Remarks

* de Donder, Lepage. (differential geometric approach to CoV).
* Vinogradov, 1978, 1984 ... (Talks to follow). C spectral sequence
* Tulczjew 1982.✓
* Tsujishita 1982. – intepretation of cohomology
* Fuchs, Gabrielov, Gelfand 1975.

"The general idea of making topological invariants local by means of explicit formula suggested by Gelfand in 1970"

- variational bicomplex✓
  - ↳ "invented by Gelfand / Losik, in the case of metrics (to appear!!)
- difference bicomplex✓
- from International Congress (1970).

In the end of this part of the report I would like to introduce a general concept of formal differential geometry. It arises when one formalises and generalises the methods of construction of Pontryagin and Chern classes (by means of metrics and connections); also in the expression of the index of a differential operator in terms of the symbol and the metric of the manifold.

* Also in the same years ~ higher order symmetries, conservation laws for soliton equations.

Generalizations

$R \subset J^2(E)$

1. $\mathcal{R} \hookrightarrow J^k(E)$, $\mathcal{R} \to M$, $J^k(E) \to M$, $M \to M$

Example $u_t = u_{xx}$. $D_x u = u_x$, $D_t u = u_{xx}$. so the structure equations change dramatically.

Prolong

$\mathcal{R}^\infty \to J^\infty(E)$, $\mathcal{R}^\infty \to M$, $J^\infty(E) \to M$.

($\mathcal{R}^{(m)} \to \mathcal{R}^{m-1} \cdots \mathcal{R}$; formal integ. Spencer Goldschmidt)

$\Omega^{*,*}(\mathcal{R}^\infty)$. variational bicomplex for differential equations

$\omega \in \Omega^{n-1,0}(\mathcal{R}^\infty)$, $d_H \omega = 0$ mod equation classical conservation laws.

$\omega \neq d_H$.

2. Let G act on $E \to M$; $\boxed{\text{pr } G \text{ acts on } J^\infty(E)}$ $\Omega^{*,*}_{\text{pr}G}(J^\infty(E))$.

symmetry grp.

invariant variational bicomplex.

$G \sim$ diffeo group $\lambda \in \Omega^{n,0}_{\text{pr}G}$, λ. a Lagrangian for a generally covariant field theory.

3. If G is a symmetry group of $\mathcal{R}$; $\Omega^{*,*}_G(\mathcal{R})$.

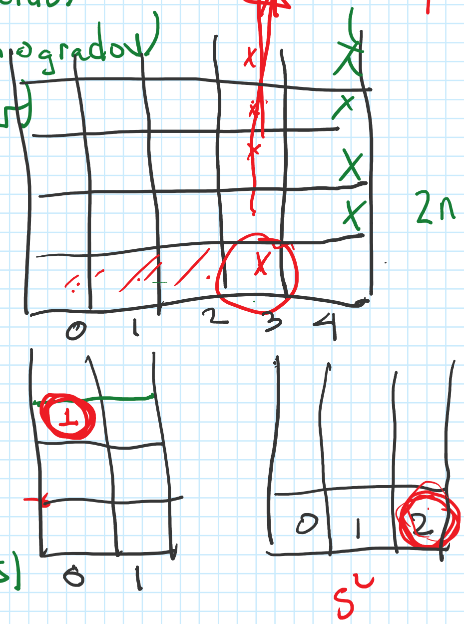
$\underline{Ex}$. $\Omega^{*,*}_{\text{diff } M}(G_{ij} = 0)$. natural variational bicomplex for Einstein Eq.

Darboux Int.

