Topological recursion, discrete surfaces and cohomological field theories

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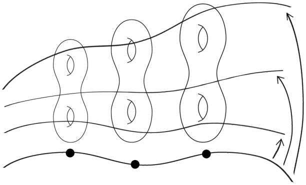
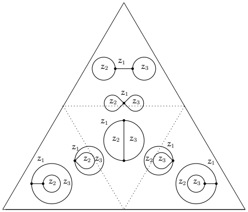
Workshop: Higher Structures Emerging from Renormalisation





Erwin Schrödinger International Institute for Mathematics and Physics
Vienna, 17th of November, 2021

Outline

1. Topological recursion (TR)
2. Witten's conjecture, Kontsevich's theorem
3. Cohomological field theories (CohFT)
4. Witten's r-spin class and the r-KdV hierarchy
   (based on joint work with R. Belliard, S. Charbonnier and B. Eynard, arXiv:2105.08035)
   - Combinatorial model: Generalised Kontsevich graphs
   - Tutte's recursion
   - Topological recursion for ciliated maps
   - From graphs to intersection numbers with Witten's class
5. Further consequences: ongoing and future

Topological recursion (TR, Chekhov–Eynard–Orantin '04-'07)

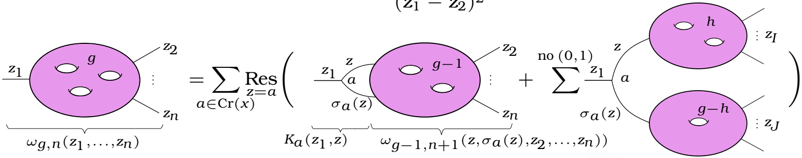
Goal: "Count surfaces $S_{g,n}$ of genus g with n boundaries (topology (g,n))."

Spectral curve

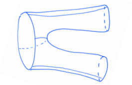
TR : $\begin{cases} \Sigma \text{ Riemann surface} \\ x\colon \Sigma \to \mathbb{C}P^1 \\ \omega_{0,1} = y\,dx \ \text{1-form (discs)} \\ \omega_{0,2} \ \ (1,1)\text{-form (cylinders)} \end{cases}$ $\rightsquigarrow$ **recursion on** $|\chi(S_{g,n})| = 2g-2+n$ $\rightsquigarrow$ Differential forms $\omega_{g,n}(z_1,\dots,z_n)$, $z_i \in \Sigma$, $\forall g,n \geq 0$.

- x finitely many simple ramification points ($\mathrm{Cr}(x)$) and y holomorphic around $a \in \mathrm{Cr}(x)$ and $dy(a) \neq 0 \Rightarrow$ Local involution σ around every ramification point: $x(z) = x(\sigma(z))$.
- $\omega_{0,2}$ symmetric bi-differential on $\Sigma \times \Sigma$ with only double poles along the diagonal and vanishing residues, that is when $z_1 \to z_2$

$$\omega_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2} + \overbrace{h(z_1,z_2)}^{\text{holomorphic}}.$$



- Terms in correspondence with the ways of cutting a **pair of pants** $(0,3)$ from $S_{g,n}$.



Properties, connections and examples

- Interesting/powerful properties: $\omega_{g,n}$ are symmetric with poles at ramifications points, controlled deformations along families, dilaton equation, symplectic invariance, modularity, integrability...

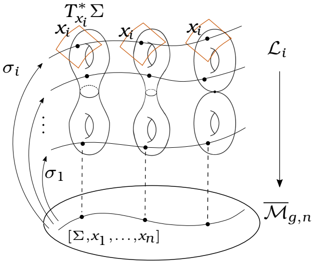
- For the Lambert curve $x = ye^{-y}$, TR provides simple Hurwitz numbers (Eynard–Mulase–Safnuk, '09, arXiv:0907.5224).
- For $y = \frac{-\sin(2\pi\sqrt{x})}{2\pi}$, TR gives Mirzakhani's recursion for Weil–Petersson volumes (of the moduli space of bordered hyperbolic surfaces), (Eynard–Orantin, '07, arXiv:0705.3600).
- TR on mirror curve of a toric CY3 computes its open Gromov–Witten theory (Bouchard–Klemm–Mariño–Pasquetti, '07, arXiv:0709.1453), (Fang–Liu–Zong, '16, arXiv:1604.07123).
- Chern–Simons theory on S^3 is governed by TR. Gopakumar–Ooguri–Vafa correspondence gives an A-model picture: GW of the resolved conifold, and B-model can be seen as TR on its Hori–Iqbal–Vafa mirror curve. (Brini, '17, hal-01474196).
- Statistical physics models on random maps: 1-hermitian matrix model, Ising model, Potts model, O(n)-loop model (Borot–Eynard, '09, arXiv:0910.5896), (Borot–Eynard–Orantin, '13, arXiv:1303.5808)...
- From modular functors to cohomological field theories to topological recursion (Andersen–Borot–Orantin, '15, arXiv:1509.01387).
- Reconstruction of formal WKB expansions, integrability, isomonodromic systems (Borot–Eynard, '11, arXiv:1110.4936), (Eynard, '17, arXiv:1706.04938), (Eynard–G-F–Marchal–Orantin, '21, arXiv:2106.04339)...
- Conjecturally, for the A-polynomial of a knot as a spectral curve, TR computes the colored Jones polynomial of the knot (Borot–Eynard, '12, arXiv:1205.2261)).
- Extension to the non-perturbative world, resurgence theory: work in progress!

Moduli space of curves $\mathcal{M}_{g,n}$

For $g, n \geq 0$, with $2g - 2 + n > 0$, we define the moduli space:

$$\mathcal{M}_{g,n} := \left\{ \begin{array}{l} \text{curves of genus } g \text{ with } n \\ \text{marked points } x_1, \ldots, x_n \end{array} \right\} \Big/ \sim \ .$$

- $\overline{\mathcal{M}}_{g,n} \rightsquigarrow$ Deligne–Mumford compactification (including nodal curves).



$$\psi_i := c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}),$$
$$i = 1, \ldots, n.$$

Intersection numbers or correlators of psi classes:

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \in \mathbb{Q},$$

which are zero unless $\sum_{i=1}^n d_i = \dim_{\mathbb{C}}(\overline{\mathcal{M}}_{g,n}) = 3g - 3 + n$.

Witten's conjecture

Generating series of intersection numbers of psi classes:

$$F(t_0, t_1, \ldots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(d_1,\ldots,d_n)} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle t_{d_1} \cdots t_{d_n}$$

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Conjecture: The series F satisfies the Korteweg–de Vries (KdV) hierarchy, the first equation of which is the classical KdV equation

$$\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3} \quad \left(U = \frac{\partial^2 F}{\partial t_0^2} \right),$$

and the string equation $\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i}$.

- Witten's motivation: Two different models of 2D quantum gravity should coincide.
- The conjecture uniquely determines F.

One explicit version of Witten's conjecture

Virasoro operators:

$$V_{-1} = -\frac{1}{2}\frac{\partial}{\partial t_0} + \frac{1}{2}\sum_{k=0}^{\infty} t_{k+1}\frac{\partial}{\partial t_k} + \frac{t_0^2}{4}, \quad V_0 = -\frac{3}{2}\frac{\partial}{\partial t_1} + \frac{1}{2}\sum_{k=0}^{\infty}(2k+1)t_k\frac{\partial}{\partial t_k} + \frac{1}{48},$$

and for $n > 0$,

$$V_n = -\frac{(2n+3)!!}{2}\frac{\partial}{\partial t_{n+1}} + \sum_{k=0}^{\infty}\frac{(2k+2n+1)!!}{2(2k-1)!!}t_k\frac{\partial}{\partial t_{k+n}} + \sum_{k_1+k_2=n-1}\frac{(2k_1+1)!!(2k_2+1)!!}{4}\frac{\partial^2}{\partial t_{k_1}\partial t_{k_2}}.$$

They satisfy the Virasoro relations:

$$[V_m, V_n] = (m-n)V_{m+n}.$$

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Theorem (equivalent to Witten's conjecture ('91))

For every integer $n \geq -1$, $V_n(\exp F) = 0$.

