Differential operators invariant with respect to the compact operators

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Themes

- Images of elliptic operators on infinite rank bundles - their topology
- Topology on cohomology groups of elliptic complexes
  $\Rightarrow$ Topological aspects of BRST/BFV-theories ("first steps")
- Hilbert C^*-modules
- Assumption *invariance of the operators*. Use Mishchenko's elliptic operators theory (Fomenko, Mishchenko [4], '70)

Motivation

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- Classical Physics
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  - Hamilton formulation via Legendre transform

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    - Theories in Physics testable by measurements $\Longrightarrow$ State spaces must have a good behaviour of limits. At least: Limit of a sequence either doesn't exist, or it is unique if it exists. Quotient space of Hausdorff topological vector spaces has uniqueness of limits if and only if the dividing space is closed.

Further motivation

2) Elliptic equations on compact manifolds (without boundary). Hodge theory

1. Cohomology of elliptic operators (or better their chain complexes) on finite rank bundles over compact manifolds are finite dimensional vector spaces.
2. Kernels of complexes' Laplacians (de Rham's Laplacian on Riemannian, Dolbeault's Laplacian on complex manifolds) are as vector spaces isomorphic to the cohomology groups of the complex.
   Not true: If compact is omitted. Not true: If finite rank is omitted. Not true: If elliptic is omitted.

Fails of Hodge theorem generalizations

- de Rham complex on $\mathbb{R}^n$. Laplace has infinite dimensional kernel: e.g., 1, $x, y\ xy, x^2 - y^2$, $x^3 - 3xy^2, y^3 - 3x^2y \ldots$. The kernel is non-trivial in each homogeneity (by Weyl thy, or separation of variables technique). Just observe $(x \pm iy)^k$ are non-zero and harmonic. (Also further solutions - $\mathrm{sh}(x + iy)$ etc.)
- Let $(e_i)_i$ be ON-basis of separable Hilbert space H of infinite dimension. Let $D : C^\infty(S^1, H) \to C^\infty(S^1, H)$, $f = \sum_{i=1}^\infty f_i e_i$ be given by $(Df)(m) = \sum_{i=1}^\infty \frac{df_i}{d\phi}(m) e_i$. Then $Df = 0 \Leftrightarrow$ $df_i/d\phi = 0 \Leftrightarrow f_i(e^{2\pi i\phi}) = a_i \in \mathbb{C} \Rightarrow f = \sum_i a_i e_i$. Thus f is any constant H-valued function. Solution space $\simeq H$ (linearly). Thus **infinite** dimensional.
- d'Alembert on Minkowski-flat torus: $f_n(x^0, x^1) = c_n e^{2\pi i n(x^0 - x^1)}$, $c_n \in \mathbb{C}$, $n \in \mathbb{N}$, independent. $*d*d\, f_n = \Box f_n = (\partial^2_{x^0} - \partial^2_{x^1}) f_n = 0$, where $*$ is the Hodge-star operator for Minkowski metric

Main steps in proving the Hodge theory

(Hodge '30, Weyl '43, de Rham '46)

- *Fredholm theory* - needs complete normed spaces
  Linear operator $D : H \to H'$ is *Fredholm* (def.) $\Leftrightarrow$ has finite rank and corank ($=$ dimension of $H'/\mathrm{Im} D$). Image is consequently closed.

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- *Atkinson's theorem*: D Fredholm op. (between complete normed spaces) $\Leftrightarrow$ exists $\check{D}$ such that $D\check{D} = \mathrm{Id}_{H'} + K_1$ and $\check{D}D = \mathrm{Id}_H + K_2$, K_i compact ops ($i = 1, 2$)

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- Rellich–Kondratchov fails for Sobolev spaces of *infinite dimensional vector space valued* functions on compact sets: $\mathrm{Id} : W^{s+1}(M, H) \to W^s(M, H)$ is not compact if $\dim H = \infty$.

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- We cannot handle elliptic differential operators $D : C^\infty(M, H) \to C^\infty(M, H')$ if H is infinite dimensional, even if M is compact (as one does in the Hodge theory).
- The idea: Any Hilbert space H is finitely generated over $B(H) = \{A : H \to H | \, A \text{ linear and bounded}\}$. Any vector in H can be achieved by dilated projection: $|v\rangle = (|v\rangle \langle f|) \, |f\rangle$ (Fix unit vector $f \in H$, $v \in H$ arbitrary. Then v is achieved by the ket-bra above.)
  We can work with $F(H)$ (*finite rank ops* $H \to H$) or $K(H)$ (*compact ops* $H \to H$) only.
- It needs Hilbert C^*-modules: generalization of Hilbert spaces.