$u_{xy} = 0$ $u_{xy} = e^u$

B. In. equations

1 Inverse Problem for ODE (Douglas)
2 Contact-Valued Conservation (Tsujishita, Todr. Fields)
3 Lower Degree Conservation Laws (Byrant-Griffiths, Vinogradov)
4 Characteristic Lagrangians (Nirenburg, Kazden-Warner)
5 Symmetric Criticality (Palais)
6 Sym + Cons Law ⇒ Variational (Takens)
7 Invariant Bicomplexes (Gelfand-Fuks)
8 Winding Number
9 Invariants of Riemannian Manifolds (Gilkey)
10 Gauss Bonnet (Chern, Bott ..... localization) $\Omega^{0,2}$
11 Variational Principles for Riemannian Metrics (Chern Simons)
. Foliations (Godbillon Vey)
. Difference Bicomplex (Fuchs, Gabrielov, Gel'fand)
. H' Principle (Gromov)



$\Delta^\alpha = \ddot{u}^\alpha - F^\alpha(x, u^\beta, \dot{u}^\beta)$

Find a matrix $M_{\alpha\gamma}(x, u^\beta, \dot{u}^\beta)$ such that

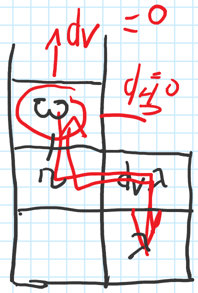
$$M_{\alpha\beta} \cdot \Delta^\beta = EL_\alpha(L) = \frac{\delta L}{\delta u^\alpha}$$

$$\omega = \frac{1}{2} A_{\alpha\beta}\, \theta^\alpha \wedge \theta^\beta + B_{\alpha\beta}\, \theta^\alpha \wedge \dot{\theta}^\beta$$

$$\theta^\alpha = du^\alpha - \dot{u}^\alpha dx$$

$$\dot{\theta}^\alpha = d\dot{u}^\alpha + F^\alpha dx .$$

Lemma. If $\Delta^\alpha = 0$ is variational the $H^{0,2}_{d_H}(\Delta) \neq 0$.



$d_V \lambda = E(L) + d_H \eta .$

$d_H f = 0$

Example's

* $u'' = u^2 + v^2, \quad v'' = u$
* $u'' = u^2 + v^2 \quad v'' = 0$

(Douglas 1939).

$H^{0,2}_{d_H} = 0.$

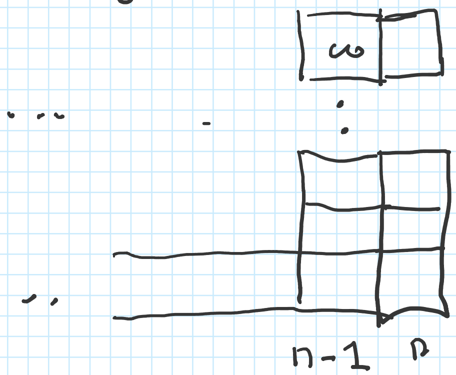
$M\,(u_t - K(x, u, u_x \ldots)) = E(L)$, M now a diff. operator

$M = a + b D_x + c D_x^2 \cdots$

Again $H^{1,2}_{d_H}(R) \neq 0.$

(Mark Fel's 2019).

$\mathcal{R}$: system of 2^{nd} order PDE.



$\omega \in \Omega^{n-1,s}(\mathcal{R}) \quad s \geq 3.$

$d_H \omega = 0$ on $\mathcal{R}$.

$\omega \neq d_H \eta$.

Wave Eq. $u_{xy} = 0$

$$\omega = dx \wedge \theta_x \wedge \theta_{xx} \wedge \theta_{xxx} \cdots$$

$$d_H \omega = dy\, D_y(\quad '' \quad) = 0$$

Tsujishita: Relate $H^{1,s}_{d_H} \neq 0$ to Darboux int.

Examples: $u_{xy} = e^u$. $\rightsquigarrow$ Toda lattice equations

$(u_y^2+1)u_{xx} - 2u_x u_y u_{xy} + (u_x^2+1)u_{yy} = 0$ Born-Infeld

$u_{xy} = \frac{2n\sqrt{pq}}{x+y}$ - Goursat.

Note: These examples are symmetric $x \longleftrightarrow y$,

---

Theorem. Let $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$ be a hyperbolic pde with $x \leftrightarrow y$ symmetry.

Then the equation is Darboux Int $\Longleftrightarrow$ $H^{1,s}_{d_H}(\mathcal{R}^\infty) \neq 0$ for some $s > 3$.