Witten's conjecture ⇝ Kontsevich's theorem

1. Kontsevich maps and matrix model

TR ('07)

2. Intersection numbers
$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$$

Witten's conjecture, '90

3. Integrable hierarchy KdV

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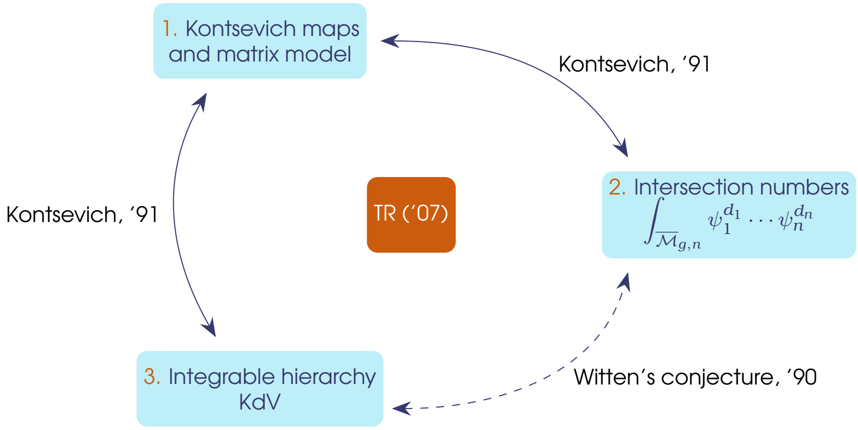

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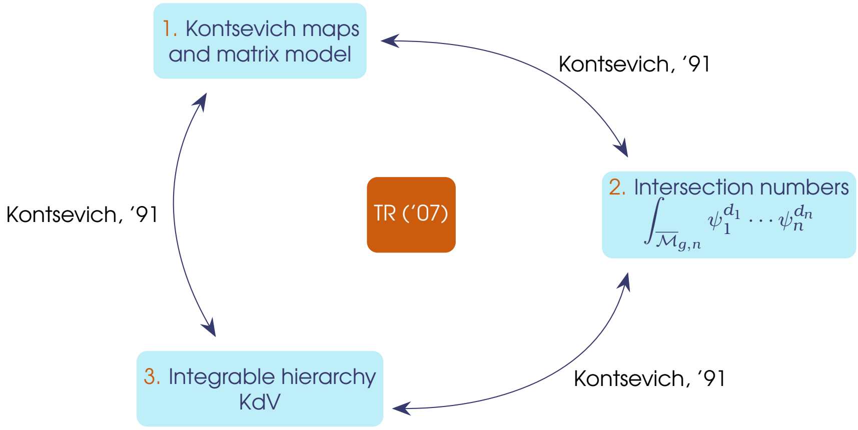

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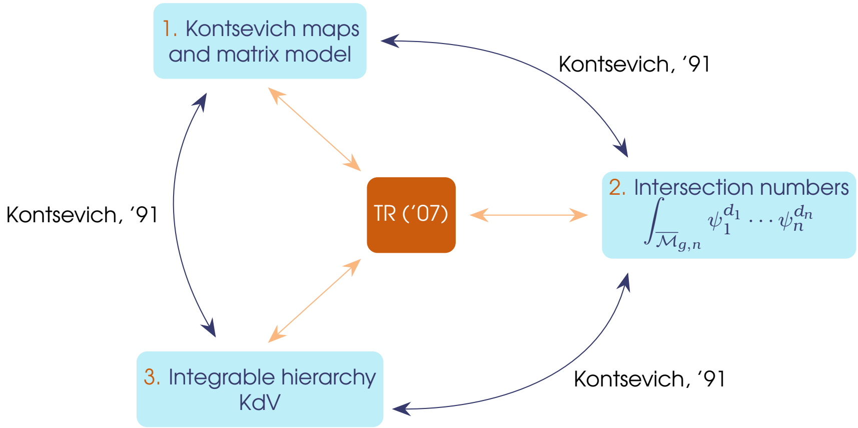

TR applied to the Airy curve $(x, y) = (\frac{z^2}{2}, z)$ produces

$$\omega_{g,n}(z_1, \ldots, z_n) = 2^{2-2g-n} \sum_{\sum_i d_i = 3g-3+n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) \prod_{i=1}^{n} \frac{(2d_i+1)!! dz_i}{z_i^{2d_i+2}}.$$

Cohomological field theories

Definition (cohomological field theory (CohFT))

V vector space with a nondegenerate symmetric bilinear form η. A CohFT $\{\Omega_{g,n}\}_{2g-2+n>0}$ over (V,η) is a collection of $\mathfrak{S}_n$-invariant morphisms

$$\Omega_{g,n}\colon V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}) \quad \text{such that}$$

given the gluing maps

$$q\colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n},$$
$$r\colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}, \quad g_1+g_2=g,\ n_1+n_2=n,$$

we have

$$q^*\Omega_{g,n}(v_1\otimes\cdots\otimes v_n) = \Omega_{g-1,n+2}(v_1\otimes\cdots\otimes v_n\otimes\eta^\dagger),$$
$$r^*\Omega_{g,n}(v_1\otimes\cdots\otimes v_n) = (\Omega_{g_1,n_1+1}\times\Omega_{g_2,n_2+1})\Big(\bigotimes_{i=1}^{n_1} v_i\otimes\eta^\dagger\otimes\bigotimes_{j=1}^{n_2} v_{n_1+j}\Big),$$

where $\eta^\dagger \in V^{\otimes 2}$ is the bivector dual to η.

Correlators: With $\sum_{i=1}^n d_i \leq \dim_{\mathbb{C}}(\overline{\mathcal{M}}_{g,n}) = 3g-3+n$,

$$\langle \tau_{d_1}(v_1)\cdots\tau_{d_n}(v_n)\rangle_g^{\Omega} := \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(v_1\otimes\cdots\otimes v_n)\prod_{i=1}^n \psi_i^{d_i}.$$

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Examples: $V=\mathbb{Q}$, $\eta(1,1)=1$. Then $\Omega_{g,n}=\Omega_{g,n}(1^{\otimes n})$.

- Trivial CohFT $\Omega_{g,n}=1 \rightsquigarrow$ Witten–Kontsevich intersection numbers.
- $\Omega_{g,n}=\exp(2\pi^2\kappa_1)$, with $\kappa_m := \pi_*(\psi_{n+1}^{m+1}) \in H^{2m}(\overline{\mathcal{M}}_{g,n}) \rightsquigarrow$ Weil–Petersson volumes (hyperbolic geometry).

- A CohFT defines a quantum product $\star$ on V by

$$\eta(v_1 \star v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3).$$

Commutative (by $\mathfrak{S}_n$-invariance) and associative (by the last two axioms).

