# Resurgent Lambert series in quantum field theory and string theory (plus hyper-asymptotics from the cosmohedron)

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**Lambert series** $\sum_{n>0} a(n)q^n/(1-q^n)$ result from a massive **3-loop banana integral** whose algebraic geometry involves a family of **K3 surfaces**. Thanks to **Ramanujan** we may handle this near the **singular** limit $q \to 1$, where **quasi-modularity** leads to a phenomenon called *Cheshire cat* **resurgence**. **Daniele Dorigoni** and I generalized this to include resurgent Lambert series with **characters**, finding applications to **topological string** observables [arXiv:2507.21352].

In a study of **combinatorics of the cosmohedron**, Ardila-Mantilla, et al. found an integer sequence, 1, 2, 10, 72, 644, 6704, 78408, 1008480, 14065744, 210682080…

Its **hyper-asymptotic** behaviour readily submits to **Jean Écalle's** theory of **resurgent** functions and **alien calculus**, which **Michael Borinsky** and I used to analyse infinite series of **Feynman integrals** [arXiv:2202.01513].

## Part 1: Ramanujan's transformation of the 3-loop banana

I first encountered **Riemann zeta values** as a school-boy, during the **Cuban missile crisis** (6–28 October 1962), learning that the **average energy** of photons from the **Sun** is given by [**Feynman's sunshine numbers**, arXiv:1004.4238]

$$< E >= \frac{\zeta(4)}{\zeta(3)} 3kT, \quad \zeta(4) = \sum_{n>0} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \zeta(3) = \sum_{n>0} \frac{1}{n^3} = 1.202056903\ldots$$

where $k$ is a constant celebrated here, in **Boltzmanngasse**, and $T \approx 5777$ **Kelvin**. Studying work by **Spencer Bloch, Matt Kerr and Pierre Vanhove** [arXiv:1406.2664] on the 3-loop banana, at **Les Houches** in 2014, I was to led to Ramanujan's efficient formula, with a rapidly convergent **Lambert series**:

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n>0} \frac{a(n)q^n}{1-q^n}, \quad a(n) = \frac{1}{n^3}, \quad q = \exp(-2\pi) < \frac{\mathbf{1}}{\mathbf{353}}$$

which I later improved to obtain Lambert series with $q = \exp(-12\pi/\sqrt{\mathbf{15}}) < \mathbf{1/16879}$.

Ramunujan's value of $\zeta(3)$ came from a family of **quasi-modular** transformations of

$$T_s(\tau) = \frac{1}{2}\zeta(s) + \sum_{n>0} \frac{1}{n^s} \frac{q^n}{1-q^n}, \quad q = \exp(2\pi \mathrm{i}\tau) = \exp(-2\pi y)$$

with $\Im\tau = \Re y > 0$. This converges well at large $y$. To access $q \to 1^-$, as $y \to 0^+$, we use a **Fricke involution** $y \to 1/y$, when $s$ is **odd**. For example, he obtained

$$\begin{aligned}
yT_3(\mathrm{i}/y) &= \frac{\pi^3}{180}\left(y^2 + 5 + \frac{1}{y^2}\right) - \frac{T_3(iy)}{y} \\
y^2T_5(\mathrm{i}/y) &= \frac{\pi^5}{3780}\left(2y^3 + 7y - \frac{7}{y} - \frac{2}{y^3}\right) + \frac{T_5(\mathrm{i}y)}{y^2} \\
y^3T_7(\mathrm{i}/y) &= \frac{\pi^7}{56700}\left(3y^4 + 10y^2 - 7 + \frac{10}{y^2} + \frac{3}{y^4}\right) - \frac{T_5(\mathrm{i}y)}{y^3}
\end{aligned}$$

which originate from **Eichler** integration of **Eisenstein series**

$$\sum_{n>0} \frac{n^s q^n}{1-q^n} = \left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^s \sum_{n>0} \frac{n^{-s} q^n}{1-q^n}.$$

