Signature cumulants and generalized Magnus expansions

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Motivation: Moments and Cumulants

Let X be a r.v. s.t. $\mathbb{E}e^{\lambda X} < \infty$ for some $\lambda > 0$.

Definition

Moment-generating function:

$$\mu(z) := \mathbb{E}[e^{zX}] = \sum_{n=0}^{\infty} \mathbb{E}[X^n] \frac{z^n}{n!}, \quad z \leq \lambda$$

Cumulant-generating function:

$$\kappa(z) := \log \mu(z) = \sum_{n=1}^{\infty} \kappa_n \frac{z^n}{n!}, \quad z \leq \lambda.$$

Under some conditions, cumulants characterize the distribution of X, e.g. $X \sim \mathcal{N}(0, \sigma^2)$ iff

$$\kappa_1 = 0, \ \kappa_2 = \sigma^2, \ \kappa_3 = \kappa_4 = \cdots = 0.$$

Also, $X \sim \text{Poisson}(\lambda)$ iff

$$\kappa_n = \lambda, \quad n \geq 1.$$

Theorem (Leonov–Shiryaev (1959))

$$\kappa_n = \mathbb{E}[X^n] - \sum_{m=1}^{n-1} \binom{n-1}{m} \kappa_m \mathbb{E}[X^{n-m}]$$

Theorem (Speed (1983), Ebrahimi-Fard–Patras–T.–Zambotti (2018))

The following relation between moments and cumulants holds:

$$\mathbb{E}[X^n] = \sum_{\pi \in \mathcal{P}(n)} \prod_{B \in \pi} \kappa_{|B|}.$$

Also a multivariate version indexed by the same lattice.

Remark

Cumulants are also related to Wick products:

$$e^{zX - \kappa(z)} = \sum_{n=0}^{\infty} {:}X^n{:} \frac{z^n}{n!}$$

Motivation: Sine–Gordon model

Let K be a positive-semidefinite kernel on $\mathbb{R}^d$ and consider X a Gaussian field with covariance K, i.e. $\mathbb{E}[X(x)X(y)] = K(x,y)$.

Formally tilt the measure by setting

$$\tilde{\mathbb{P}}(\mathrm{d}X) := \frac{1}{Z}\exp\left(2\alpha\int\cos(\beta X(x))\,\mathrm{d}x\right)\mathbb{P}(\mathrm{d}X).$$

Since X is a distribution, this does not make sense.

Consider a regularised kernel K_t and the corresponding field $X_t(x)$ with $\mathbb{E}[X_s(x)X_t(y)] = K_{t\wedge s}(x,y)$.

This gives rise to the martingale

$$M_t := \int\cos(\beta X_t(x))\mathrm{e}^{\frac{\beta^2}{2}K_t(x,x)}\,\mathrm{d}x$$

One can show that $\mathbb{E}[\mathrm{e}^{\alpha M_t}]$ equals

$$\sum_{n=0}^{\infty}\frac{\alpha^n}{2^n n!}\sum_{\lambda\in\{+1,-1\}^n}\int\exp\left(-\beta^2\sum_{i<j}\lambda_i\lambda_j K_t(x_i,x_j)\right)\prod_{i=1}^{n}\mathrm{d}x_i.$$

Let $\mathbb{K}_t^{(n)}$ be the “martingale cumulants” and define

$$Z_t[f] := \mathbb{E}[f(X)\mathrm{e}^{\alpha M_t}]\mathrm{e}^{-\sum_{k=1}^{n-1}\frac{\alpha^{2k}}{(2k)!}\mathbb{K}_t^{(2k)}}.$$

Theorem (Lacoin–Rhodes–Vargas (2019))

For $\beta < 2d$, the sequence Z_t has a limit as $t\to\infty$. Moreover,

$$\tilde{\mathbb{P}}(\mathrm{d}X) := \lim_{t\to\infty}\frac{1}{Z_t[1]}\mathrm{e}^{\alpha M_t-\sum_{k=1}^{n-1}\frac{\alpha^{2k}}{(2k)!}\mathbb{K}_t^{(2k)}}\,\mathbb{P}(\mathrm{d}X)$$

defines a probability measure.

Motivation: Diamond expansions

Let $\mathscr{S}$ be the space of semimartingales on a filtered probability space $(\mathcal{F}_t)_{t\geq 0}$.

Definition (Diamond product)

Let $X, Y \in \mathscr{S}$. Define

$$(X \diamond Y)_t(T) := \mathbb{E}_t[\langle X^c, Y^c\rangle_{t,T}] \in \mathscr{S}.$$

Theorem (Gatheral–Radoičić (2018), Friz–Gatheral–Radoičić (2020), Lacoin–Rhodes–Vargas (2019))

Let $A_T \in \mathcal{F}_T$ sufficiently integrable. Set $\mu_t(T) := \mathbb{E}_t[\mathrm{e}^{zA_T}]$ and $\mathbb{K}_t(T) := \log \mu_t(T)$. If $\mathbb{K}_t(T) = z\mathbb{E}_t[A_T] + \sum_{n\geq 2} z^n \mathbb{K}_t^{(n)}(T)$, we have the recursion

$$\mathbb{K}_t^{(1)}(T) := \mathbb{E}_t[A_T]$$

$$\mathbb{K}_t^{(n+1)}(T) = \frac{1}{2}\sum_{k=1}^{n} (\mathbb{K}^{(k)} \diamond \mathbb{K}^{(n-k)})_t(T).$$

Proof.

