

# Type-independent canonical mirror models for cominuscule homogeneous spaces

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## Landau-Ginzburg models

For given  $X$  a pair  $(X^\vee, \mathcal{W}_q : X^\vee \rightarrow \mathbb{C})$  s.t.  
 $qH^*(X)_{(q)} \cong \mathbb{C}[X^\vee \times (\mathbb{C}^*)^r] / \langle \partial \mathcal{W}_q \rangle$ .

## Homogeneous spaces

Projective  $X = G/P$  with  $G$  a linear alg. group over  $\mathbb{C}$ . Such  $P$  stabilize a set  $\{\varpi_i \mid i \in I\}$  of fundamental weights, denoted  $P = P_I$ . If  $I = \{k\}$  with  $\varpi_k$  s.t.

$$\langle \alpha_0^\vee, \varpi_k \rangle = 1,$$

then  $X$  is called *cominuscule*.

## Langlands dual hom. spaces

$G$  and  $P$  determine *Langlands dual groups*  $G^\vee$  and  $P^\vee$  exchanging roots and coroots. These give the “Langlands dual hom. space”

$$\mathbb{X}^\vee = P^\vee \backslash G^\vee.$$

If  $X$  is cominuscule, there is an embedding

$$\mathbb{X}^\vee \hookrightarrow \mathbb{P}(V(\varpi_k^\vee)^*)$$

yielding proj. coordinates  $(p_w \mid w \triangleleft w^P)$  called (*generalized*) *Plücker coordinates* indexed by *order ideals* of the *minuscule poset*  $w^P$ , which can be drawn as a (*skew*) *Young diagram* with the smallest “box” top-left and ordered by adjacency.

## The cominuscule families

There are five infinite families of cominuscule spaces, as well as an exceptional family consisting of two members:

1)  $X = \text{Gr}(k, n) = \text{SL}_n/P_k = A_{n-1}^{\text{sc}}/P_k$   
 $\mathbb{X}^\vee = \text{Gr}(n-k, n) = P_k^\vee \backslash \text{PSL}_n = P_k^\vee \backslash A_{n-1}^{\text{ad}}$

$$w^P = \begin{array}{cccc} 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \end{array}$$

2)  $X = Q_{2n-1} = \text{Spin}_{2n+1}/P_1 = B_n^{\text{sc}}/P_1$   
 $\mathbb{X}^\vee = \mathbb{P}^n = P_1^\vee \backslash \text{PSP}_{2n} = P_1^\vee \backslash C_n^{\text{ad}}$

$$w^P = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \end{array}$$

3)  $X = Q_{2n} = \text{Spin}_{2n}/P_1 = D_n^{\text{sc}}/P_1$   
 $\mathbb{X}^\vee = Q_{2n} = P_1^\vee \backslash \text{PSO}_{2n} = P_1^\vee \backslash D_n^{\text{ad}}$

$$w^P = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 \\ & 6 & 4 & 3 & 2 & 1 \end{array}$$

4)  $\text{LG}(n, 2n) = \text{Sp}_{2n}/P_n = C_n^{\text{sc}}/P_n$   
 $\text{OG}(n, 2n+1) = P_n^\vee \backslash \text{PSO}_{2n+1} = P_n^\vee \backslash B_n^{\text{ad}}$

$$w^P = \begin{array}{ccccc} 5 & & & & \\ 4 & 5 & & & \\ 3 & 4 & 5 & & \\ 2 & 3 & 4 & 5 & \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

5)  $X = \text{OG}(n, 2n) = \text{Spin}_{2n}/P_n = D_n^{\text{sc}}/P_n$   
 $\mathbb{X}^\vee = \text{OG}(n, 2n) = P_n^\vee \backslash \text{PSO}_{2n} = P_n^\vee \backslash D_n^{\text{ad}}$

$$w^P = \begin{array}{ccccc} 6 & & & & \\ 4 & 5 & & & \\ 3 & 4 & 6 & & \\ 2 & 3 & 4 & 5 & \\ 1 & 2 & 3 & 4 & 6 \end{array}$$

6<sub>1</sub>)  $X = \mathbb{O}\mathbb{P}^2 = E_6^{\text{sc}}/P_6, \mathbb{X}^\vee = \mathbb{O}\mathbb{P}^2 = P_6^\vee \backslash E_6^{\text{ad}}$

$$w^P = \begin{array}{cccccc} 6 & 5 & 4 & 3 & 1 & \\ & 2 & 4 & 3 & & \\ & & 5 & 4 & 2 & \\ & & & 6 & 5 & 4 & 3 & 1 \end{array}$$