The **Schwinger** parametrization of the equal-mass **3-loop banana** integral is

$$J(t)=\int_0^\infty\int_0^\infty\int_0^\infty\left(\frac{1}{(1+a+b+c)(1+1/a+1/b+1/c)-t}\right)\frac{\mathrm{d}a}{a}\frac{\mathrm{d}b}{b}\frac{\mathrm{d}c}{c}$$

and for real $t<16$ it is given by the **Bessel moment**

$$J(t)=8\int_0^\infty I_0(\sqrt{t}x)K_0^4(x)x\,\mathrm{d}x.$$

**Stefano Laporta** and I discovered the **on-shell** evaluation

$$J(1)=\frac{\pi^3}{2}\left(1-\frac{1}{\sqrt{5}}\right)\left(\sum_{n=-\infty}^{\infty}\exp(-\sqrt{\mathbf{15}}\pi n^2)\right)^4=\frac{\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)}{30\sqrt{5}}.$$

A **modular parametrization** is achieved using the **Dedekind eta** function

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)=\sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24}=\frac{\eta(-1/\tau)}{\sqrt{-\mathrm{i}\tau}}.$$

Denoting $\eta_n := \eta(n\tau)$, let

$$t = -64\left(\frac{\eta_2\eta_6}{\eta_1\eta_3}\right)^6, \quad \psi = \frac{\eta_1^2\eta_3^2}{\eta_2\eta_6}, \quad h = \frac{\eta_2^{16}}{\eta_1^8} - 9\frac{\eta_6^{16}}{\eta_3^8}.$$

Then $\psi^2$ satisfies the homogeneous **Picard-Fuchs** equation of a K3 surface and

$$\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^3 \frac{J(t)}{\psi^2} = 24h = 24\sum_{n=1}^{\infty}\frac{n^3(q^n - 8q^{3n} + q^{5n})}{1 - q^{6n}}.$$

This is easily integrated to give **Lambert series** in

$$\frac{J(t)}{\psi^2} = H(\tau) = 24T_3(\tau) - 3T_3(2\tau) - 8T_3(3\tau) + T_3(6\tau)$$

which converges well for $|t| \le 8$. The **Fricke** involution $\tau \to -1/(6\tau)$ gives $t \to 64/t$ and Ramanujan delivers

$$3\tau^2 H\left(\frac{-1}{6\tau}\right) = 2(\pi\mathrm{i}\tau)^3 + 3T_3(\tau) - 6T_3(2\tau) - T_3(3\tau) + 2T_3(6\tau)$$

for fast evaluation of $J(t)$ with $|t| \ge 8$.

**Cheshire cat resurgence:**

*I've often seen a cat without a grin, thought Alice, but a grin without a cat!*
*It's the most curious thing I ever saw in my life!*

Ramanujan's very useful formula

$$T_{2k+1}\left(\frac{-1}{\tau}\right) = \frac{\mathrm{i}}{\pi}\sum_{j=0}^{k+1}\frac{\zeta(2j)\zeta(2k+2-2j)}{\tau^{2j-1}} + \frac{T_{2k+1}(\tau)}{\tau^{2k}}$$

has a **terminating** Laurent series and an **exponentially suppressed** tail, as $\tau \to \mathrm{i}\infty$. By contrast, Lambert series $\sum_{n>0} n^{-s}q^n/(1-q^n)$ for **even** s, give **divergent** series, requiring directional **Borel re-summation** whose ambiguities are eventually resolved by careful treatment of the exponentially suppressed tail.

In `arXiv:2001.11035`, **Daniele Dorigoni** and **Axel Kleinschmidt** described the **quasi-modularity** of Ramanujan's formula for **odd integer** $s$, as an example of what they called "Cheshire cat resurgence".

Amusingly, we know of a Cheshire cat with **even integer** $s$.