Semi-simplicity, classification and Witten's class

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Theorem (Givental–Teleman classification, Teleman '12)

Let Ω be a semi-simple CohFT with flat unit and ω the associated TFT (degree 0 part). Then there exists a unique R-matrix such that

$$\Omega = R.\omega.$$

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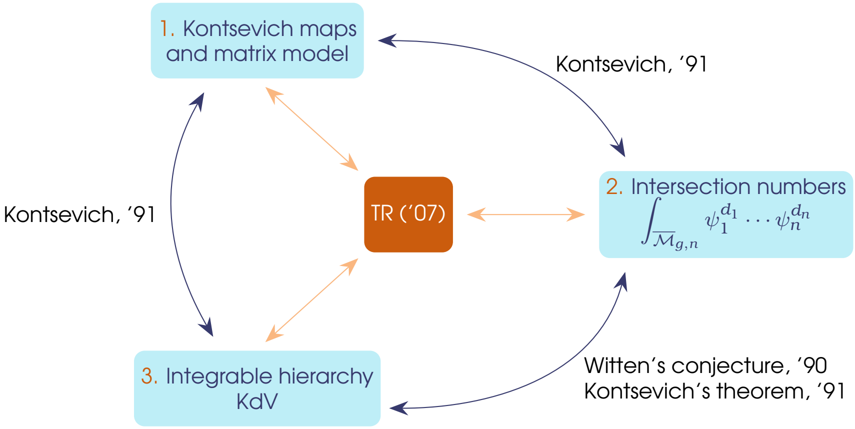
Example (non semi-simple)

$V = \langle e_0, e_1, \ldots, e_{r-2}\rangle_{\mathbb{Q}}$, $\eta(e_a, e_b) = \delta_{a+b,r-2}$. Witten's r-spin CohFT:

$$c_W^r(a_1, \ldots, a_n) = \Omega_{g,n}(e_{a_1}, \ldots, e_{a_n}),$$

of degree $\frac{(r-2)(g-1)+\sum_{i=1}^n a_i}{r}$, with $a_1, \ldots, a_n \in \{0, \ldots, r-2\}$.

Witten's conjecture ⇝ Kontsevich's theorem



Theorem (Eynard '11, Dunin-Barkowski–Orantin–Shadrin–Spitz '14)

TR for spectral curves with simple ramification points $\leftrightarrow$ *Semi-simple CohFTs.*

r-spin Witten conjecture, '93 (proved by Faber–Shadrin–Zvonkine)

1. Generalised Kontsevich maps and matrix model

Higher TR ('13)

2. Intersection numbers
$$\int_{\overline{\mathcal{M}}_{g,n}} c_W^r(a_1, \ldots, a_n)\psi_1^{d_1} \cdots \psi_n^{d_n}$$

Witten, '93

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Kontsevich $r = 2$ (Mirzakhani, . . .)
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Can we complete the picture in the general r case? Combinatorial side?

Definition

A *map* of genus g with n *boundaries* is an embedding of a graph Γ into an oriented surface X of genus g such that

$$X \setminus \Gamma \cong \bigsqcup \mathbb{D} \quad \text{(\textit{faces}), with } n \text{ marked faces (\textit{boundaries}).}$$

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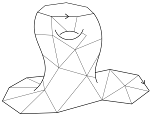

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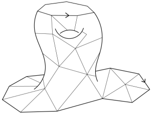

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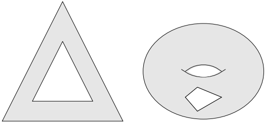

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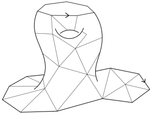

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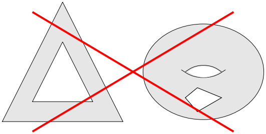

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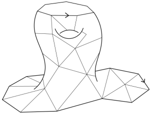

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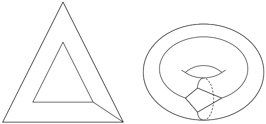

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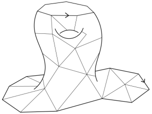
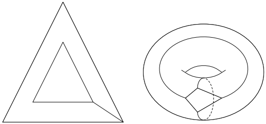
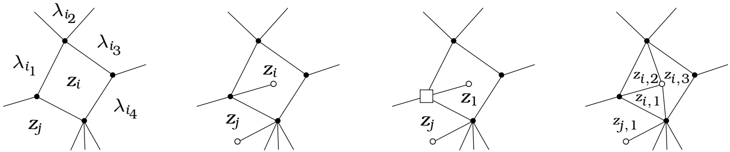


Topology $(g, n) = (1, 2$ boundaries).



(Ciliated) Kontsevich maps: Degree of black vertices $v \rightsquigarrow 3 \leq d_v \leq r + 1$.
Maximum one white vertex per boundary. $\{\lambda_1, \ldots, \lambda_N\} \rightsquigarrow$ internal faces.

$\mathcal{F}^{[r]}_{g,n}(z_1, \ldots, z_n)$ $\quad$ $\mathcal{W}^{[r]}_{g,n}(z_1, \ldots, z_n)$ $\quad$ $\mathcal{U}^{[r]}_{g,n}(z_1, \ldots, z_n)$ $\quad$ $\mathcal{S}^{[r]}_{g,(k_1,\ldots,k_n)}(S_1, \ldots, S_n)$



Generalised Kontsevich graphs

Definition

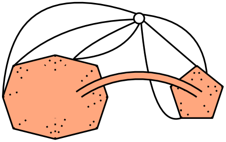
A *map* of genus g with n *boundaries* is an embedding of a graph Γ into an oriented surface X of genus g such that

$$X \setminus \Gamma \cong \bigsqcup \mathbb{D} \quad \text{(\textit{faces}), with } n \text{ marked faces (\textit{boundaries}).}$$

$/\sim$

White vertices $\rightsquigarrow$ star constraint.

No star constraint $\rightsquigarrow$



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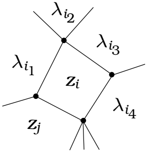
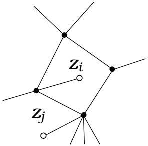
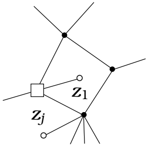
$\mathcal{F}^{[r]}_{g,n}(z_1, \ldots, z_n)$



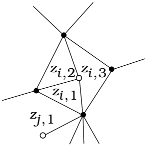
$\mathcal{W}^{[r]}_{g,n}(z_1, \ldots, z_n)$



$\mathcal{U}^{[r]}_{g,n}(z_1, \ldots, z_n)$



$\mathcal{S}^{[r]}_{g,(k_1,\ldots,k_n)}(S_1, \ldots, S_n)$



Map degrees and local weights

- Degree: $\deg G = (r+1)\left(\#\mathcal{E}(G) - \#\mathcal{V}(G)\right) = (r+1)\left(\#\mathcal{F}(G) - 2 + 2g(G)\right)$.

Fixed a degree $\delta = \deg G/(r+1)$ and a topology (g,n), the sets

$$\mathcal{F}_{g,n}^{[r],\delta}(z_1,\ldots,z_n),\ \mathcal{W}_{g,n}^{[r],\delta}(z_1,\ldots,z_n),\ \mathcal{U}_{g,n}^{[r],\delta}(u;z_1,\ldots,z_n)\ \text{ and }\ \mathcal{S}_{g,\underline{k}}^{[r],\delta}(S_1,\ldots,S_n)$$

are finite.

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The potential of the model is a polynomial $V \in \mathbb{C}[z]$ of degree $r+1$:

$$V(z) = \sum_{j=1}^{r+1} \frac{v_j}{j}\, z^j.$$

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$$V(z) = \sum_{j=1}^{r+1} \frac{v_j}{j} z^j.$$

With $a_i \in \{\lambda_1,\ldots,\lambda_N\} \cup \{z_1,\ldots,z_n\}$, we define the weight per:

- **Edge** bounding faces decorated by a_1, a_2

$$\mathcal{P}(a_1,a_2) := \frac{a_1 - a_2}{V'(a_1) - V'(a_2)},$$

  and $\mathcal{P}(a_1,a_1) = \lim_{a_2 \to a_1} \mathcal{P}(a_1,a_2) = \frac{1}{V''(a_1)}$.