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- **idea:** Field, complex/real numbers, is "replaced" by algebra (here $B(H), F(H), K(H)$). Choice: a C^*-algebra (but there are also modules for H^*-algebras)
- We explain parts of Mishchenko–Fomenko [4] generalization of Fredholm theory based on "analytic" modules over C^*-algebras

Hilbert A-modules

Rieffel [12], Paschke [11]

- *Pre-Hilbert A-module* H is a *complex vector space* which is a right module over a C^*-algebra A; together with a hermitian-symmetric A-valued *sesquilinear form* $(\,,\,)_H : H \times H \to A$ which is positive definite (the *C^*-product*).

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- *Positive elements* in A are those which have non-negative spectrum (in the unitalization) of A. If $||_A$ denotes the norm in A, $v \ni H \mapsto |v|_H := \sqrt{|(v,v)_H|_A}$, $v \in H$, is a norm on H (the C^**-norm*).

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  A pre-Hilbert A-module is a normed space.
- *Spectrum* of element $a \in A$. All complex λ such that $a - \lambda 1$ does not have an inverse in A or in $A \oplus \mathbb{C}$ (unitalization), unit $e = (0, 1)$. Algebra is unital if it contains an element e (unit) such that $ea = ae = a$ for all $a \in A$. ($e = e'e = ee' = e'$.)

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- Definition: Hilbert A-module = pre-Hilbert A-module whose C^*-norm is complete (the space is complete normed vector space; a Banach space)

Examples of (pre-)Hilbert A-modules

- $H = A$, $v \cdot a = va$, $(v, w)_H = v^* w$, where $a, v, w \in A$. Operation $*$ is from A. The C^*-module norm is $|v|_H^2 = |(v, v)_H|_A = |v^* v|_A = |v|_A^2$ (by C^*-identity in A). (*Tautological module*) (brackets)

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- $A = \mathbb{C}$, $H = C^\infty(S^1)$ with $(f, g)_H = \int_{S^1} \overline{f} g \mathrm{vol}_g$
  $|f|_H^2 = |(f, f)_H|_{\mathbb{C}} = \int_{S^1} |f|^2 \mathrm{vol}_g = \int_{x=0}^1 |f(e^{2\pi i x})|^2 d\lambda_{\mathbb{R}}(x)$, $S^1 \subseteq \mathbb{C}$. Not Cauchy complete. *Pre-Hilbert A-module.* (It is Cauchy complete if considered as metric space generated by classical Fréchet "seminorms".)

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- Also $H = A^n = A \oplus \ldots \oplus A$ with diagonal action $(a_1, \ldots, a_n) \cdot a = (a_1 a, \ldots, a_n a)$ and $((a_1, \ldots, a_n), (b_1, \ldots, b_n))_H = \sum_{i=1}^n a_i^* b_i$

Morphisms of Hilbert C^*-modules

- $\ell^2(A) = \{(a_i)_{i=1}^{\infty} | \sum_{i=1}^{\infty} a_i^* a_i$ converges in $A\}$. Action is diagonal and product is $((a_i)_i, (b_i)_i)_{\ell^2(A)} = \sum_i a_i^* b_i$. Called *basic A-module*. $\ell^2(A) = A \otimes \ell^2(\mathbb{N})$ (projective tensor product [5]).

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- Morphisms: $\Theta_{e,f}(h) = e \cdot (f, h)_H$ for $e \in H'$ and $f, h \in H$ (*elementary operators*). Note $(f, h)_H \in A$. *A-Compact* operators (def.) are in the closure in $B_A(H)$ of finite linear sums of elementary operators (Kasparov).