## A characterful Cheshire cat, at sunrise

The 2-loop **sunrise diagram** produces Lambert series with $s = 2$. So how can it **avoid** a difficult trans-series? For the sunrise integral $I(w^2) = 4\int_0^\infty I_0(wx)K_0^3(x)x\,dx$, the **modular** parameterization

$$w = 3\frac{\eta_2^2\eta_3^4}{\eta_1^4\eta_6^2}\,, \quad f = \frac{\eta_1^6\eta_6}{\eta_2^3\eta_3^2}\,, \quad g = \frac{\eta_3^9}{\eta_1^3} + \frac{\eta_6^9}{\eta_2^3}$$

neatly reduces the problem to solving

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \frac{I(w^2)}{f} = 6g = 6\sum_{n=1}^{\infty}\frac{n^2(q^n - q^{5n})}{1-q^{6n}}$$

and the integration produces, inter alia, a **Lambert series with a character**

$$\mathcal{L}_2(\chi_{6,5};q) = \sum_{n=1}^{\infty}\frac{\chi_{6,5}(n)}{n^2}\left(\frac{q^n}{1-q^n}\right)$$

with $\chi_{6,5}(n) = \pm 1$ for $n = \pm 1 \bmod 6$. Then quasi-modularity allows one to expand about any of the singular points $w \in \{0, 1, 3, \infty\}$ corresponding to cusps of modular forms on $\Gamma_0(6)$.

**Two types of divisor sums:**

$$\begin{aligned}\mathcal{L}_s(\chi_r;\tau) &= \sum_{n>0} \frac{\chi_r(n)}{n^s}\frac{q^n}{1-q^n}\\ &= \sum_{n>0} \frac{\sigma'_{s,\chi_r}(n)\,q^n}{n^s}, \quad \sigma'_{s,\chi_r}(n) = \sum_{d|n} \chi_r(n/d)d^s\\ \widetilde{\mathcal{L}}_s(\chi_r;\tau) &= \sum_{n>0} \frac{\sigma_{s,\chi_r}(n)\,q^n}{n^s}, \quad \sigma_{s,\chi_r}(n) = \sum_{d|n} \chi_r(d)d^s.\end{aligned}$$

If $\chi_r$ is a primitive **quadratic** character with fundamental discriminant $D$, then we have quasi-modularity for **odd** $s$ when $D > 1$, and for **even** $s$, when $D < 0$. With conductor $N = |D|$, the modularity gap is given by Bernoulli polynomials

$$\sqrt{D}\left(2\mathcal{L}_s(\chi_D;\tau) + \sum_{n>0} \frac{\chi_D(n)}{n^s}\right) + 2(-\tau)^{s-1}\widetilde{\mathcal{L}}_s\left(\chi_D;\frac{-1}{N\tau}\right) =$$

$$\frac{(2\pi \mathrm{i})^s}{(s+1)!}\sum_{j=0}^{\lfloor s/2\rfloor} \tau^{2j-1}\binom{s+1}{2j} B_{2j}(0) \sum_{m=1}^{N-1} \chi_D(m) B_{s+1-2j}(m/N).$$

**Generalization:** We generalized by including **pairs** of characters and indices in

$$\Xi_{s_1,s_2}(\chi_{r_1},\chi_{r_2};\tau)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{\chi_{r_1}(n_1)}{n_1^{s_1}}\frac{\chi_{r_2}(n_2)}{n_2^{s_2}}q^{n_1 n_2}.$$

With $\tau = \mathrm{i}y$ and $y \to 0^+$ we found that resummed perturbative terms require, in general, an **exponentially suppressed** tail that is a **sum** of transformed terms:

$$\sum_{n=0}^{\infty}\frac{(s_1)_n(s_2)_n}{n!}\frac{(r_1 r_2\tau)^{s_1+s_2+n-1}}{(-2\pi i)^n}\,\Xi_{s_1+n,s_2+n}\left(\bar\chi_{r_2},\bar\chi_{r_1};\frac{-1}{r_1 r_2\tau}\right)$$

