrow \qquad \big[Q(g)\,,\; I_k(g)\big] \;=\; 0$$

a conserved charge of order $\frac{1}{2}n(n-1)(2g-1)$ algebraically independent of $\{I_k, F_\ell\}$

seeming other option $g \in \mathbb{N}+\frac{1}{2}$ fails:

$$M(g-1)\cdots M(\tfrac{3}{2})M(\tfrac{1}{2})M(-\tfrac{1}{2})M(-\tfrac{3}{2})\cdots M(1-g) \;=\; \big(\mathcal{R}(g)\big)^{g-\frac{1}{2}}$$

$$H(0) \xrightarrow[\cong]{M(0)} H(1) \xrightarrow{M(1)} H(2) \xrightarrow{M(2)} \ldots\ H(g) \xrightarrow{M(g)} H(g+1) \xrightarrow{M(g+1)} \ldots$$

$$H(1) \xrightarrow{\mathbb{M}(g)} H(g)$$

$$\begin{array}{ccccccc}
H(1-g) & \xrightarrow{\mathbb{M}(g)^*} & H(0) & \xrightarrow[\cong]{M(0)} & H(1) & \xrightarrow{\mathbb{M}(g)} & H(g) \\
\| & & & & & & \| \\
H(g) & \longrightarrow & \longrightarrow & \xrightarrow{Q(g)} & \longrightarrow & \longrightarrow & H(g)
\end{array}$$

check the square of the new Liouville charge:

$$\begin{aligned}
Q(g)^2 &= M(g-1)\cdots M(3-g)M(2-g)\underbrace{M(1-g)M(g-1)}M(g-2)\cdots M(1-g) \\
&= M(g-1)\cdots M(3-g)M(2-g)\mathcal{R}(g-1)M(g-2)M(g-3)\cdots M(1-g) \\
&= M(g-1)\cdots M(3-g)\underbrace{M(2-g)M(g-2)}\mathcal{R}(g-2)M(g-3)\cdots M(1-g) \\
&= M(g-1)\cdots M(3-g)\big(\mathcal{R}(g-2)\big)^2 M(g-3)\cdots M(1-g) \\
&\;\;\vdots \\
&= M(g-1)M(1-g)\big(\mathcal{R}(1-g)\big)^{2g-2} = \big(\mathcal{R}(1-g)\big)^{2g-1} = \big(\mathcal{R}(g)\big)^{2g-1}
\end{aligned}$$

again a polynomial in the Liouville integrals, so formally $Q = \mathcal{R}^{g-\frac{1}{2}}$ for $g \in \mathbb{N}$

nonlinear ($\mathbb{Z}_2$ graded) algebra of $2n$ conserved charges:

$$[I_k, I_\ell] = 0 \qquad [I_k, F_\ell] = \mathcal{A}_{k,\ell}(I) \qquad [F_k, F_\ell] = \mathcal{B}_{k,\ell}(I, F)$$

$$[Q, I_\ell] = 0 \qquad [Q, F_\ell] = (2g-1)\, Q\, \mathcal{C}_\ell(I) \qquad Q^2 = \big(\mathcal{R}(I)\big)^{2g-1}$$

with some definite polynomials $\mathcal{A}_{k,\ell}$, $\mathcal{B}_{k,\ell}$, $\mathcal{C}_\ell$ and $\mathcal{R}$

example: three particles ($N=3$)

$$M(g) = \mathrm{res}\Big(\pi_{12}(g)\pi_{23}(g)\pi_{31}(g)\Big) \qquad \Delta = x_{12}x_{23}x_{31}$$

$$= \Delta^{g}\Big(p_{12}\,p_{23}\,p_{31} - \frac{\mathrm{i}g}{x_{12}}p_{12}^2 - \frac{\mathrm{i}g}{x_{23}}p_{23}^2 - \frac{\mathrm{i}g}{x_{31}}p_{31}^2 + \frac{2g}{x_{12}^2}p_{12} + \frac{2g}{x_{23}^2}p_{23} + \frac{2g}{x_{31}^2}p_{31}\Big)\,\Delta^{-g}$$

