Differential operators invariant with respect to the compact operators

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Themes

- Images of elliptic operators on infinite rank bundles - their topology
- Topology on cohomology groups of elliptic complexes
  ⇒ Topological aspects of BRST/BFV-theories ("first steps")
- Hilbert $C^*$-modules
- Assumption *invariance of the operators*. Use Mishchenko’s elliptic operators theory (Fomenko, Mishchenko [4], ’70)
1) Constrained systems in Physics

- **Classical Physics**
  
  Configuration space known, exact potential function for the constraints not known or too complicated (ball falling in gravity field to a desk + repulsive force of a desk etc.)
Motivation

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  - Hamilton formulation via Legendre transform
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Theories in Physics testable by measurements $\Rightarrow$ State spaces must have a good behaviour of limits. At least: Limit of a sequence either doesn’t exist, or it is unique if it exists. Quotient space of Hausdorff topological vector spaces has uniqueness of limits if and only if the dividing space is closed.
2) Elliptic equations on compact manifolds (without boundary). Hodge theory

1. Cohomology of elliptic operators (or better their chain complexes) on finite rank bundles over compact manifolds are finite dimensional vector spaces.

2. Kernels of complexes’ Laplacians (de Rham’s Laplacian on Riemannian, Dolbeault’s Laplacian on complex manifolds) are as vector spaces isomorphic to the cohomology groups of the complex. Not true: If compact is omitted. Not true: If finite rank is omitted. Not true: If elliptic is omitted.
Fails of Hodge theorem generalizations

- de Rham complex on $\mathbb{R}^n$. Laplace has infinite dimensional kernel: e.g., $1, x, y, xy, x^2 - y^2, x^3 - 3xy^2, y^3 - 3x^2y \ldots$. The kernel is non-trivial in each homogeneity (by Weyl thy, or separation of variables technique). Just observe $(x \pm iy)^k$ are non-zero and harmonic. (Also further solutions - sh($x + iy$) etc.)

- Let $(e_i)_i$ be ON-basis of separable Hilbert space $H$ of infinite dimension. Let $D : C^\infty(S^1, H) \to C^\infty(S^1, H)$, $f = \sum_{i=1}^{\infty} f_i e_i$ be given by $(Df)(m) = \sum_{i=1}^{\infty} \frac{df_i}{d\phi}(m)e_i$. Then $Df = 0 \iff df_i/d\phi = 0 \iff f_i(e^{2\pi i\phi}) = a_i \in \mathbb{C} \Rightarrow f = \sum_i a_i e_i$. Thus $f$ is any constant $H$-valued function. Solution space $\simeq H$ (linearly). Thus infinite dimensional.

- d’Alembert on Minkowski-flat torus:
  $f_n(x^0, x^1) = c_n e^{2\pi in(x^0 - x^1)}$, $c_n \in \mathbb{C}$, $n \in \mathbb{N}$, independent.
  $\ast d \ast d f_n = \Box f_n = (\partial_{x^0}^2 - \partial_{x^1}^2)f_n = 0$, where $\ast$ is the Hodge-star operator for Minkowski metric
Main steps in proving the Hodge theory

(Hodge ’30, Weyl ’43, de Rham ’46)

- *Fredholm theory* - needs complete normed spaces
  Linear operator $D : H \rightarrow H'$ is *Fredholm* (def.) $\iff$ has finite rank and corank ($= \text{dimension of } H'/\text{Im}D$). Image is consequently *closed*. 
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- **Atkinson’s theorem**: \( D \) Fredholm op. (between complete normed spaces) ⇔ exists \( \tilde{D} \) such that \( D\tilde{D} = \text{Id}_{H'} + K_1 \) and \( \tilde{D}D = \text{Id}_H + K_2 \), \( K_i \) compact ops (\( i = 1, 2 \))
A. S. Mishchenko’s idea

- Rellich–Kondratchov fails for Sobolev spaces of *infinite dimensional vector space valued* functions on compact sets: \( \text{Id} : W^{s+1}(M, H) \to W^{s}(M, H) \) is not compact if \( \text{dim } H = \infty \).
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- The idea: Any Hilbert space \( H \) is finitely generated over \( B(H) = \{ A : H \rightarrow H | A \text{ linear and bounded} \} \). Any vector in \( H \) can be achieved by dilated projection: \( |v \rangle = (|v \rangle \langle f|) |f \rangle \) (Fix unit vector \( f \in H \), \( v \in H \) arbitrary. Then \( v \) is achieved by the ket-bra above.)