with only one term surviving in the single character cases

$$\begin{aligned}\mathcal{L}_s(\chi_r;\tau)&=\Xi_{s,0}(\chi_r,\chi_{1,1};\tau)=\Xi_{0,s}(\chi_{1,1},\chi_r;\tau)\,,\\ \widetilde{\mathcal{L}}_s(\chi_r;\tau)&=\Xi_{0,s}(\chi_r,\chi_{1,1};\tau)=\Xi_{s,0}(\chi_{1,1},\chi_r;\tau)\,.\end{aligned}$$

relevant to work by **Veronica Fantini** and **Claudia Rella**, on quantum modularity of resurgent **topological strings** [arXiv:2212.10606; 2404.10695; 2404.11550; 2506.08265].

In a case with the odd character $\chi_{3,2}(n) = \pm 1$ for $n = \pm 1 \bmod 3$, the behaviour of

$$\log\left[\mathrm{Tr}(\rho_{\mathbb{P}^2})\right] = -\frac{1}{2}\log(3^{\frac{5}{2}}\tau) - \frac{\pi i}{4} + \frac{3}{2}\left(\widetilde{\mathcal{L}}_1(\chi_{3,2};\tau) - i\sqrt{3}\mathcal{L}_1(\chi_{3,2};-\frac{1}{3\tau})\right)$$

at large or small $y = -\mathrm{i}\tau$ requires resummation of an infinite number of perturbative terms, with the exponentially supressed terms in

$$\begin{pmatrix}\mathcal{L}_1(\chi_{3,2};\tau)\\ \widetilde{\mathcal{L}}_1(\chi_{3,2};\tau)\end{pmatrix} = \begin{pmatrix}\mathcal{S}_{\mp}\left[\mathcal{L}_1^{\mathrm{Pert}}\right](\chi_{3,2};\tau)\\ \mathcal{S}_{\mp}\left[\widetilde{\mathcal{L}}_1^{\mathrm{Pert}}\right](\chi_{3,2};\tau)\end{pmatrix} \pm i\begin{pmatrix}0 & \frac{1}{\sqrt{3}}\\ \sqrt{3} & 0\end{pmatrix}\begin{pmatrix}\mathcal{L}_1(\chi_{3,2};-\frac{1}{3\tau})\\ \widetilde{\mathcal{L}}_1(\chi_{3,2};-\frac{1}{3\tau})\end{pmatrix}$$

resolving the ambiguity of directional Borel resummation, indicated by $S_{\mp}$.

**Spectral trace of local** $\mathbb{P}^{m,n}$ : Here we encounter the conductor $N = m + n + 1$. With $(m, n) = (2, 1)$, $\widetilde{\mathcal{L}}_1(\chi_{4,3};\tau) - 2i\mathcal{L}_1(\chi_{4,3};-1/(4\tau))$ occurs.

With $(m, n) = (3, 2)$, there are three characters to consider: the real **even** character $\chi_{5,4}$ and a complex pair $\chi_{5,2} = \overline{\chi}_{5,3}$. The former gives a perturbative contribution that **terminates**. Yet its non-perturbative tail **persists**, like the grin of Lewis Carroll's **Cheshire cat**.

## Part 2: A simple trans-series from the cosmohedron

In a recent article entitled **combinatorics of the cosmohedron** [arXiv:2603.03425] **Federico Ardila-Mantilla, Nima Arkani-Hamed, Carolina Figueiredo** and **Francisco Važ̃ao** encountered the integer sequence

`1, 2, 10, 72, 644, 6704, 78408, 1008480, 14065744, 210682080...`

enumerating **nestings of polygons** within polygons and satisfying a **recursion**

$$A_n = \sum_{k=1}^{n-1} (k+1) A_k A_{n-k} \text{ for } n > 1, \text{ with } A_1 = 1,$$

resulting from patterns of energies in denominators of **old-fashioned, time-ordered** perturbation theory, applied to **scalar field theory** in the background of a **Friedmann-Lemaître-Robertson-Walker** metric evolving in **cosmological** time. They conjecture that there is a **finite constant** $S$ such that

$$\lim_{n\to\infty} \frac{A_n}{(n+4)!} = S \approx 0.005428\ldots$$

and suggest that this lies on "the **border** between decidability and **undecidability**". In fact, this integer sequence yields **readily** to the methods of **Jean Écalle**.