$$= p_{12}\,p_{23}\,p_{31} + \frac{2\mathrm{i}g}{x_{12}}\,p_{23}\,p_{31} + \frac{2\mathrm{i}g}{x_{23}}\,p_{31}\,p_{12} + \frac{2\mathrm{i}g}{x_{31}}\,p_{12}\,p_{23}$$
$$+ \Big(\frac{g(g-1)}{x_{31}^2} - \frac{4g^2}{x_{12}x_{23}}\Big)p_{31} + \Big(\frac{g(g-1)}{x_{12}^2} - \frac{4g^2}{x_{23}x_{31}}\Big)p_{12} + \Big(\frac{g(g-1)}{x_{23}^2} - \frac{4g^2}{x_{31}x_{12}}\Big)p_{23}$$
$$- \frac{6\mathrm{i}\,g^2(g+1)}{x_{12}x_{23}x_{31}} + 2\mathrm{i}g(g-1)(g+2)\Big(\frac{1}{x_{12}^3} + \frac{1}{x_{23}^3} + \frac{1}{x_{31}^3}\Big)$$

$$M^*M \equiv \mathcal{R} = -3I_3^2 + 6I_3I_2I_1 - \tfrac{4}{3}I_3I_1^3 + \tfrac{1}{2}I_2^3 - \tfrac{7}{2}I_2^2I_1^2 + \tfrac{3}{2}I_2I_1^4 - \tfrac{1}{6}I_1^6$$

$$
\begin{aligned}
Q(g{=}2) \;=\; & \tfrac{1}{6}\, p_{12}^3\, p_{23}^3\, p_{31}^3 \\
& + \frac{3}{x_{12}^2}\left(p_{12}^3\, p_{23}^2\, p_{31}^2 + 2\, p_{12}\, p_{23}^3\, p_{31}^3\right) \\
& + \frac{12\mathrm{i}}{x_{12}^3}\left(p_{12}^2\, p_{23}^3\, p_{31} + p_{23}^3\, p_{31}^3 + 4\, p_{12}^2\, p_{23}^2\, p_{31}^2\right) \\
& - \left(\frac{12}{x_{12}^4} - \frac{24}{x_{12}^2 x_{31}^2}\right) p_{12}^3\, p_{23}^2 + \left(\frac{264}{x_{23}^4} - \frac{180}{x_{12}^4} - \frac{168}{x_{12}^2 x_{23}^2}\right) p_{12}\, p_{23}^2\, p_{31}^2 \\
& + \mathrm{i}\left(\frac{1440}{x_{12}^5} - \frac{720}{x_{12})^3 x_{31}^2} - \frac{720}{x_{12})^2 x_{31}^3}\right) p_{12}^3\, p_{23} \\
& + \mathrm{i}\left(\frac{1080}{x_{12}^5} - \frac{360}{x_{31}^5} - \frac{360}{x_{12}^3 x_{31}^2} - \frac{1080}{x_{12}^2 x_{31}^3}\right) p_{12}^2\, p_{23}^2 \\
& + \left(\frac{4200}{x_{12}^6} + \frac{3360}{x_{23}^6} - \frac{1920}{x_{12}^3 x_{23}^3} + \frac{1200}{x_{23}^3 x_{31}^3} + \frac{2880}{x_{12}^2 x_{31}^4}\right) p_{12}^3 \\
& - \frac{4320}{x_{12}^2 x_{31}^4}\, p_{12}^2\, p_{23} - \frac{5760}{x_{12}^2 x_{31}^4}\, p_{12}\, p_{23}\, p_{31} \\
& + \mathrm{i}\left(\frac{25200}{x_{12}^7} - \frac{10080}{x_{23}^7} - \frac{7200}{x_{12}^5 x_{23}^2} - \frac{5760}{x_{12}^4 x_{23}^3} + \frac{10080}{x_{12}^3 x_{23}^4} - \frac{1440}{x_{12}^2 x_{23}^5}\right) p_{12}^2 \\
& - \left(\frac{90720}{x_{12}^8} + \frac{198720}{x_{12}^7 x_{23}} - \frac{129600}{x_{12}^6 x_{23}^2} + \frac{34560}{x_{12}^5 x_{23}^3} - \frac{17280}{x_{12}^3 x_{23}^5}\right) p_{12} \\
& - \mathrm{i}\left(\frac{181440}{x_{12}^9} + \frac{60480}{x_{12}^7 x_{23} x_{31}}\right) \;+\; \text{all permutations in } (123)
\end{aligned}
$$