- **Black vertex** of degree $3 \le d \le r+1$ adjacent to faces decorated with $a_1,\ldots,a_d$

$$\mathcal{V}_d(a_1,\ldots,a_d) := \sum_{i=1}^{d} \frac{-V'(a_i)}{\prod_{j\neq i}(a_i - a_j)}.$$

- **White vertex**: 1.

Generating series

Weight of a map G:

$$w(G) = \prod_{\substack{e \in \mathcal{E}(G) \\ e=(f_1,f_2)}} \mathcal{P}(a_{f_1}, a_{f_2}) \prod_{\substack{v \in \mathcal{V}(G) \\ \text{black}}} \mathcal{V}_{d_v}(\{a_f\}_{f \mapsto v}) \prod_{\substack{v \in \mathcal{V}(G) \\ \text{square}}} \prod_{f \mapsto v} \frac{1}{u - a_f}.$$

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Generating series of unciliated maps of topology (g, n):

$$F^{[r]}_{g,n}(z_1, \ldots, z_n; \lambda; v_j; \alpha) = \sum_{G \in \mathcal{F}^{[r]}_{g,n}(z_1,\ldots,z_n)} \frac{w(G)}{\#\mathrm{Aut}\, G} \alpha^{-\deg G}$$

$$= \sum_{\delta \geq (2g+n-2)} \alpha^{-(r+1)\delta} \sum_{G \in \mathcal{F}^{[r],\delta}_{g,n}(z_1,\ldots,z_n)} \frac{w(G)}{\#\mathrm{Aut}\, G} \in \mathbb{Q}[\{z_i^{-1}, \lambda_j^{-1}, v_k\}][[\alpha^{-1}]],$$

$i \in [\![1, n]\!]$, $j \in [\![1, N]\!]$, $k \in [\![1, r+1]\!]$.

- Analogously:

$$W^{[r]}_{g,n}(z_1, \ldots, z_n; \lambda; v_j; \alpha),\ U^{[r]}_{g,n}(z_1, \ldots, z_n; \lambda; v_j; \alpha),\ S^{[r]}_{g,\underline{k}}(S_1, \ldots, S_n; \lambda; v_j; \alpha).$$

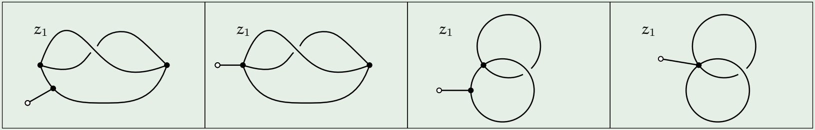
Torus with one boundary

Weight of a map G:

$$w(G) = \prod_{\substack{e\in\mathcal{E}(G)\\ e=(f_1,f_2)}} \mathcal{P}(a_{f_1}, a_{f_2}) \prod_{\substack{v\in\mathcal{V}(G)\\ \text{black}}} \mathcal{V}_{d_v}(\{a_f\}_{f\mapsto v}) \prod_{\substack{v\in\mathcal{V}(G)\\ \text{square}}} \prod_{f\mapsto v} \frac{1}{u-a_f}.$$

Example (topology $(1,1)$ and case $\lambda_j = \infty$, i.e. without internal faces:)

$\deg G = (r+1)(2g-2+n) = r+1 \rightsquigarrow \mathcal{W}_{1,1}^{[r]}(z_1)$ has 4 graphs:



$$W_{1,1}^{[r]}(z_1) = \alpha^{-(r+1)} \sum_{G\in\mathcal{W}_{1,1}^{[r],1}(z_1)} \frac{w(G)}{\#\mathrm{Aut}\,G} = \alpha^{-(r+1)}\Big[\mathcal{P}(z_1,z_1)^5\mathcal{V}_3(z_1,z_1,z_1)^3$$

$$+ 2\mathcal{P}(z_1,z_1)^4\mathcal{V}_3(z_1,z_1,z_1)\mathcal{V}_4(z_1,z_1,z_1,z_1) + \mathcal{P}(z_1,z_1)^3\mathcal{V}_5(z_1,z_1,z_1,z_1,z_1)\Big]$$

$$= \alpha^{-(r+1)}\left[\frac{1}{6}\frac{V^{(3)}(z_1)V^{(4)}(z_1)}{V''(z_1)^4} - \frac{1}{8}\frac{V^{(3)}(z_1)^3}{V''(z_1)^5} - \frac{1}{24}\frac{V^{(5)}(z_1)}{V''(z_1)^3}\right], \quad \mathcal{V}_m(z,\ldots,z) = \frac{-V^{(m)}(z)}{(m-1)!}.$$

Theorem

$$W_{g,n}^{[r]}(z_1,\dots,z_n)=\frac{1}{V''(z_1)}\frac{\partial}{\partial z_1}\cdots\frac{1}{V''(z_n)}\frac{\partial}{\partial z_n}F_{g,n}^{[r]}(z_1,\dots,z_n)$$
$$+\delta_{g,0}\delta_{n,2}\left(\frac{1}{V''(z_1)V''(z_2)(z_1-z_2)^2}-\frac{1}{(V'(z_1)-V'(z_2))^2}\right)$$
$$+\delta_{g,0}\delta_{n,1}\sum_{j=1}^{N}\left(\frac{1}{V''(z_1)(z_1-\lambda_j)}-\frac{1}{(V'(z_1)-V'(\lambda_j))}\right).$$

For $(g,n)\neq(0,1)$:

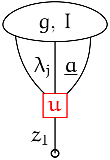
$$S_{g;\underline{k}}^{[r]}(S_1,\dots,S_n)=\sum_{j_1=1}^{k_1}\cdots\sum_{j_n=1}^{k_n}\frac{W_{g,n}^{[r]}(z_{1,j_1},\dots,z_{n,j_n})}{\prod_{m=1}^{n}\alpha^{k_m(r+1)}\prod_{\substack{i_m=1\\ i_m\neq j_m}}^{k_m}\left(V'(z_{m,i_m})-V'(z_{m,j_m})\right)}.$$

$$-\operatorname*{Res}_{u=\infty}du\;V'(u)\,(u-z_1)U_{g,n}^{[r]}(u;z_1,\dots,z_n)=\frac{V'(z_1)}{V''(z_1)}W_{g,n}^{[r]}(z_1,\dots,z_n)$$
$$+\delta_{g,0}\delta_{n,1}\left(\frac{N}{V''(z_1)}\right).$$

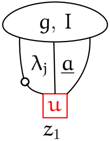
Idea: Erase the first cilium from $G \in \mathcal{U}^{[r]}_{g,n}(u; z_1, \ldots, z_n)$ and introduce a bivalent white vertex on the following edge around the square vertex in the clockwise direction $\rightsquigarrow$ Recursion for $U^{[r]}_{g,n}$ and $W^{[r]}_{g,n}$ (à la Tutte).

$(g, n) \neq (0, 1) \rightsquigarrow$ 4 cases. $I = \{z_2, \ldots, z_n\}$, $I_j = I \backslash \{z_j\}$, $J \sqcup J' = I$, $h + h' = g$.