I now show how to develop the **asymptotic expansion**

$$A_n \sim S \sum_{k>0} \Gamma(n+\mathbf{6}-k)B_k = S\,(n+4)!\left(\mathbf{1} - \frac{\mathbf{8}}{n+4} + \frac{\mathbf{4}}{(n+4)(n+3)} + O\left(\frac{1}{n^3}\right)\right)$$

with **decreasing factorials** and **rational coefficients** $B_k$ forming the sequence

$$1, -8, 4, -36, -388, -\tfrac{22992}{5}, -\tfrac{894152}{15}, -\tfrac{87775088}{105}, -\tfrac{1318809728}{105} - \tfrac{7025275808}{35}, -\tfrac{1781593980128}{525} \ldots$$

This is the first of an infinite series of **hyper-asymptotic** expansions. The **next** is

$$B_n \sim 2S \sum_{k>0} \Gamma(n+\mathbf{6}-k)C_k = 2S\,(n+4)!\left(-\mathbf{1} + \frac{\mathbf{16}}{n+4} - \frac{\mathbf{74}}{(n+4)(n+3)} + O\left(\frac{1}{n^3}\right)\right)$$

whose coefficients $C_k$ form the sequence

$$-1, 16, -74, 160, 168, \tfrac{17584}{5}, \tfrac{762688}{15}, \tfrac{1921536}{105}, \tfrac{264968176}{21}, \tfrac{22528464128}{105}, \tfrac{669680351104}{175}, \tfrac{58928184776704}{825} \ldots$$

with asymptotic behaviour $C_n \sim 3S \sum_{k>0} \Gamma(n+\mathbf{6}-k)D_k$ with $D_1 = \mathbf{3/2}, D_2 = -\mathbf{37}$.

*Large A's need smaller B's, especially to guide them,*
*and larger B's need smaller C's, and so ad infinitum.*

**Step 1: a non-linear differential equation:**
The **formal** power series $G_0(x) = \sum_{n>0} A_n x^{n-1} = 1 + 2x + 10x^2 + 72x^3 + O(x^4)$ solves

$$G = 1 + G\frac{\mathrm{d}}{\mathrm{d}x}(x^2 G).$$

**Step 2: the first instanton correction:**
We posit a solution of the form

$$G(x) = G_0(x) + \sigma x^{-\boldsymbol{\beta}} \mathrm{e}^{-\boldsymbol{\mu}/x} G_1(x) + O(\sigma^2)$$

with an **arbitrary** constant $\boldsymbol{\sigma}$ and a power series $G_1(x) = \sum_{n>0} B_n x^{n-1}$. Then

$$x^2 G_1' + \left(\boldsymbol{\mu} + (2-\boldsymbol{\beta})x - \frac{1}{G_0^2}\right) G_1 = 0$$

is soluble by a formal power series $G_1 = 1 + O(x)$ if and only if $\boldsymbol{\mu} = \mathbf{1}$ and $\boldsymbol{\beta} = \mathbf{6}$.
**Step 3: the complete trans-series** has the form

$$G(x) = G_0(x) + \sum_{m>0} \left(\sigma x^{-\mathbf{6}} \mathrm{e}^{-\mathbf{1}/x}\right)^m G_m(x).$$

**Step 4: numerical evaluation of a Stokes constant:** For a generic problem we expect $A_n \sim S\boldsymbol{\mu}^{-n}\sum_{k>0}\Gamma(n+\boldsymbol{\beta}-k)B_k$. In the present case, we have $\mu = 1$ and $\boldsymbol{\beta} = \mathbf{6}$. Then it takes about **one second** to obtain **500 decimal digits** of $S = \mathtt{0.005428317993266202636748034138132075286101589263688307574601631285}\ldots$ by truncating the series $A_n \sim S\sum_{k>0}\Gamma(n+\mathbf{6}-k)B_k$ at $k \approx n/2 \approx 850$, using

$$B_1 = 1, \quad (k-1)B_k = \sum_{:j=1}^{k-1} t_{j+2}B_{k-j} \text{ for } k > 1, \quad \sum_{n>0} t_n x^{n-1} = \frac{1}{G_0^2},$$