We can work with \( F(H) \) (finite rank ops \( H \rightarrow H \)) or \( K(H) \) (compact ops \( H \rightarrow H \)) only.

- It needs Hilbert \( C^* \)-modules: generalization of Hilbert spaces.
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- We explain parts of Mishchenko–Fomenko [4] generalization of Fredholm theory based on “analytic” modules over $C^*$-algebras
Hilbert $A$-modules

Rieffel [12], Paschke [11]

- **Pre-Hilbert $A$-module** $H$ is a *complex vector space* which is a right module over a $C^*$-algebra $A$; together with a hermitian-symmetric $A$-valued *sesquilinear form* $(\cdot, \cdot)_H : H \times H \to A$ which is positive definite (the $C^*$-product).
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- **Positive elements** in $A$ are those which have non-negative spectrum (in the unitalization) of $A$. If $|\cdot|_A$ denotes the norm in $A$, $v \ni H \mapsto |v|_H := \sqrt{|(v, v)_H|_A}$, $v \in H$, is a norm on $H$ (the $C^*$-norm).
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- **Spectrum** of element $a \in A$. All complex $\lambda$ such that $a - \lambda 1$ does not have an inverse in $A$ or in $A \oplus \mathbb{C}$ (unitalization), unit $e = (0, 1)$. Algebra is unital if it contains an element $e$ (unit) such that $ea = ae = a$ for all $a \in A$. ($e = e'e = ee' = e'$.)
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Definition: Hilbert $A$-module = pre-Hilbert $A$-module whose $C^*$-norm is complete (the space is complete normed vector space; a Banach space)
Examples of (pre-)Hilbert $A$-modules

- $H = A$, $v \cdot a = va$, $(v, w)_H = v^*w$, where $a, v, w \in A$. Operation $\ast$ is from $A$. The $C^*$-module norm is
  \[ |v|_H^2 = |(v, v)_A|_A = |v^* v|_A = |v|_A^2 \] (by $C^*$-identity in $A$).
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- $A = \mathbb{C}$, $H = C^\infty(S^1)$ with $(f, g)_H = \int_{S^1} \bar{f} g \text{vol}_g$
  $|f|^2_H = |(f, f)_H|_{\mathbb{C}} = \int_{S^1} |f|^2 \text{vol}_g = \int_{x=0}^{1} |f(e^{2\pi i x})|^2 d\lambda_{\mathbb{R}}(x)$,
  $S^1 \subseteq \mathbb{C}$. Not Cauchy complete. Pre-Hilbert $A$-module. (It is Cauchy complete if considered as metric space generated by classical Fréchet “seminorms”.)
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- Also $H = A^n = A \oplus \ldots \oplus A$ with diagonal action
  $(a_1, \ldots, a_n) \cdot a = (a_1 a, \ldots, a_n a)$ and
  $((a_1, \ldots, a_n), (b_1, \ldots, b_n))_H = \sum_{i=1}^n a_i^* b_i$
Morphisms of Hilbert $C^*$-modules

- $\ell^2(A) = \{(a_i)_{i=1}^{\infty} | \sum_{i=1}^{\infty} a_i^* a_i \text{ converges in } A\}$. Action is diagonal and product is $((a_i), (b_i))_{\ell^2(A)} = \sum_i a_i^* b_i$. Called basic $A$-module. $\ell^2(A) = A \otimes \ell^2(\mathbb{N})$ (projective tensor product [5]).
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- Morphism of Hilbert $C^*$-modules is any adjointable map between Hilbert $C^*$-modules, i.e., $L : H \to H'$ and exists $L' : H' \to H$ such that $(Lh, h')_{H'} = (h, L'h')_H$. Notation $L' = L^*$. It is automatically $A$-linear. Denoted as $B_A(H, H')$, with operator norm topology. Denote $B_A(H)$ if $H = H'$. 
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- Morphisms: $\Theta_{e,f}(h) = e \cdot (f, h)_H$ for $e \in H'$ and $f, h \in H$ (elementary operators). Note $(f, h)_H \in A$. $A$-Compact operators (def.) are in the closure in $B_A(H)$ of finite linear sums of elementary operators (Kasparov).
Example: $C^*$-Fredholm with non-closed image