For a system of pde. $\Delta = \Delta_\beta(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij})$

set

$$\mathcal{L}_\Delta(\rho) = \frac{\partial\Delta_\beta}{\partial u^\alpha}\rho^\alpha + \frac{\partial \Delta_\beta}{\partial u^\alpha_i} D_i\rho^\alpha + \frac{\partial\Delta_\beta}{\partial u^\alpha_{ij}} D_{ij}\rho^\alpha \to P^{ij}_{\beta\alpha}$$

$$\rho = \rho(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij} \dots).$$

Symmetries. $\mathcal{L}_\Delta(\rho) = 0$ on $\Delta = 0$.

Conservation laws: ($H^{n-1,0}_{d_H}(\mathcal{R})$) $\mathcal{L}^*_\Delta(\rho) = 0$ (+ other conditions).

Lower degree conservation laws:

$\square\square \dots \quad \square \quad \square_{n-1} \quad \square$ $\qquad d_H\varpi\ 0$ on $\mathcal{R}$.

$r \quad d_H(\omega) = 0$

Put $\eta = d_V\varpi$,

$d_H\eta = 0 \mod \Delta = 0$, $\quad d_V\Delta = 0$

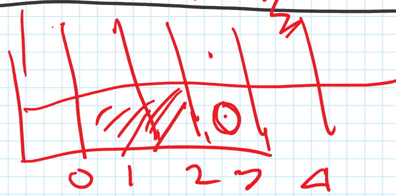
⇒ $d_H\eta \equiv \rho^\alpha d_V\Delta_\alpha + \rho^{\alpha i} d_V D_i\Delta_\alpha + \dots \rho^{\alpha i_1\dots i_k} d_V D_{i_1}\dots D_{i_k}\Delta_\alpha$ *.

⇒ $\boxed{\rho^{\alpha(i_1\dots i_k} P^{hk)}_{\alpha\beta} = 0}$

Algebraic Spencer Equations for Lower degree conservation laws.

A solution $\rho^{\alpha i\dots i_k}$ is trivial if ⇒ integrate by parts.

Eins Eq



Einstein Equations

Monday, August 23, 2021 10:04 AM

Nirenburg Problem.

When is a smooth map $k: S^2 \to \cancel{\mathbb{R}}$ ~~the~~ the Gaussian curvature of a metric on S^2,

$\mathcal{R}$: $\quad K(g) = k$. $\quad$ If $\quad g = e^{2u} g_0 \qquad \Delta u = 1 - k e^{2u}$.

Characteristic Lagrangians

Given $\iota: R \subset J^k(E)$, find

$$\lambda \in \Omega^{n,0}(J^\infty(E)), \quad \lambda \neq d_H \eta.$$

$$\boxed{\iota^*(\lambda) = d_H \xi} \quad *$$

$$\Rightarrow \quad \int_M \lambda = 0.$$

Theorem. If λ satisfies $*$ for $\Delta u = 1 - k e^{2u}$, the

$$\lambda = \lambda_{G.B} + \lambda_{K.W}$$

$$\lambda_{GB} = e^{2u} k \cdot \nu_0$$

$$\lambda_{K.W} = \frac{1}{2} A^i k_{|i} e^{2u} \nu_0 \qquad A_{i|j} + A_{j|i} = 0$$

Other Examples ???

Let G be a Lie symmetry group of $\lambda \in \Omega^{n,0}(J^\infty(E))$.
Let $\tilde{\Delta} = E(\lambda)(j^\infty(s))$ be the reduced diff equations for the G invariant sections. $g \cdot s(x) = s(g \cdot x)$.

Is there reduced Lagrangian $\tilde{\lambda}$ for $\tilde{\Delta}$.

Example.