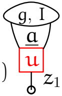
1 Following edge is adjacent to a face decorated with $\lambda_j, j \in \{1, \ldots, N\}$:

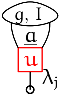


$$= \frac{\mathcal{P}(z_1,z_1)}{\mathcal{P}(z_1,\lambda_j)} \times \frac{1}{u-z_1} \times$$



$$= \frac{1}{(u-z_1)\alpha^{r+1}} \frac{1}{V''(z_1)} \frac{V''(z_1)\;[\text{diagram}]\; - V''(\lambda_j)\;[\text{diagram}]}{V'(z_1) - V'(\lambda_j)}$$





Contribution:

$$\frac{\alpha^{-(r+1)}}{u - z_1} \frac{1}{V''(z_1)} \sum_{j=1}^{N} \frac{V''(z_1) U^{[r]}_{g,n}(u; z_1, I) - V''(\lambda_j) U^{[r]}_{g,n}(u; \lambda_j, I)}{V'(z_1) - V'(\lambda_j)}.$$

Tutte's recursion

Idea: Erase the first cilium from $G \in \mathcal{U}^{[r]}_{g,n}(u; z_1, \dots, z_n)$ and introduce a bi-valent white vertex on the following edge around the square vertex in the clockwise direction $\rightsquigarrow$ Recursion for $U^{[r]}_{g,n}$ and $W^{[r]}_{g,n}$ (à la Tutte).

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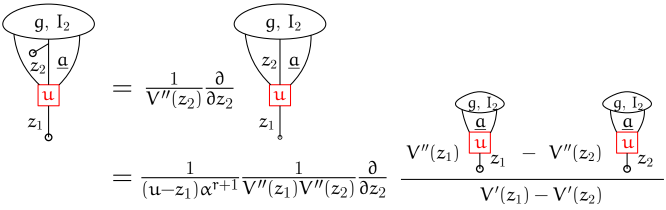
2 Following edge is adjacent to a face decorated with z_m, $m \in \{2, \dots, n\}$:



Contribution:

$$\frac{\alpha^{-(r+1)}}{u - z_1} \frac{1}{V''(z_1)} \sum_{m=2}^{n} \frac{1}{V''(z_m)} \frac{\partial}{\partial z_m} \frac{V''(z_1) U^{[r]}_{g,n-1}(u; z_1, I_m) - V''(z_m) U^{[r]}_{g,n-1}(u; z_m, I_m)}{V'(z_1) - V'(z_m)}.$$

Idea: Erase the first cilium from $G \in \mathcal{U}_{g,n}^{[r]}(u; z_1, \ldots, z_n)$ and introduce a bivalent white vertex on the following edge around the square vertex in the clockwise direction $\rightsquigarrow$ Recursion for $U_{g,n}^{[r]}$ and $W_{g,n}^{[r]}$ (à la Tutte).

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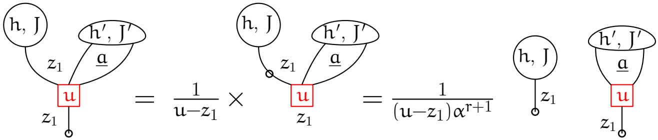
3. Following edge is adjacent to the first marked face:



Contribution:

$$\frac{\alpha^{-(r+1)}}{u - z_1} \sum_{\substack{h+h'=g \\ J \sqcup J' = I}} W_{h,1+\#J}^{[r]}(z_1, J) U_{h',1+\#J'}^{[r]}(u; z_1, J').$$

Idea: Erase the first cilium from $G \in \mathcal{U}^{[r]}_{g,n}(u; z_1, \ldots, z_n)$ and introduce a bivalent white vertex on the following edge around the square vertex in the clockwise direction $\rightsquigarrow$ Recursion for $U^{[r]}_{g,n}$ and $W^{[r]}_{g,n}$ (à la Tutte).

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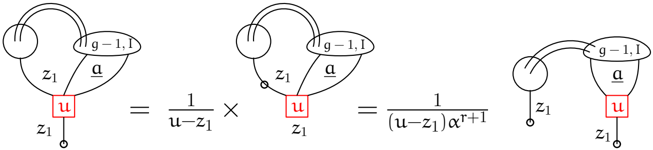

Contribution:

$$\frac{\alpha^{-(r+1)}}{u - z_1} U^{[r]}_{g-1,n+1}(u; z_1, z_1, I).$$

Tutte's equation and spectral curve

Tutte's equation $\rightsquigarrow$ recursive relation on $2g+n+\delta$. For $(g,n) \neq (0,1)$:

$$U^{[r]}_{g,n}(u;z_1,I) = \frac{\alpha^{-(r+1)}}{u-z_1}\Bigg(\frac{1}{V''(z_1)}\sum_{j=1}^{N}\frac{V''(z_1)U^{[r]}_{g,n}(u;z_1,I)-V''(\lambda_j)U^{[r]}_{g,n}(u;\lambda_j,I)}{V'(z_1)-V(\lambda_j)}$$

$$+\frac{1}{V''(z_1)}\sum_{m=2}^{n}\frac{1}{V''(z_m)}\frac{\partial}{\partial z_m}\frac{V''(z_1)U^{[r]}_{g,n-1}(u;z_1,I_m)-V''(z_m)U^{[r]}_{g,n-1}(u;z_m,I_m)}{V'(z_1)-V(z_m)}$$

$$+\sum_{\substack{h+h'=g\\ J\sqcup J'=I}} W^{[r]}_{h,1+J}(z_1,J)\,U^{[r]}_{h',1+J'}(u;z_1,J')+U^{[r]}_{g-1,n+1}(u;z_1,z_1,I)\Bigg).$$

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Towards the **spectral curve**:

$$x(z) := V'(z), \quad y(z) := z + \alpha^{-(r+1)}W^{[r]}_{0,1}(z) + \alpha^{-(r+1)}\sum_{j=1}^{N}\frac{1}{V'(z_1)-V'(\lambda_j)}.$$

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Theorem

$\exists$ polynomial Q of degree r, such that if ζ is the implicit function defined by

$$Q(\zeta)=x(z),\quad \zeta \underset{z\to\infty}{=} z+\mathcal{O}(1),\quad \text{then}\quad y(\zeta)=\zeta+\alpha^{-(r+1)}\sum_{j=1}^{N}\frac{1}{Q'(\xi_j)(\zeta-\xi_j)},$$

where $Q(\xi_i)=V'(\lambda_i)$. Q is a formal power series in $\alpha^{-(r+1)}$ and determined by:

$$V'(y(\zeta))-Q(\zeta)\underset{\zeta\to\infty}{=}\mathcal{O}(1/\zeta).$$

Proof of TR for ciliated maps

1. Recursion for $U_{g,n}^{[r]}$ and $W_{g,n}^{[r]}$ (à la Tutte).

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1. Recursion for $U_{g,n}^{[r]}$ and $W_{g,n}^{[r]}$ (à la Tutte).
2. Base topologies $(0,1)$ and $(0,2)$ give us the spectral curve.

$$\mathcal{S}: \begin{cases} x(\zeta) = Q(\zeta), \text{ with } Q(\xi_i) = V'(\lambda_i), \\ y(\zeta) = \zeta + \alpha^{-(r+1)} \sum_{i=1}^{N} \frac{1}{Q'(\xi_i)(\zeta - \xi_i)}, \\ \omega_{0,1}^{[r]}(\zeta) = \alpha^{r+1} y(\zeta) \mathrm{d}x(\zeta), \\ \omega_{0,2}^{[r]}(\zeta_1, \zeta_2) = \frac{\mathrm{d}\zeta_1 \, \mathrm{d}\zeta_2}{(\zeta_1 - \zeta_2)^2}. \end{cases}$$