Example: C^*-Fredholm with non-closed image

- D is A-Fredholm if **(def.)** D is invertible modulo an A-compact operator: $D\check{D} = \mathrm{Id} + K_1$.
- $A = H = C^0([0,1])$ with pointwise multiplication, $f^* = \overline{f}$, and sup norm, ie., $|f|_A = \sup\{|f(x)|, x \in [0,1]\}$, $(f,g)_H = \overline{f}g \in A$, $f, g \in H = A$. The C^*-norm on H, $|f|_H = |f|_A$. $Df = xf$,

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  Thus, $D = \Theta_{1,x}$. (Invertible modulo A-compacts: $D\mathrm{Id}_H = \mathrm{Id}_H D = \mathrm{Id}_H + (D - \mathrm{Id}_H)$.)
- Easy compute $D^* = D$. If $\mathrm{Im}\, D$ were closed, then $H = \mathrm{Im}\, D \oplus \mathrm{Ker}\, D$. But easily $\mathrm{Ker}\, D = 0$. Thus $\mathrm{Im}\, D = H$. But $D(f)(x) = (xf)(x) = xf(x) = 1$ (for constant function 1) has solution $f(x) = 1/x$, not in $C^0([0,1])$. So 1 is not in $\mathrm{Im}\, D$, D is not surjective, and thus $\mathrm{Im}\, D$ is not closed.

Hilbert C^*-bundles

- Let A be a C^*-algebra. A *Hilbert A-bundle* $\mathcal{H}$ with fibre H is a locally trivial Banach bundle with fibre the Banach space H, where each fibre is a Hilbert A-module, equipped with a maximal smooth atlas whose transition functions are bijective maps in $B_A(H)$. *Finitely generated* Hilbert A-bundle means the **fibre** is finitely generated A-module.
- An *(Hilbert) A-differential operator* between Hilbert A-bundles is any A-linear map on section spaces such that locally it is of form $\sum_{|\alpha|\leq r} m_\alpha \partial^\alpha$, where $m_\alpha \in B_A(H, H')$ Hilbert A-module morphisms and ∂^α are Fréchet derivatives of functions with values in the appropriate normed vector space of bounded linear maps $B_A(H, H')$ (operator norm).
- For each $\xi \in T^*M$, the *symbol* $\sigma(D, \xi)$ of operator D in direction ξ defined as in the classical case. **Elliptic operator** = symbol is isomorphism for any non-zero cotangent vector ξ.

Some topological results on Hilbert and Banach bundles

- Each holomorphic *finite rank* bundle over 2-sphere is a direct sum of holomorphic line bundles. All of the line bundles, up to one, are non-trivial. Grothendieck [6].
- If H is infinite dimensional Hilbert space and $U(H)$, equipped with the strong operator topology, is contractible $\Longrightarrow$ the first Čech cohomology group is trivial by Dixmier, Douady [2]/Kuiper [10].
  $\Longrightarrow$
  $\exists$ (continuous) bundle isomorphism $J : H \to M \times \mathcal{H}$ for any Hilbert bundle $\mathcal{H} \to M$ with fibre an *infinite rank Hilbert space*.
- There is a holomorphic *Banach* fibre bundle over 2-sphere such that the quotient topology on the sheaf cohomology of the bundle holomorphic sections spaces is non-Hausdorff. Erat [3].

Mishchenko–Fomenko's result

- **Theorem 1:** Let $\mathcal{H} \to M$ be a **finitely** generated A-Hilbert bundle over a **compact** manifold, where A is a C^*-algebra with unit and D is an A-elliptic operator on sections of $\mathcal{H}$. Then the kernel of D is a finitely generated Hilbert A-module. There is an A-compact op. K such that $D + K$ has closed image.
  Proof. Mishchenko–Fomenko [4].
- The index of D is computed by the Chern class of $\mathcal{H}$ and the Todd class of the symbol of D.
- The extensions of D to completions of smooth sections of $\mathcal{H}$ are A-**Fredholm**
- If f is a solution of $\widetilde{D}f = 0$ for the extension of D to a Sobolev space W^r, then f is smooth, i.e., C^∞ (*regularity as for the Laplace operator on functions*)
- But note that D is A-Fredholm does not imply D has closed image (example above), which we wanted.

Hodge theory in case of compactly invariant operators

Theorem 2: Let A be a C^*-algebra and $(\mathcal{H}^i \to M)_i$ be a sequence of finitely generated Hilbert A-bundles over a compact manifold M. Suppose that images of the complex's Laplacians of an elliptic A-invariant complex in sections $\Gamma(\mathcal{H}^i)$ are closed. Then the cohomology groups of the complex are finitely generated Hilbert A-modules.

Proof. SK [7].

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Proof. SK [7].