I determined **5000 digits** of $S$ in **40 minutes**, finding no fit to products of powers of rationals, $\pi$, e, $\Gamma(1/3)$, $\Gamma(1/4)$, etc. **Stokes constants** are **rarely** determined by simpler transcendentals.

**Step 5: continuation of the trans-series:** For $m > 1$, I determine $G_m(x)$ from an **inhomogeneous** differential equation of the form

$$x^2 G_m' + \left(m + (2-6m)x - \frac{1}{G_0^2}\right) G_m = \frac{T_m}{G_0^{m+1}}$$

with $G_2 = \sum_{n>0} C_n x^{n-1}$, $G_3 = \sum_{n>0} D_n x^{n-1}$, etc.

The **inhomogeneous** term $T_m$ is **polynomial** in $\{G_k | k < m\}$. For example, we have

$$T_2 = -G_1^2, \quad T_3 = -2G_2G_1G_0 + G_1^3, \quad T_4 = -2(G_3G_1 + G_2^2)G_0^2 + 3G_2G_1^2G_0 - G_1^4,$$
$$T_5 = -2(G_4G_1 + G_3G_2)G_0^3 + 3(G_3G_1^2 + G_2^2G_1)G_0^2 - 4G_2G_1^3G_0 + G_1^5,$$

whose monomials correspond to **partitions** of $m$. For $m > 0$, the **leading** terms

$$G_m(0) = \frac{(-m)^{m-1}}{m!}$$

are the Taylor coefficients of the principal branch of the **Lambert W** function, $W(x) = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \frac{125}{24}x^5 + O(x^6)$, satisfying $W \exp(W) = x$.

**Conclusion:** The trans-series for this cosmohedron problem is a **textbook example** of Écalle's methods. However, Jean's books contain **1100 pages** of specialized vocabulary and I relied on my mentors **Ovidiu Costin** and **Gerald Dunne** to decode them.

**Closer to the border** is a trans-series from a **third** order **Dyson-Schwinger** equation with **quartic** non-linearity and **six Stokes constants**, whose hyper-asymptotics were determined by **Michael Borinsky** and me.

**Dirk Kreimer** and I derived the **Dyson-Schwinger** equation

$$(g(x)P-1)(g(x)P-2)(g(x)P-3)g(x)=-3, \quad P=x\left(2x\frac{\mathrm{d}}{\mathrm{d}x}+1\right)$$

from an infinite series of Feynman integrals from $\phi^3$ theory in 6 dimensions. Its formal perturbative solution

$$g_0(x)\sim\sum_{n=0}^{\infty}A_nx^n=\frac{1}{2}+\frac{11}{24}x+\frac{47}{36}x^2+\frac{2249}{384}x^3+\frac{356789}{10368}x^4+\frac{60819625}{248832}x^5+O(x^6)$$

has coefficients that grow factorially:

$$A_n=S_1\Gamma\left(n+\mathbf{\frac{35}{12}}\right)\left(1-\mathbf{\frac{97}{48}}\left(\frac{1}{n}\right)+O\left(\frac{1}{n^2}\right)\right).$$