Proof of TR for ciliated maps

1. Recursion for $U_{g,n}^{[r]}$ and $W_{g,n}^{[r]}$ (à la Tutte).
2. Base topologies $(0,1)$ and $(0,2)$ give us the spectral curve.
3. Combinatorial interpretation of certain universal expressions:

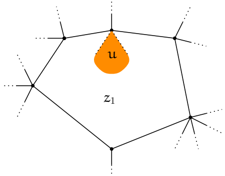
$$\check{H}_{g,n}^{[r]}(u;\zeta_1,I) := v_{r+1}\sum_{k=0}^{r-1}(-1)^k u^{r-1-k}\alpha^{-(k-1)(r+1)}\sum_{\underline{t}\subseteq_k (x^{-1}(x(\zeta_1))\setminus\{\zeta_1\})}\mathcal{E}^{(k)}W_{g,n}^{[r]}(\underline{t};I),$$

$$\check{P}_{g,n}^{[r]}(u;\zeta_1,I) := v_{r+1}\sum_{k=0}^{r}(-1)^k u^{r-k}\alpha^{-(k-1)(r+1)}\sum_{\underline{t}\subseteq_k x^{-1}(x(\zeta_1))}\mathcal{E}^{(k)}W_{g,n}^{[r]}(\underline{t};I),$$

where $I=\{\zeta_2,\ldots,\zeta_n\}$ and

$$\mathcal{E}^{(k)}W_{g,n}^{[r]}(\underline{t};I) := \sum_{\mu\in\mathcal{S}(\underline{t})}\sum_{\bigsqcup_{i=1}^{\ell(\mu)}J_i=I}\sum_{\sum_{i=1}^{\ell(\mu)}g_i=h+\ell(\mu)-k}\left[\prod_{i=1}^{l(\mu)}\widetilde{W}_{g_i,|\mu_i|+|J_i|}^{[r]}(\mu_i,J_i)\right].$$

$$H_{g,n}^{[r]}(u;\zeta_1,I) := V''(\zeta_1)\left[V'(u)U_{g,n}^{[r]}(u;I)\right]_+$$
$$= \check{H}_{g,n}^{[r]}(u;\zeta_1,I).$$



Proof of TR for ciliated maps

1. Recursion for $U_{g,n}^{[r]}$ and $W_{g,n}^{[r]}$ (à la Tutte).
2. Base topologies $(0, 1)$ and $(0, 2)$ give us the spectral curve.
3. Combinatorial interpretation of certain universal expressions (for a large class of spectral curves).
4. Analytic properties: polar structure of $W_{g,n}^{[r]}(z_1, \ldots, z_n)$.
5. 3 ⇒ Loop equations. $I = \{\zeta_2, \ldots, \zeta_n\}$. $Q(\zeta)$ polynomial of degree r, so the equation $Q(\zeta) = Q(\zeta_0)$ has r solutions denoted $\zeta_0 = \zeta_0^{(0)}, \zeta_0^{(1)}, \ldots, \zeta_0^{(r-1)}$.

Linear:

$$\sum_{k=0}^{r-1} \omega_{g,n}^{[r]} \left(\zeta_1^{(k)}, I\right) = \delta_{g,0}\delta_{n,1}\left(- \frac{v_r \alpha^{r+1}}{v_{r+1}} + \sum_{j=1}^{N} \frac{1}{x(\zeta_1) - x(\xi_j)}\right) \mathrm{d}x(\zeta_1)$$
$$+ \delta_{g,0}\delta_{n,2} \frac{\mathrm{d}x(\zeta_1)\,\mathrm{d}x(\zeta_2)}{(x(\zeta_1) - x(\zeta_2))^2}.$$

Quadratic:

$$\sum_{k=0}^{r-1} \left[\omega_{g-1,n+1}^{[r]} \left(\zeta_1^{(k)}, \zeta_1^{(k)}, I\right) + \sum_{\substack{h+h'=g \\ J \sqcup J' = I}} \omega_{h,1+\#J}^{[r]} \left(\zeta_1^{(k)}, J\right) \omega_{h',1+\#J'}^{[r]} \left(\zeta_1^{(k)}, J'\right) \right]$$

is a differential in $x(\zeta_1)$ without poles at the ramification points of x.

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6. 2, 4 and 5 $\Rightarrow$ **Topological recursion**

$$\omega_{g,n}^{[r]}(\zeta_1, \ldots, \zeta_n) = W_{g,n}^{[r]}(z_1, \ldots, z_n)\mathrm{d}x(\zeta_1) \cdots \mathrm{d}x(\zeta_n).$$

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---

7. Consider the family of spectral curves with $V'_\varepsilon(z) = z^r - r\varepsilon^{-r-1}z$, which for $\varepsilon \neq 0$ have $r-1$ simple ramification points. Take the limit $\varepsilon \to 0$ and obtain
   - **topological recursion** (admitting ramification points of higher order) for $\omega^{[r],0}_{g,n}$ with spectral curve with $V'_0(z) = z^r$ (with one ramification point of order $r-1$);
   - $\lim_{\varepsilon\to 0}\omega^{[r],\varepsilon}_{g,n}(\zeta_1,I) = \omega^{[r],0}_{g,n}(\zeta_1,I)$.

Witten's conjecture r-spin, '93 (proved by Faber–Shadrin–Zvonkine)

1. Generalized maps and matrix model

Higher TR ('13)

2. Intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} c_W^r(a_1, \ldots, a_n)\psi_1^{d_1} \cdots \psi_n^{d_n}$$

3. Hierarchy r-KdV

Faber–Shadrin–Zvonkine, '10

Generalized Kontsevich maps and TR

1. Generalized maps and matrix model

Belliard–Charbonnier–Eynard–G-F, '21

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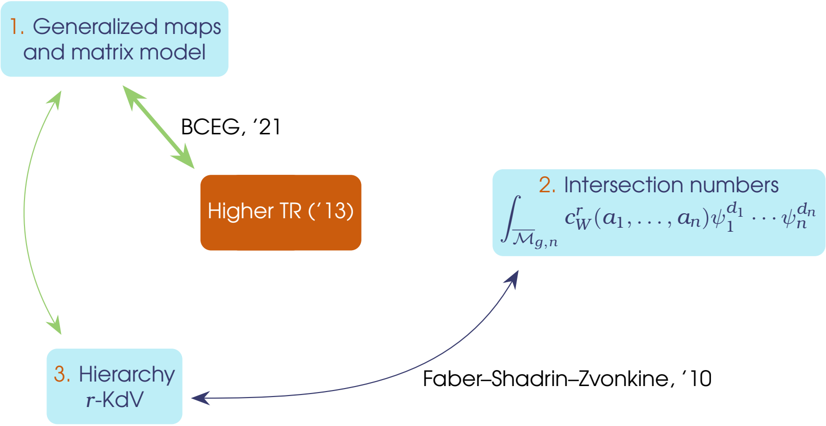
Theorem (Belliard–Charbonnier–Eynard–G-F, '21)

Generalised (Kontsevich) maps satisfy topological recursion.