Vertical intertwiners and a network

Chalykh, Feigin, Veselow (1998) generalize quantum integrable Calogero model:

$$H_N^{+1}(g) \;:=\; H_N(g) + \tfrac{1}{2}p_{N+1}^2$$

$$\widetilde{H}_{N+1}(g) \;:=\; H_N(g) + \tfrac{1}{2}p_{N+1}^2 + \sum_{i=1}^{N} \frac{\hbar^2\, g}{(x_i - \sqrt{g-1}\, x_{N+1})^2}$$

$\widetilde{H}_{N+1}(g)$ algebraically integrable since $\exists\, \widetilde{W}$ with $\widetilde{W}(g)\, H_N^{+1}(g) \;=\; \widetilde{H}_{N+1}(g)\, \widetilde{W}(g)$

but trivializes in the classical limit $\hbar \to 0$ with $\hbar\, g \to \gamma$

we have $\widetilde{H}_{N+1}(2) = H_{N+1}(2)$ hence $\widetilde{W}(2)\, H_N^{+1}(2) \;=\; H_{N+1}(2)\, \widetilde{W}(2)$

this suggests existence of “vertical intertwiner” $W_N(g)$ such that

$$W_N(g)\, H_N^{+1}(g) \;=\; H_{N+1}(g)\, W_N(g) \qquad \forall g \in \mathbb{N}$$

where $W_N(1) = \mathbb{1}$ and $W_N(2) = \widetilde{W}(2)$

together with horizontal intertwiners $M_N(g)\,H_N(g) = H_N(g{+}1)\,M_N(g)$ have

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\begin{array}{ccccccc}
H_N^{+1}(1)\ \ldots & \xrightarrow{M_N(g-1)} & H_N^{+1}(g) & \xrightarrow{M_N(g)} & H_N^{+1}(g+1) & \xrightarrow{M_N(g+1)} & \ldots \\
\Vert & & \downarrow{\scriptstyle W_N(g)} & & \downarrow{\scriptstyle W_N(g+1)} & & \\
H_{N+1}(1)\ \ldots & \xrightarrow{M_{N+1}(g-1)} & H_{N+1}(g) & \xrightarrow{M_{N+1}(g)} & H_{N+1}(g+1) & \xrightarrow{M_{N+1}(g+1)} & \ldots \\
H_{N+1}^{+1}(1)\ \ldots & \xrightarrow{M_{N+1}(g-1)} & H_{N+1}^{+1}(g) & \xrightarrow{M_{N+1}(g)} & H_{N+1}^{+1}(g+1) & \xrightarrow{M_{N+1}(g+1)} & \ldots \\
\Vert & & \downarrow{\scriptstyle W_{N+1}(g)} & & \downarrow{\scriptstyle W_{N+1}(g+1)} & & \\
H_{N+2}(1)\ \ldots & \xrightarrow{M_{N+2}(g-1)} & H_{N+2}(g) & \xrightarrow{M_{N+2}(g)} & H_{N+2}(g+1) & \xrightarrow{M_{N+2}(g+1)} & \ldots
\end{array}
$$

hence a recursion $W_N(g{+}1)\,M_N(g) \;=\; M_{N+1}(g)\,W_N(g)$

$\Rightarrow\quad W_N(g) \;=\; \prod_{i=1}^{N}(p_i - p_{N+1})^{g-1} \;+\; O(g)$ is an operator of order $N(g-1)$

proof of existence or construction of the vertical intertwiner is a factorization problem:

either recursively $M_{N+1}(g)\, W_N(g) \;=\; W_N(g+1)\, M_N(g)$

or directly from $\mathbb{M}_{N+1}(g) \equiv \mathbb{M}_{N+1}(g)\, W_N(1) \;=\; W_N(g)\, \mathbb{M}_N(g)$

examples:

$$W_2(2) \;=\; p_{13}\,p_{23} + \frac{2\,\mathrm{i}}{x_{13}}\,p_{23} + \frac{2\,\mathrm{i}}{x_{23}}\,p_{13} - \frac{4}{x_{13}\,x_{23}} - \frac{2}{x_{12}^2}$$

$$\begin{aligned} W_2(3) \;=\;& p_{13}^2 p_{23}^2 + \frac{6}{x_{13}}\,\mathrm{i}\,p_{13}\,p_{23}^2 + \frac{6}{x_{23}}\,\mathrm{i}\,p_{13}^2 p_{23} \\ &- \frac{12}{x_{13}^2}\,p_{23}^2 - \frac{12}{x_{23}^2}\,p_{13}^2 - \Big(\frac{36}{x_{23}x_{13}} + \frac{12}{x_{12}^2}\Big)\,p_{13}\,p_{23} \\ &- \Big(\frac{72}{x_{13}^2 x_{23}} + \frac{36}{x_{12}^2 x_{13}} + \frac{12}{x_{12}^3}\Big)\,\mathrm{i}\,p_{23} - \Big(\frac{72}{x_{23}^2 x_{13}} + \frac{36}{x_{12}^2 x_{23}} - \frac{12}{x_{12}^3}\Big)\,\mathrm{i}\,p_{13} \\ &+ \frac{144}{x_{12}^2}\Big(\frac{1}{x_{13}^2} + \frac{1}{x_{23}^2} - \frac{1}{x_{13}x_{23}}\Big) \end{aligned}$$

have computed also $W_2(4)$ and $W_2(5)$ and proved the existence of $W_2(g{\le}7)$

Conclusions

- the rational Calogero model for N unrestricted realizes a $W_{1+\infty}$ algebra
- for particle number N fixed it becomes a nonlinear dynamical W_N algebra
- conserved charges = commutant of H in $\mathcal{U}(W_N)$, is easily characterized
- smallest W_3 Casimir is of order 9 in 9 generators and of order 12 in (x_i, p_j)
- classical limit also via solving a system of PDEs (using Poisson brackets)
- horizontal intertwiners relate $H_N(g)$ to $H_N(g{+}1)$ $\Rightarrow$ isosprectrality
- extra "odd' conserved charge for integral coupling $\Rightarrow$ algebraic integrability
- new vertical intertwiners couple an additional particle $\Rightarrow$ $H_N(g)$ to $H_{N+1}(g)$
- a network relates all $H_N(g{\in}\mathbb{N})$ to free particles $\Rightarrow$ extend to cons'd charges?

THANK YOU FOR

YOUR ATTENTION !

HAPPY

BIRTHDAY

HARALD !

Some history

- 1923 Burchnall & Chaundy:
  odd-order ordinary differential operators commuting with 1d Hamiltonian
- 1978 Krichever:
  their existence is tied to algebro-geometric, or ‘finite-gap’, nature of Hamiltonian
- 1989 Dunkl:
  commuting operators combining partial differentials and Coxeter reflections
- 1990 Chalykh & Veselov:
  “commutative rings of partial differential operators and Lie algebras”
  1st examples of 2d finite-gap Hamiltonians, construction of intertwiners for $g{=}2$
- 1991 Heckman:
  uses Dunkl operators to construct intertwiners for any multiplicity $g{-}1$
- 1990s Berest, Chalykh, Etingof, M. Feigin, Ginzburg, Styrkas, Veselov:
  extension to higher dimension $N{-}1$ and multiplicity $g{-}1$, in particular:
  construction of Baker-Akhiezer functions, explicit formulæ for add’l charges,
  including via Darboux dressing with intertwiners (only for $N{=}3$, $g{=}2$)