- $D$ is $A$-Fredholm if (def.) $D$ is invertible modulo an $A$-compact operator: $D \tilde{D} = \text{Id} + K_1$.

- $A = H = C^0([0, 1])$ with pointwise multiplication, $f^* = \overline{f}$, and sup norm, i.e., $|f|_A = \sup\{ |f(x)|, x \in [0, 1] \}$, $(f, g)_H = \overline{f}g \in A$, $f, g \in H = A$. The $C^*$-norm on $H$, $|f|_H = |f|_A$. $Df = xf$, 

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  Thus, $D = \Theta_{1,x}$. (Invertible modulo $A$-compacts: $D\text{Id}_H = \text{Id}_H D = \text{Id}_H + (D - \text{Id}_H)$.)
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- Easy compute \(D^* = D\). If \(\text{Im} \ D\) were closed, then \(H = \text{Im} \ D \oplus \text{Ker} \ D\). But easily \(\text{Ker} \ D = 0\). Thus \(\text{Im} \ D = H\).

But \(D(f)(x) = (xf)(x) = xf(x) = 1\) (for constant function 1) has solution \(f(x) = 1/x\), not in \(C^0([0, 1])\). So 1 is not in \(\text{Im} \ D\), \(D\) is not surjective, and thus \(\text{Im} \ D\) is not closed.
Hilbert $C^*$-bundles

- Let $A$ be a $C^*$-algebra. A *Hilbert $A$-bundle* $\mathcal{H}$ with fibre $H$ is a locally trivial Banach bundle with fibre the Banach space $H$, where each fibre is a Hilbert $A$-module, equipped with a maximal smooth atlas whose transition functions are bijective maps in $B_A(H)$. *Finitely generated* Hilbert $A$-bundle means the fibre is finitely generated $A$-module.

- An *(Hilbert) $A$-differential operator* between Hilbert $A$-bundles is any $A$-linear map on section spaces such that locally it is of form $\sum_{|\alpha| \leq r} m_\alpha \partial^\alpha$, where $m_\alpha \in B_A(H, H')$ Hilbert $A$-module morphisms and $\partial^\alpha$ are Fréchet derivatives of functions with values in the appropriate normed vector space of bounded linear maps $B_A(H, H')$ (operator norm).

- For each $\xi \in T^*M$, the *symbol* $\sigma(D, \xi)$ of operator $D$ in direction $\xi$ defined as in the classical case. *Elliptic operator* $\equiv$ symbol is isomorphism for any non-zero cotangent vector $\xi$. 
• Each holomorphic *finite rank* bundle over 2-sphere is a direct sum of holomorphic line bundles. All of the line bundles, up to one, are non-trivial. Grothendieck [6].

• If \( H \) is infinite dimensional Hilbert space and \( U(H) \), equipped with the strong operator topology, is contractible \( \Rightarrow \) the first Čech cohomology group is trivial by Dixmier, Douady [2]/Kuiper [10].

\[ \Rightarrow \]

\( \exists \) (continuous) bundle isomorphism \( J : H \to M \times \mathcal{H} \) for any Hilbert bundle \( \mathcal{H} \to M \) with fibre an *infinite rank Hilbert space*.

• There is a holomorphic *Banach* fibre bundle over 2-sphere such that the quotient topology on the sheaf cohomology of the bundle holomorphic sections spaces is non-Hausdorff. Erat [3].
Theorem 1: Let $\mathcal{H} \to M$ be a finitely generated $A$-Hilbert bundle over a compact manifold, where $A$ is a $C^*$-algebra with unit and $D$ is an $A$-elliptic operator on sections of $\mathcal{H}$. Then the kernel of $D$ is a finitely generated Hilbert $A$-module. There is an $A$-compact op. $K$ such that $D + K$ has closed image.