Generalized Kontsevich maps and integrable hiearchy



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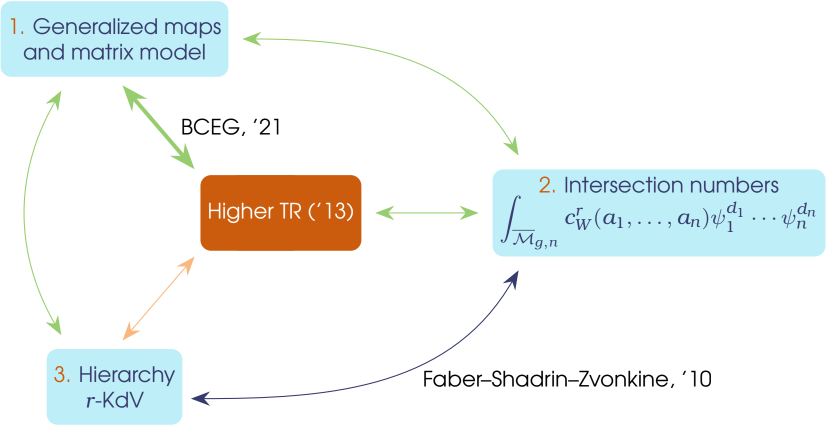
Generalized Kontsevich maps and r-spin intersection numbers



Theorem (Belliard–Charbonnier–Eynard–G-F, '21)

TR (allowing ramification points of higher order) applied to the spectral curve $(x, y) = (z^r, z)$ produces r-spin intersection numbers.

Generalized Kontsevich matrix model (GKM)

Kontsevich graphs are Feynman graphs of the hermitian matrix model with external field $\lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$:

$$Z = \int_{\mathcal{H}_N + \lambda} \mathrm{d}M \; e^{-N \mathrm{Tr}\left(\frac{M^3}{3} - M\lambda^2\right)}.$$

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Natural generalisation $\rightsquigarrow$ GKM:

$$Z(V;\lambda) = \int_{\mathcal{H}_N + \lambda} \mathrm{d}M \; e^{-N\alpha^{r+1} \,\mathrm{Tr}\, (V(M) - MV'(\lambda))}, \quad V(z) = \sum_{j=1}^{r+1} v_j \frac{z^j}{j}.$$

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Re-writing $M = \lambda + \widetilde{M}$ to eliminate the linear term:

$$C(\lambda) \int_{\mathcal{H}_N} d\widetilde{M} e^{-N\alpha^{r+1}\left(\frac{1}{2}\sum_{i,j=1}^{N} \widetilde{M}_{i,j}\widetilde{M}_{j,i}\frac{1}{\mathcal{P}(\lambda_i,\lambda_j)} - \sum_{\ell=3}^{r+1}\frac{1}{\ell}\sum_{i_1,\ldots,i_\ell=1}^{N} \widetilde{M}_{i_1,i_2}\widetilde{M}_{i_2,i_3}\cdots\widetilde{M}_{i_\ell,i_1}\mathcal{V}_\ell(\lambda_{i_1},\ldots,\lambda_{i_\ell})\right)}.$$

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$$\log\frac{Z}{Z_0} = \sum_{g\geq 0} \sum_{G \in \mathcal{F}^{[r]}_{g,0}} \frac{N^{-\frac{\deg G}{r+1}}\alpha^{-\deg G}}{\#\mathrm{Aut}\,G} \prod_{\substack{e\in\mathcal{E}(G)\\ e=(f_1,f_2)}} \mathcal{P}(\lambda_{f_1},\lambda_{f_2}) \prod_{v\in\mathcal{V}(G)} \mathcal{V}_{d_v}\left(\{\lambda_f\}_{f\mapsto v}\right).$$

For $i_1 \neq \cdots \neq i_n$, connected correlation functions $\rightsquigarrow$ ciliated maps (1):

$$\langle \widetilde{M}_{i_1,i_1}\cdots\widetilde{M}_{i_n,i_n}\rangle_c = \frac{1}{(N\alpha^{r+1})^n}\frac{\partial}{\partial\Lambda_{i_1}}\cdots\frac{\partial}{\partial\Lambda_{i_n}}\log Z = \sum_{g\geq 0} N^{2-2g-n} W^{[r]}_{g,n}(\lambda_{i_1},\ldots,\lambda_{i_n}).$$

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$\left\langle \mathrm{Tr}\frac{1}{x_1 - M} \cdots \mathrm{Tr}\frac{1}{x_n - M}\right\rangle_c$ admit topological expansions computed by TR applied to the spectral curve (y, x) (Eynard–Orantin,'07,'09).

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From maps to r-spin intersection numbers

r-spin intersection numbers:

$$\langle \tau_{d_1,a_1} \cdots \tau_{d_n,a_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} c_W^r(a_1, \ldots, a_n) \prod_{i=1}^{n} \psi_i^{d_i}.$$

1. The partition function Z of the GKM provides the only solution $\mathcal{I}_N(t_1, t_2, \ldots)$ of the r-KdV hiearchy that satisfies the string equation, with $t_k = \frac{1}{k}\mathrm{Tr}\big(\alpha N^{\frac{1}{r+1}} \lambda\big)^{-k}$ (using Adler–van Moerbeke, '92).

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$$= \frac{(-1)^g r^{g-1+n}}{\alpha^{(r+1)(2g-2+n)}} \sum_{\substack{d_1,\ldots,d_n \geq 0 \\ 0 \leq j_1,\ldots,j_n \leq r-1}} \prod_{\ell=1}^{n} \frac{c_{d_\ell+1,j_\ell}}{\Lambda_{i_\ell}^{d_\ell+1+\frac{j_\ell+1}{r}}} \left\langle \prod_{i=1}^{n} \tau_{d_i,j_i} e^{\sum_{d,j} t_{d,j}\tau_{d,j}} \right\rangle_g,$$

with $t_{d,j} = c_{d,j} \sum_{k=1}^{N} \Lambda_k^{-d-\frac{j+1}{r}}$ and $c_{d,j} = (-1)^d \frac{\Gamma\left(d+\frac{j+1}{r}\right)}{\Gamma\left(\frac{j+1}{r}\right)}$.

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Remark (ELSV-type formula)

ELSV-like (Ekedahl–Lando–Shapiro–Vainshtein, '01) formulas relate combinatorial problems with intersection theory over $\overline{\mathcal{M}}_{g,n}$.

r-spin intersection numbers for topology $(1,1)$

From the enumeration of ciliated maps of topology $(1,1)$:

$$W_{1,1}^{[r]}(z_1) = \alpha^{-(r+1)} \left[\frac{1}{6} \frac{V^{(3)}(z_1) V^{(4)}(z_1)}{V''(z_1)^4} - \frac{1}{8} \frac{V^{(3)}(z_1)^3}{V''(z_1)^5} - \frac{1}{24} \frac{V^{(5)}(z_1)}{V''(z_1)^3} \right] .$$

In the case $V(z) = \frac{z^{r+1}}{r+1}$, we get

$$W_{1,1}^{[r]}(z_1) = -\frac{\alpha^{-(r+1)}}{24} \frac{r^2-1}{r^2} \frac{1}{z_1^{2r+1}},$$

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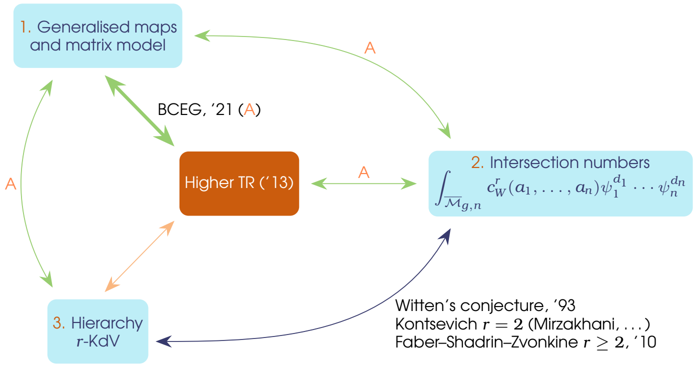
From our ELSV-type formula:

$$\omega_{1,1}^{[r]}(z_1) = -\frac{\alpha^{-(r+1)}}{24} \frac{r^2-1}{r} \frac{\mathrm{d}z_1}{z_1^{r+2}} = -\alpha^{-(r+1)} \frac{r+1}{r} \langle \tau_{1,0} \rangle_1 \frac{\mathrm{d}z_1}{z_1^{r+2}} .$$