**Michael Borinsky** and I developed its **trans-series**

$$g(x)=\sum_{m=0}^{\infty}\left(x^{-\frac{35}{12}}\,\mathrm{e}^{-\frac{1}{x}}\right)^m\sum_{i=0}^{\lfloor m/2\rfloor}\sum_{j=0}^{\lfloor (m-2i)/3\rfloor}\sigma_1^s\widehat{\sigma}_2^i\widehat{\sigma}_3^j x^{5(i+j)}\sum_{n\geq 0}a_{i,j}^{(m)}(n)x^n,$$

$$s=m-2i-3j\geq 0, \quad \widehat{\sigma}_2=\sigma_2+\tfrac{21265}{2304}\sigma_1^2\log(x), \quad \widehat{\sigma}_3=\sigma_3+\tfrac{21265}{2304}\sigma_1^3\log(x).$$

With $m = s + 2i + 3j$, a conjecturally complete description of **resurgence** is given by

$$
\begin{aligned}
a^{(m)}_{i,j}(n) \sim -(s+1)\mathbf{S_1} \sum_{k\geq 0} a^{(m+1)}_{i,j}(k)\Gamma(n + \tfrac{35}{12} - k) \\
+ S_1 \sum_{k\geq 0} \Big(4(i+1)a^{(m+1)}_{i+1,j}(k) + 6(j+1)a^{(m+1)}_{i,j+1}(k)\Big)\,\Gamma(n - \tfrac{25}{12} - k)\left(\tfrac{21265}{4608}\psi(n - \tfrac{25}{12} - k) + \mathbf{d_1}\right) \\
+ \tfrac{1}{4}\mathbf{S_3} \sum_{k\geq 0} \Big(4(s+1)a^{(m-1)}_{i-1,j}(k) + 6(j+1)a^{(m-1)}_{i-2,j+1}(k)\Big)\,(-1)^{n-k}\Gamma(n + \tfrac{25}{12} - k) \\
- 2(s - 2i - 1)S_3 \sum_{k\geq 0} a^{(m-1)}_{i,j}(k)(-1)^{n-k}\Gamma(n - \tfrac{35}{12} - k)\left(\tfrac{21265}{4608}\psi(n - \tfrac{35}{12} - k) + \mathbf{f_1}\right) \\
- S_3 \sum_{k\geq 0} \Big(8(i+1)a^{(m-1)}_{i+1,j}(k) + 6(j+1)a^{(m-1)}_{i,j+1}(k)\Big)\,(-1)^{n-k}\Gamma(n - \tfrac{95}{12} - k)\mathbf{Q}(n - \tfrac{95}{12} - k) \\
- (f_1 - c_1)S_3 \sum_{k\geq 0} \Big(2(i+1)a^{(m-1)}_{i+1,j-1}(k) + 6(i+j)a^{(m-1)}_{i,j}(k)\Big)\,(-1)^{n-k}\Gamma(n - \tfrac{35}{12} - k),
\end{aligned}
$$

$$
\mathbf{Q}(z) = \left(\tfrac{21265}{4608}\right)^2\Big(\psi^2(z) + \psi'(z)\Big) + 2\mathbf{c_1}\left(\tfrac{21265}{4608}\right)\psi(z) + \mathbf{c_2}, \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.
$$

There are **17 resurgent terms**, tested at **high precision**, for $m \leq 8$. The **Stokes constants**

$$\begin{aligned}
S_1 &= 0.08759555290917912448379544742126299062738801740682\,2\ldots \\
S_3 &= 2.17178531405909902116086012279038923024794641930 27\ldots \\
d_1 &= -43.33263472825075592450071739031938070346072802227 8\ldots \\
f_1 &= -40.90369250922851500381447912690135478526366955301 4\ldots \\
c_1 &= -41.03195676430271058392106810154550945370489789818 8\ldots \\
c_2/c_1^2 &= 1.00020164721319925958228053808383241880115723042 76\ldots
\end{aligned}$$

have been determined at **1000-digit** precision.

**Envoi:** This week marks the end of my 51 years of affiliation with the Open University of the United Kingdom, after 12 years of retirement from teaching. Henceforth, I shall be an independent researcher, `djbroadhurst@yahoo.co.uk`, in which capacity I hope to attend many more workshops and conferences. I have greatly enjoyed 4 weeks of this ESI programme, in one of my favourite cities, and sincerely thank all concerned.