$\lambda = (u_x^2 + u_y^2 + u_z^2)\, dx\, dy\, dz \in \Omega^{3,0}([x, y, z, u, u_x \ldots])$

$E(\lambda) = u_{xx} + u_{yy} + u_{zz}$

$G = SO(3) \qquad u = v(r)$. $\quad r = \sqrt{x^2+y^2+z^2}$

$\tilde{\Delta} = v_{rr} + \frac{2}{r} v_r$

$\tilde{\lambda} = \frac{1}{2} r^2 v_r^2\, dr \in \Omega^{1,0}([r, v, v_r, v_{rr}])$.

Using symmetries to reduce field equations from pde to ode.

?Palais: Are symmetric critical pts = critical pts of symmetric variables

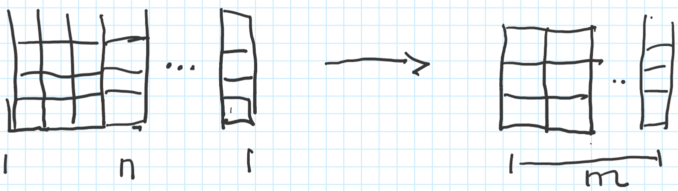
. We studied this problem thru V.B.
, we obtained new obstructions to Lagrangian reduction.

So we have 2 bicomplexes

$$\Omega^{r,s}_G(J^\infty(E)) \qquad\qquad \Omega^{r,s}(J^\infty(\bar{E}))$$

$$x^i, u^\alpha, u^\alpha_i \ldots \qquad\qquad y^a, v^A, v^A_a \ldots$$

$u^\alpha = f^\alpha(y^a, v^A)$ G inv. sections



$m = n -$ orbit dim G.

r ''

"Dimensional Reduction".

Theorem: There is a G invariant multi-vector.

$$\chi = X_1 \wedge X_2 \wedge \cdots X_r$$

inducing a map of bicomplexes which commutes with d_H, EL, Iparts if and only if $H^n(\Gamma, \Gamma_0) \neq 0$.

$\omega \in \Omega^{r,s}_G(J^\infty(E))$

$\downarrow$

$\chi \lrcorner \omega$

$\downarrow$

$$\mathcal{L}_X \lambda = X \lrcorner \, EL(\lambda) + d_H \, "X \lrcorner \Theta"$$

$$\Delta \in \Omega^{n,1}(J^\infty(E)).$$

$$\mathcal{L}_X \Delta = X \lrcorner \, \mathcal{H}(\Delta) + EL(X \lrcorner \Delta).$$

X symm of Δ $\quad \mathcal{L}_X \Delta = 0$

X a cons. law generator $EL(X \lrcorner \Delta)$.

$\Rightarrow \quad X \lrcorner \, \mathcal{H}(\Delta) = 0$

Is Δ has "lots" of symmetries
and lots of conservation laws
$\overset{??}{\Rightarrow}$ $\mathcal{H}(\Delta) = 0$, ie Δ is locally variation.

Anderson, Pohjanpelto (94, 95, 96).
Dasfinger (2020).

$A^{ij} = A^{ij}(g, \partial g, \partial^2 g, \ldots)$ rank 2 natural tensor on bundle of metrics

$A^{ij}{}_{|j} = 0 \Rightarrow A^{ij} = \frac{\delta L}{\delta g_{ij}}$. (classification of div. free tensors)

Invariant Bicomplex

Let G be a finite dim Lie group acting on $\pi: E \to M$, with infinitesimal generators Γ.
Let $\mathcal{U} < J^\infty(E)$, Γ acting freely. $\Gamma_0 = \{0\}$.

$\Rightarrow$ the interior rows $\Omega^{*,s}(J^\infty(\mathcal{U}))_\Gamma$ are exact

$\Rightarrow$ $H^*(\mathcal{E}^*(J^\infty(\mathcal{U}))_\Gamma = H^*(\mathfrak{g})$.

(Anderson, Pohjanpelto).

Can one do better, to include jets of invariant sections.

$J^\infty(\mathbb{R}, \mathbb{R}^2)$

$x, u, v, \dot{u}, \dot{v}, \ddot{u}, \ddot{v} \dots$

$\dot{u}^2 + \dot{v}^2 \neq 0.$

G : Euclidean group in u-v plane.

: diffeos $\bar{x} = \phi(x)$.