Therefore,

$$\boxed{\langle \tau_{1,0} \rangle_1 = \frac{r-1}{24} .}$$

Work so far and further consequences ⇝ in progress and future



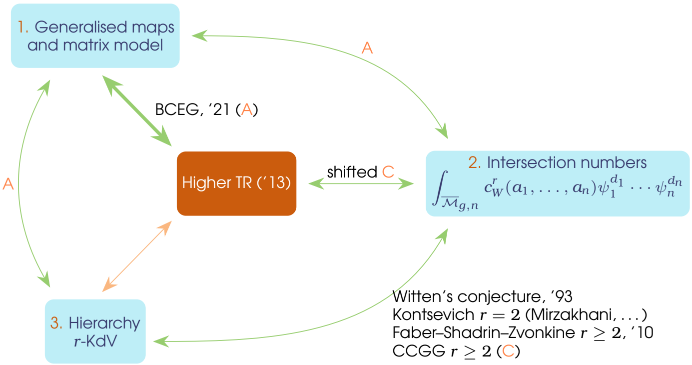
So far:

A *Topological recursion for generalised Kontsevich graphs and r-spin intersection numbers*, with R. Belliard, S. Charbonnier and B. Eynard, arXiv:2105.08035.

B Consequences in combinatorics (and free probability): **Conjecture from 2017 (now theorem):** If (x, y) is the spectral curve for ordinary maps (1-hermitian matrix model), then fully simple maps (non self-intersecting disjoint boundaries) satisfy TR with spectral curve (y, x).

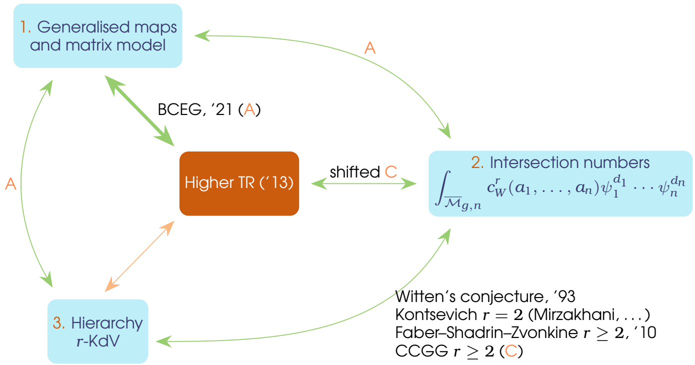
Topological recursion for fully simple maps from ciliated maps, with G. Borot and S. Charbonnier, arXiv:2106.09002.

Work so far and further consequences ⇝ in progress and future



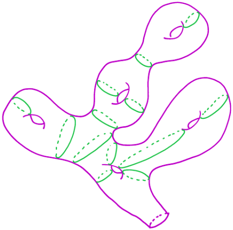
Work in progress ⇝

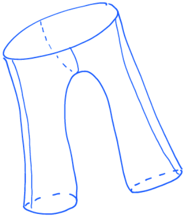
- C Use the power of TR to study the intersection of Witten's class when varying the spectral curve (via Eynard–DOSS) (with S. Charbonnier, N. Chidambaran and A. Giacchetto).
- D Use the non-perturbative extension of TR, its relation to integrability and resurgence to get the large genus asymptotics of r-spin intersection numbers (with B. Eynard, P. Gregori and D. Lewański).
- E Use the duality coming from TR for ordinary and fully simple maps (B) to establish relations between moments and free cumulants (with G. Borot, S. Charbonnier, F. Leid and S. Shadrin).



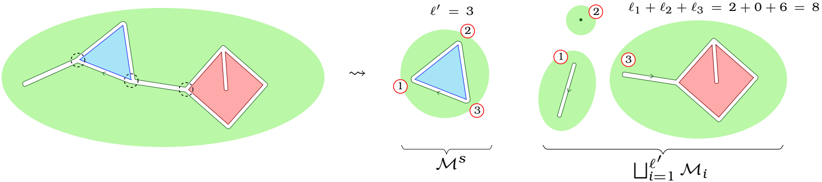
Future work ⇝

- Better understanding of Witten's class making use of our graphs? Can we establish the relation 1. ↔ 2. directly?
- Symplectic invariance for a large class of spectral curves? **Conjecture:** If two spectral curves $\mathcal{S}$ and $\mathcal{S}'$ are symplectically equivalent, i.e. $|\mathrm{d}x \wedge \mathrm{d}y| = |\mathrm{d}x' \wedge \mathrm{d}y'|$, then $\omega_{g,0}[\mathcal{S}] = \omega_{g,0}[\mathcal{S}']$. Under exploration for the symplectic transformation $x \leftrightarrow y$.
- Express the relations between ordinary and fully simple maps in terms of operadic languange? More generally, modular operads for topological recursion?





Vielen Dank für Ihre Aufmerksamkeit!



Witten's class

$V = \langle e_0, e_1, \ldots, e_{r-2} \rangle_{\mathbb{Q}}$, $\eta(e_a, e_b) = \delta_{a+b,r-2}$. Witten's r-spin CohFT :

$$c_W^r(a_1, \ldots, a_n) = \Omega_{g,n}(e_{a_1}, \ldots, e_{a_n}),$$

of degree $D_{g,n}^r := \frac{(r-2)(g-1)+\sum_{i=1}^n a_i}{r}$, with $a_1, \ldots, a_n \in \{0, \ldots, r-2\}$.

For $[\Sigma, x_1, \ldots, x_n] \in \overline{\mathcal{M}}_{g,n}$, $\exists\, \mathcal{T}$ line bundle over Σ such that

$$\mathcal{T}^{\otimes r} \cong \omega_\Sigma\Big(- \sum_{i=1}^n a_i x_i \Big), \text{ with } [\Sigma, x_1, \ldots, x_n] \in \overline{\mathcal{M}}_{g,n},$$

with ω_Σ the canonical bundle. Every r-th root of this fiber (r-spin structure) $\rightsquigarrow$ point in $\overline{\mathcal{M}}_{g,n}^{1/r}(a_1, \ldots, a_n)$:

$$\pi \colon \overline{\mathcal{M}}_{g,n}^{1/r}(a_1, \ldots, a_n) \to \overline{\mathcal{M}}_{g,n}.$$

- **Genus** $0 \rightsquigarrow$ Witten. For $[\Sigma, x_1, \ldots, x_n, \mathcal{T}] \in \overline{\mathcal{M}}_{0,n}^{1/r}(a_1, \ldots, a_n)$, $U = H^1(\Sigma, \mathcal{T}) \rightsquigarrow$ vector bundle $\mathcal{U} \to \overline{\mathcal{M}}_{0,n}^{1/r}(a_1, \ldots, a_n)$ (U has constant dimension, since $H^0(\Sigma, \mathcal{T}) = 0$).

$$c_W^r(a_1, \ldots, a_n) := \pi_* e(\mathcal{U}^*) \in H^{2D_{0,n}^r}(\overline{\mathcal{M}}_{0,n}).$$

- **For** $g > 0$, existence non-trivial and construction complicated (Polishchuk–Vaintrob '04, Chiodo '06, Fan–Jarvis–Ruan '13...).