6<sub>2</sub>)  $X = \text{Fr} = E_7^{\text{sc}}/P_7, \mathbb{X}^\vee = \text{Fr} = P_7^\vee \backslash E_7^{\text{ad}}$

$$w^P = \begin{array}{ccccccc} 7 & 6 & 5 & 4 & 3 & 1 & \\ & 2 & 4 & 3 & & & \\ & & 5 & 4 & 2 & & \\ & & & 6 & 5 & 4 & 3 & 1 \\ & & & & 7 & 6 & 5 & 4 & 3 \\ & & & & & 2 & 4 & & \\ & & & & & & 5 & & \\ & & & & & & & 6 & \\ & & & & & & & & 7 \end{array}$$

## Thm: Canonical LG-models

For  $X = G/P_k$  cominuscule we give a fully combinatorial construction of an LG-model  $(X_{\text{can}}^\vee, \mathcal{W}_{\text{can},q})$  with  $X_{\text{can}}^\vee = \mathbb{X}^\vee \setminus D$  for an anticanonical divisor  $D$  and with  $\mathcal{W}_{\text{can},q}$  a rational function in Plücker coordinates, using order ideals of  $w^P$ .

## The combinatorics

Let  $\sigma_i : [n] \rightarrow [n]$  be the (possibly trivial) involution of the Dynkin diagram of  $G^\vee$  with vertex  $i$  removed, and set  $\sigma_i(i) = i$ .

For any  $i_* \in [n] \setminus \{k\}$ , let:

- $i_{-1} = k, i_0 = i_*$  and  $i_j = \sigma_{i_{j-1}}(i_{j-2})$ ;
- $w_{i_0} = \emptyset$ , and  $w_{i_j} \triangleleft w^P$  max. obtained by adding all possible  $\square$  with  $i \neq i_j$  to  $w_{i_{j-1}}$ ;
- $C_*$  s.t.  $w_{i_{c_*+1}} = w^P$ ;
- $P_{i_*}$  the poset of  $C_*$ -tuples of ideals obtained from  $(w_{i_1}, \dots, w_{i_{c_*}})$  by moving boxes;
- $\phi_{i_*}$  be the alternating sum of  $p_w p_{w'} \dots$  for  $(w, w', \dots) \in P_{i_*}$ .

Moreover, let  $\phi_\emptyset = p_\emptyset$  and  $\phi_k = p_{w^P}$ , then  $D = \sum_{i=0}^{c_*} \{\phi_i = 0\}$ .

For  $i \in [n]$  and  $w \triangleleft w^P$ , if  $\square$  can be added to  $w$ , let  $\delta_i(p_w) = p_{w+\square}$  where  $w+\square$  is the resulting ideal; else, let  $\delta_i(p_w) = 0$ ; extend  $\delta_i$  to polynomials as a derivation.

Let  $w'' \triangleleft w^P$  maximal containing *exactly one*  $\square$ , and let  $w' \triangleleft w^P$  be its complement. Define for  $i_* \in [n] \setminus \{k\}$ :

$$\mathcal{W}_\emptyset = \frac{p_\emptyset}{p_\emptyset}, \mathcal{W}_k = q \frac{p_{w^P}}{p_{w^P}}, \mathcal{W}_{i_*} = \frac{\delta_{i_1}(\phi_{i_*})}{\phi_{i_*}},$$

then  $\mathcal{W}_{\text{can},q} = \sum_{i=0}^{c_*} \mathcal{W}_i$ .