$\Omega^{1,0}_G$ = natural Lagrangians for regular plane curves

$= \lambda = L(\kappa, \dot{\kappa}, \ddot{\kappa}, \dots)ds$ κ = curvature

$\dot{} = \frac{d}{ds}$

Theorem $E(\lambda) = 0$

$\Rightarrow \quad \lambda = a\kappa\, ds + d_H f.$

Cheung (87).

Invariant Variational Bicomplex for local diffeomorphism

$J^\infty(\mathbb{R}^n, \mathbb{R}^n)$

$(x^i, u^i, u^i_j, u^i_{jk} \dots)$ $\quad \dfrac{\partial u^i}{\partial x^j}$

$\det(u^i_j) \neq 0$

G: $\bar{x}^i = \varphi(x^i)$, local diffeo.

$u^i \sim$ scalars.

The interior rows of $\Omega_G^{*,s}(J^\infty(E))$ are ?? exact.

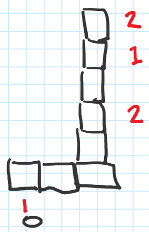
Theorem: $H^*(\mathcal{E}L(J^\infty(E))_G) =$ Gelfand Fuk cohom. of formal vector fields.

Example $n = 1$, $X_k = x^k \partial_x$, $[X_k, X_l] = (k-l) X_{k+l-1}$

$$\omega(a\partial_x, b\partial_x, c\partial_x) = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}, \quad \delta\omega = 0 \quad \omega \in H^3(W_1).$$

Bounds: $H^p = 0 \propto p < 2n+1$, $\quad H^p > n(n+2)$

Example: $n = 2$.



Remark: The isomorphism itself is easy to establish. But a direct calculation via VB is still a long story.

Gilkey's Theorem

$E = S^2(T^*M) \to M$

G = local diff.

Better

$E = \mathbb{R}^n \times S^2(T^*\mathbb{R}^n)$

(x^i, g_{ij}) , $\det g_{\cdot} \neq 0$.

$\Omega_G^{*,*}(J^\infty(E))$.

$\lambda \in \Omega_G^{n,0}(J^\infty(E))$

$\lambda = L(g_{ij}, R_{ijhk}, \nabla R_{ijhk} \cdots)\,\nu.$

Theorem

$$H^r(\mathcal{E}^*(J^\infty(E)_G)) = \begin{cases} \mathrm{Pont}(R) \\ \mathrm{Pf}(R) \end{cases}$$

Remark:

$$\lambda[g, \Gamma] = \frac{1}{\sqrt{g}} K_n(g, \Gamma).$$

Let g_t, Γ_t be a curve of metric, connections.
The first variational formula with boundary calculated from d_H homotopy operators
$\Rightarrow$ Chern's proof of generalized Gauss-Bonnet.

Other localization results also follow.

Cotton tensors

$P = P(a)$, $gl(n,\mathbb{R})$ invariant poly of degree $2m$.

$P^h_k = \frac{\partial P}{\partial a^k_h}$.

$n = 4m-1$.

$C^{ihj}_P(g,\partial g,\partial^2 g)\, dx^1 \wedge dx^2 \dots dx^n$
$= g^{ik}\, P^h_k(\Omega)\, dx^j$
$\quad 4m-2 \quad + 1 \quad 4m-1$

order 3

$C^{ij}_P(g,\partial g,\partial^2 g,\partial^3 g) = (C^{ihj} + C^{jhi})_{|h}$

Generalized Cotton Tensors

- C^{ij}_P third order
- trace-free $C^{ij}_P g_{ij} = 0$
- div. free $C^{ij}_P{}_{|j} = 0$
- locally variational.

$C^{ij} = \frac{\delta L_{c.s}}{\delta g_{ij}}$ require add fields (Frame Bund)

- no invariant Lagrangian
- $n=3$. g conf flat iff $C^{ij}_P = 0$ (York-tensor).