Theorem 3: If A is the C^*-algebra of *compact operators* on a Hilbert space and $(\mathcal{H}^i \to M)_i$ is a sequence of finitely generated Hilbert A-bundles over a compact manifold M, then cohomology groups of an elliptic A-linear complex in sections $\Gamma(\mathcal{H}^i)$ are finitely generated projective Hilbert A-modules, and are Hausdorff also in the Fréchet topology on the section spaces.
Proof. SK [8].

- Proof uses results considering transfer from $K(H)$-linearity to linearity with respect to Hilbert–Schmidt operators which are dense in $K(H)$. Moreover, they are Hilbert spaces (described, e.g., in Bakić, D., Guljaš, B., Hilbert C^*-modules over C^*-algebras of compact operators. Acta Sci. Math. (Szeged) 68 (2002), no. 1-2, 249–269).

Supplement: Algebra of compact ops. contra *CCR*-algebra as observable algebras

- Compact operators considered as observable algebra in Quantum theory, instead of the CCR algebra.
- Poisson algebra of linear functions $\sum_i^n (a_i q^i + b^i p_i) + ct$, i.e., $(\mathbb{R}^{2n+1})^*$, $a_i, b^i, c \in \mathbb{C}$.
- The associative product is the point-wise multiplication and the linear Poisson (Lie and Leibniz structure) bracket $\{,\}$ obeys $\{q^i, q^j\} = \{p_i, p_j\} = 0$ and $\{p_i, q^j\} = \delta_i^j$.
- For the Lie algebra structure, there is a Lie group – the Heisenberg group
- $(v, t) \cdot (w, t) = (v + w, t + s + \frac{1}{2}\omega(v, w))$ where $v, w \in \mathbb{R}^{2n}$, $t, s \in \mathbb{R}$ and ω is the standard symplectic form

Supplement: Compact operators versus CCR-relations

- There is an irreducible unitary representation of the Heisenberg group on the Hilbert space $H = L^2(\mathbb{R}^n)$ – the Schrödinger representation
- The differentiation on smooth vectors of this representation gives the so called CCR-relations for generators of the Lie algebra of the Poisson algebra above.
- However, on the CCR-algebra there are 'too' many states See, e.g., Feintzeig, B. [On the choice of the Algebra for Quantization, Phil. Science, 85, Vol. 1, 2018] for other advantages of choosing rather the algebra $K(H)$ than the CCR-algebra, regarding states and observables, and references there. See also Jorgensen, P. Tian, F.: arxiv601.01482v2; Buchholtz, D., Grundling, H.: arxiv0705.188v3 .

[1] Dubois-Violette, M., Michor, P., Connections on central bimodules in noncommutative differential geometry, J. Geom. Phys. 20 (1996), Issue 2–3, 218–232.

[2] Dixmier, J., Douady, A., Champs continus d'espaces hilbertiens et de C^*-algebres. Bull. Soc. Math. France 91 (1963), 227–284.

[3] Erat, M., The cohomology of Banach space bundles over 1-convex manifolds is not always Hausdorff, Math. Nachr. 248/249 (2003), 97–101.

[4] Fomenko, A., Mishchenko, A., The index of elliptic operators over C^*-algebras, Izv. Akad. Nauk SSSR, Ser. Mat. 43 (1979) No. 4, 831–859, 967, translation of the Russian orig.

[5] Grothendieck, A., Produits tensoriels topologiques et espaces nucléaires, Mem. Amer. Math. Soc. 16 (1955).

[6] Grothendieck, A., Sur la classification des fibrés holomorphes sur la sphere de Riemann, American Journal of Mathematics. 79 (1), 121–138.

[7] Krýsl, S., Hodge theory for self-adjoint parametrix possessing complexes over C^*-algebras, Annals Glob. Anal. Geom. 47 (2015) No. 4, 359–372.

[8] Krýsl, S., Elliptic complexes over C^*-algebras of compact operators, Journal of Geometry and Physics 101 (2016), 27–37.

[9] Krýsl, S., Induced C^*-complexes in metaplectic geometry, Commun. Math. Phys. 365 (2019) No. 1, 61–91.

[10] Kuiper, N., The homotopy type of the unitary group of Hilbert space. Topology 3 (1965), 19–30.

[11] Paschke, W., Inner product modules over B^*-algebras. Trans. Amer. Math. Soc. 182 (1973), 443–468.

[12] Rieffel, M., Induced representations of C^*-algebras. Advances in Math. 13 (1974), 176–257.

[13] Schick, T., L^2-index theorems, KK-theory, and connections, New York J. Math. 11 (2005), 387–443.