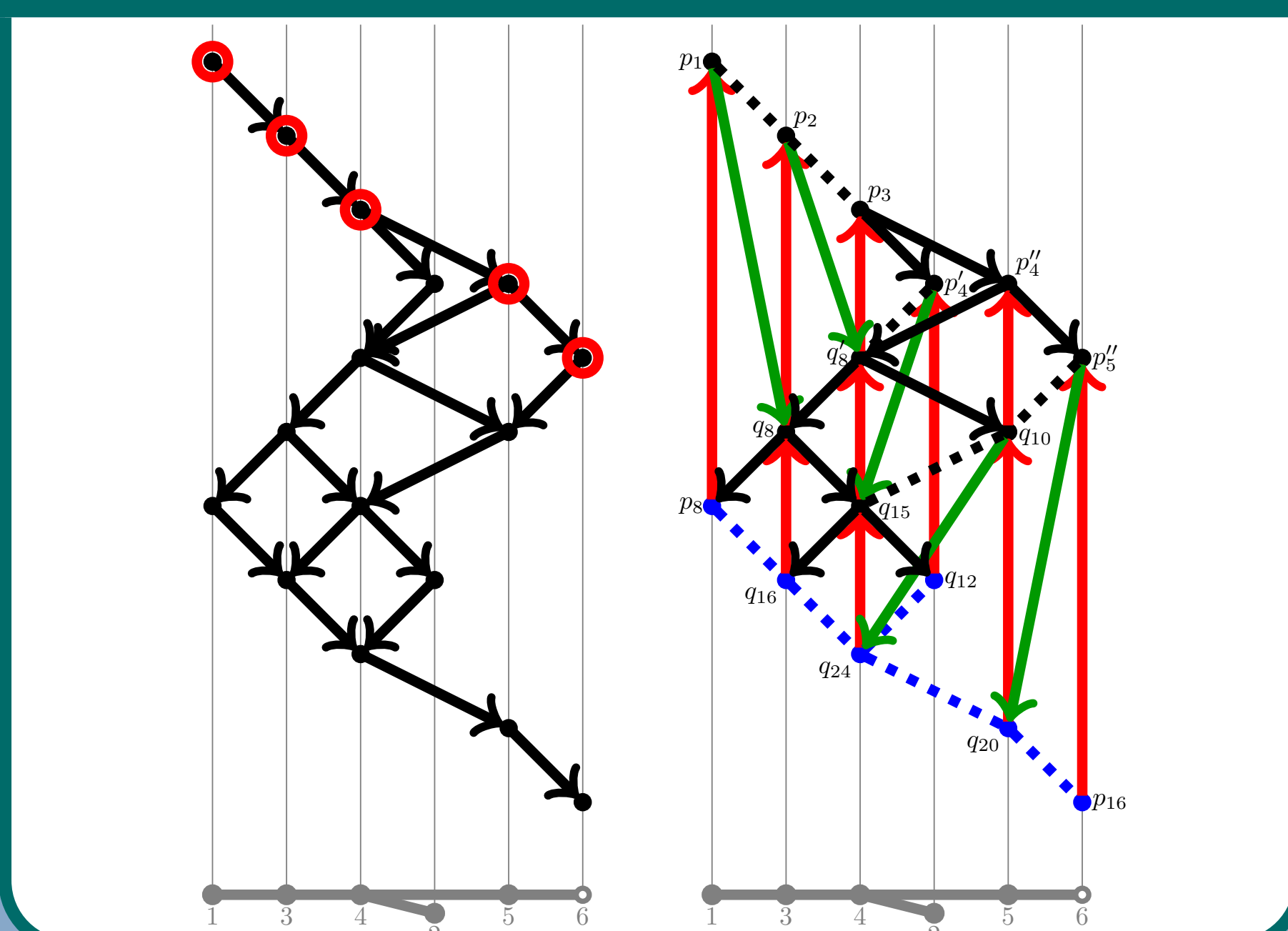
## Further properties

- $\{\phi_i \mid i \in \{0\} \cup [n]\}$  forms the set of frozen variables for a cluster structure on  $\mathbb{X}^\vee$  obtained from [1].
- The canonical models are up to relabeling identical to earlier type-dependent constructions for certain cominuscule families given in [2,3,4,5].
- The canonical models are isomorphic to the Lie-theoretic models given in [6] and hence give explicit projective coordinate expressions for these.
- On a dense algebraic torus in  $X_{\text{can}}^\vee$ , the canonical models restrict to a Laurent polynomial model isomorphic to [7].