Proof. Mishchenko–Fomenko [4].

The index of $D$ is computed by the Chern class of $\mathcal{H}$ and the Todd class of the symbol of $D$.

The extensions of $D$ to completions of smooth sections of $\mathcal{H}$ are $A$-Fredholm.

If $f$ is a solution of $\tilde{D}f = 0$ for the extension of $D$ to a Sobolev space $W^r$, then $f$ is smooth, i.e., $C^\infty$ (regularity as for the Laplace operator on functions).

But note that $D$ is $A$-Fredholm does not imply $D$ has closed image (example above), which we wanted.
**Theorem 2:** Let $A$ be a C*-algebra and $(\mathcal{H}^i \to M)$; be a sequence of finitely generated Hilbert $A$-bundles over a compact manifold $M$. Suppose that images of the complex’s Laplacians of an elliptic $A$-invariant complex in sections $\Gamma(\mathcal{H}^i)$ are closed. Then the cohomology groups of the complex are finitely generated Hilbert $A$-modules.

*Proof.* SK [7].
Theorem 2: Let $A$ be a $C^*$-algebra and $(\mathcal{H}^i \to M)$ be a sequence of finitely generated Hilbert $A$-bundles over a compact manifold $M$. Suppose that images of the complex’s Laplacians of an elliptic $A$-invariant complex in sections $\Gamma(\mathcal{H}^i)$ are closed. Then the cohomology groups of the complex are finitely generated Hilbert $A$-modules.

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Theorem 3: If $A$ is the $C^*$-algebra of compact operators on a Hilbert space and $(\mathcal{H}^i \to M)$ is a sequence of finitely generated Hilbert $A$-bundles over a compact manifold $M$, then cohomology groups of an elliptic $A$-linear complex in sections $\Gamma(\mathcal{H}^i)$ are finitely generated projective Hilbert $A$-modules, and are Hausdorff also in the Fréchet topology on the section spaces.

Proof. SK [8].
Proof uses results considering transfer from $K(H)$-linearity to linearity with respect to Hilbert–Schmidt operators which are dense in $K(H)$. Moreover, they are Hilbert spaces (described, e.g., in Bakić, D., Guljaš, B., Hilbert $C^*$-modules over $C^*$-algebras of compact operators. Acta Sci. Math. (Szeged) 68 (2002), no. 1-2, 249–269).
Compact operators considered as observable algebra in Quantum theory, instead of the CCR algebra.

Poisson algebra of linear functions \( \sum_i^n (a_i q^i + b_i p_i) + ct \), i.e., \((\mathbb{R}^{2n+1})^*, a_i, b_i, c \in \mathbb{C}\).

The associative product is the point-wise multiplication and the linear Poisson (Lie and Leibniz structure) bracket \( \{, \} \) obeys \( \{q^i, q^j\} = \{p_i, p_j\} = 0 \) and \( \{p_i, q^j\} = \delta^j_i \).

For the Lie algebra structure, there is a Lie group – the Heisenberg group

\[ (v, t) \cdot (w, t) = (v + w, t + s + \frac{1}{2} \omega(v, w)) \] where \( v, w \in \mathbb{R}^{2n}, t, s \in \mathbb{R} \) and \( \omega \) is the standard symplectic form.
Supplement: Compact operators versus CCR-relations

- There is an irreducible unitary representation of the Heisenberg group on the Hilbert space $H = L^2(\mathbb{R}^n)$ – the Schrödinger representation.

- The differentiation on smooth vectors of this representation gives the so-called CCR-relations for generators of the Lie algebra of the Poisson algebra above.

- However, on the CCR-algebra there are 'too' many states. See, e.g., Feintzeig, B. [On the choice of the Algebra for Quantization, Phil. Science, 85, Vol. 1, 2018] for other advantages of choosing rather the algebra $K(H)$ than the CCR-algebra, regarding states and observables, and references there. See also Jorgensen, P. Tian, F.: arxiv601.01482v2; Buchholtz, D., Grundling, H.: arxiv0705.188v3.