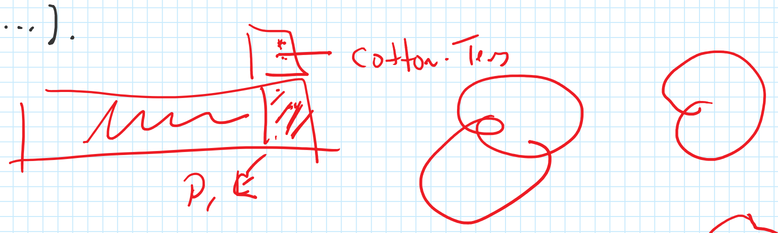
Let $A^{ij}(g,\partial g,\partial^2 g \dots)$ be natural and $Helm(A) = 0$.

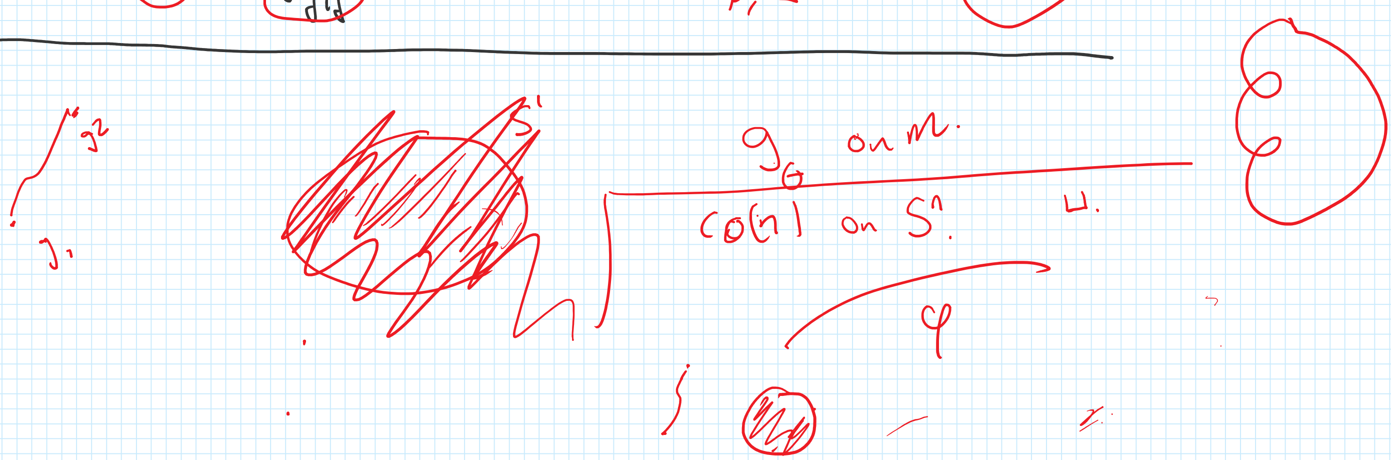
For $n \neq 4m-1$

$A^{ij} = \frac{\delta L}{\delta g_{ij}}$, $L = L(g, R, \nabla R, \dots)$.

For $n = 4m-1$,

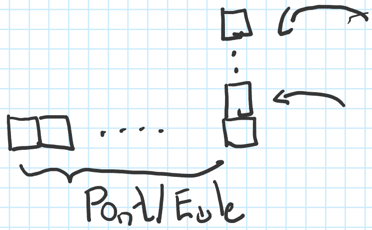
$A^{ij} = C^{ij}_P + \frac{\delta L}{\delta g_{ij}}$





Questions

1. $C_p^{ij} = 0$ on M^{4m-1}

2. 

? generalized cotton tensors.

3. ~~Skip~~

4. $H^*(\mathcal{E}^*(M)_G) = H^*(G.F, so(n))$. ?

5. Natural V.B for g, X ?

g, X, Y ?

g, Killing vector ?

g, ω. ?

Conclusions

- Many interesting/important results can be intrepreted in terms of $\Omega^{*,*}_{G}(\mathbb{R}^{\infty})$.
- Powerful variational calculus.
- Detailed computations can be difficult.

- Thank-you.
- Hope to one day met in person in "City of Music" \ "City of Dreams".