•  $\mathcal{W}_{\text{can},q}$  can be seen as a *log derivative of D* by  $\mathcal{W}_{\text{can},q} = \frac{\Delta(\Phi)}{\Phi}$  with  $\Phi = \prod_{i=0}^{c_*} \phi_i$  and  $\Delta$  a “quantum” extension of  $\sum_{i=1}^{c_*} \delta_i$

- [1] Geiß, Leclerc, Schröer, *Part. flag v. and preproj. alg.*, Ann.Four.58(2008)3  
Marsh, Pech, Rietsch, Spacek, Wang, Williams:  
[2] M.R. *The B-model conj. & mirror sym. for Gr.*, Adv.Math.366(2020)  
[3] P.R,Wi. *On LG-models for quadrics (...)*, Adv. Math. 300 (2016)  
[4] P.R. *An LG-model for LG (...)*, arXiv:math.AG/1304.4958  
[5] S.Wa. *(...) LG-models for (...) the exc. com. family*, J.Alg.630(2023)  
[6] R. *A mirror symmetric construction of qH\**, Adv. Math. 217 (2008)  
[7] S. *Laurent pol. LG-models for com. hom. spaces*, Tr.Gps.27(2022)4

## Quivers of $E_6/P_6$



## An example: the Lagrangian Grassmannian $X = \text{LG}(4, 8)$

$\mathbb{X}^\vee$  of type  $B_4$  with  $k = 4$   $w^P = \begin{array}{cccc} 4 & & & \\ 3 & 4 & & \\ 2 & 3 & 4 & \\ 1 & 2 & 3 & 4 \end{array}$   
Dynkin diagram  $\circ - \circ - \circ - \circ$

$i_* = 1: i_1 = 3, i_2 = 4, i_3 = 1$  and

$$w_{i_1} = \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array}, w_{i_2} = \begin{array}{cc} 4 & \\ 3 & 4 \\ 2 & \\ 1 & \end{array}, w_{i_3} = \begin{array}{cccc} 4 & & & \\ 3 & 4 & & \\ 2 & 3 & 4 & \\ 1 & 2 & 3 & 4 \end{array} = w^P \text{ so } P_1 = \left\{ \left( \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array}, \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \right), \left( \emptyset, \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \right) \right\}, \mathcal{W}_1 = \frac{p_\square p_\square - p_\emptyset p_\square}{p_\square p_\square - p_\emptyset p_\square}$$

$i_* = 2: i_1 = 2, i_2 = 4, i_3 = 2$  and

$$w_{i_1} = \begin{array}{cc} 4 & \\ 3 & 4 \\ 2 & 3 \\ 1 & 2 \end{array}, w_{i_2} = \begin{array}{ccc} 4 & & \\ 3 & 4 & \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{array}, w_{i_3} = \begin{array}{cccc} 4 & & & \\ 3 & 4 & & \\ 2 & 3 & 4 & \\ 1 & 2 & 3 & 4 \end{array} = w^P, \text{ so}$$

$$P_2 = \left\{ \left( \begin{array}{cc} 4 & \\ 3 & 4 \\ 2 & 3 \\ 1 & 2 \end{array}, \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \right), \left( \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array}, \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \right), \left( \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array}, \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \right), \left( \emptyset, \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \right) \right\}, \mathcal{W}_2 = \frac{p_\square p_\square - p_\square p_\square}{p_\square p_\square - p_\square p_\square + p_\square p_\square - p_\emptyset p_\square}$$

$i_* = 3: i_1 = 1, i_2 = 4, i_3 = 3$  and

$$w_{i_1} = \begin{array}{ccc} 4 & & \\ 3 & 4 & \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{array}, w_{i_2} = \begin{array}{ccc} 4 & & \\ 3 & 4 & \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{array}, w_{i_3} = \begin{array}{cccc} 4 & & & \\ 3 & 4 & & \\ 2 & 3 & 4 & \\ 1 & 2 & 3 & 4 \end{array} = w^P \text{ so } P_3 = \left\{ \left( \begin{array}{ccc} 4 & & \\ 3 & 4 & \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{array}, \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \right), \left( \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array}, \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \right) \right\},$$

$$\mathcal{W}_3 = \frac{p_\square p_\square - p_\square p_\square}{p_\square p_\square - p_\square p_\square}$$

Finally,  $w'' = \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array}$ , so  $w' = \begin{array}{ccc} 4 & & \\ 3 & 4 & \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{array}$ , and we conclude:

$$\mathcal{W}_{\text{can},q} = \frac{p_\square}{p_\square} + \frac{p_\square p_\square - p_\emptyset p_\square}{p_\square p_\square - p_\emptyset p_\square} + \frac{p_\square p_\square - p_\square p_\square}{p_\square p_\square - p_\square p_\square + p_\square p_\square - p_\emptyset p_\square} + \frac{p_\square p_\square - p_\square p_\square}{p_\square p_\square - p_\square p_\square} + q \frac{p_\square}{p_\square}$$