Let $M_t := \mathbb{E}_t[A_T]$, $\Lambda_t^T = \sum_{k\geq 2} z^n \mathbb{K}_t^{(n)}(T)$. Then $\mathrm{e}^{zM_t + \Lambda_t^T}$ is a martingale, so $zM_t + \Lambda_t^T + \frac{1}{2}\langle zM_t + \Lambda_t^T\rangle_t$ is also a martingale by Itô's formula. In particular

$$\mathbb{E}_t\left\{\mathbb{K}_T(T) + \frac{1}{2}\langle \mathbb{K}(T)\rangle_{t,T}\right\} = 0.$$

□

Motivation: Signatures

Denote by $T(d)$ the tensor algebra over $\{1,\dots,d\}$. The dual $T((d)) := T(d)^*$ is identified with formal word (tensor) series.

For $\mathbf{S} \in T((d))$ we write

$$\mathbf{S} = \sum_{w} S^{w} w = \sum_{n=0}^{\infty} \mathbf{S}^{(n)}.$$

Definition (Cauchy product)

For $\mathbf{R}, \mathbf{S} \in T((d))$,

$$\mathbf{RS} = \sum_{w} \left(\sum_{uv=w} R^{u} S^{v} \right) w = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathbf{R}^{(k)} \mathbf{S}^{(n-k)}$$

Maps exp and log are defined via the usual power series.

Denote by $G \subset T_1$ the set of series such that $S^{u} S^{v} = S^{u \sqcup\!\sqcup v}$, and $\mathfrak{g} \subset \mathfrak{t}$ the (linear) space of series such that $S^{u \sqcup\!\sqcup v} = 0$ for non-empty u, v.

Theorem

The map $\exp\colon \mathfrak{g} \longrightarrow G$ *is a bijection, with inverse* $\log\colon G \longrightarrow \mathfrak{g}$.

Denote by T_1 (resp. $\mathfrak{t}$) the sets of series with $S^{e} = 1$ (resp. $S^{e} = 0$).

Theorem

The map $\exp\colon \mathfrak{t} \longrightarrow T_1$ *is a bijection, with inverse* $\log\colon T_1 \longrightarrow \mathfrak{t}$.

Remark

Both $\mathfrak{t}$ and $\mathfrak{g}$ are Lie algebras wrt the commutator bracket.

Motivation: Signatures

Definition (Chen (1953), Lyons (1998))

For an absolutely continuous path X in $\mathbb{R}^d$, its signature is the formal tensor series

$$\mathrm{Sig}(X)_{s,t} := \sum_w \left(\int \cdots \int_{s<u_1<\cdots<u_n<t} \dot{X}_{u_1}^{w_1} \cdots \dot{X}_{u_n}^{w_n} \, \mathrm{d}u_1 \cdots \mathrm{d}u_n \right) w \in T((d))$$

Theorem (see e.g. Friz–Victoir (2010))

The signature satisfies the ODE

$$\mathrm{d}_t \mathbf{S}_{s,t} = \mathbf{S}_{s,t} \dot{X}_t, \quad \mathbf{S}_{s,s} = 1$$

Theorem (Chen–Fliess series, Fliess (1981))

Let Y solve $\mathrm{d}Y = f_i(Y)\,\mathrm{d}X^i$. Then

$$Y_t = \sum_{|w| \le N} f_w(Y_s)\, \mathrm{Sig}(X)_{s,t}^w + \mathrm{O}(|t-s|^{N+1}).$$

Theorem (Chen (1953))

Given $0 \le s < u < t \le T$ we have

$$\mathrm{Sig}(X)_{s,u}\, \mathrm{Sig}(X)_{u,t} = \mathrm{Sig}(X)_{s,t}.$$

Theorem (Ree (1954))

We have $\mathrm{Sig}(X)_{s,t} \in G$, i.e. for all words u, v

$$\mathrm{Sig}(X)_{s,t}^u\, \mathrm{Sig}(X)_{s,t}^v = \mathrm{Sig}(X)_{s,t}^{u \sqcup\!\sqcup v}$$

Motivation: Magnus expansion

Theorem (Hausdorff (1906))

Let $\Omega_{s,t} := \log \mathrm{Sig}_{s,t}(X) \in \mathfrak{g}$. *Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_{s,t} = H(-\operatorname{ad}\Omega_{s,t})\dot{X}_t, \quad \Omega_{s,s} = 0$$

where

$$H(z) := \frac{z}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

Theorem (Magnus (1954))

Expand $\Omega_{s,t} = \sum_n \Omega_{s,t}^{(n)}$. *Then*

$$\Omega_{s,t}^{(1)} = \int_s^t \dot{X}_u \,\mathrm{d}u = X_t - X_u$$

$$\Omega_{s,t}^{(n+1)} = \sum_{k=1}^{n} \frac{(-1)^k B_k}{k!} \sum_{\ell \vdash n} \int_s^t [\Omega_{s,u}^{(\ell_1)}, [\Omega_{s,u}^{(\ell_2)}, \cdots, [\Omega_{s,u}^{(\ell_k)}, \dot{X}_u] \cdots] \,\mathrm{d}u$$

Tensor-valued semimartingales

Definition

A real-valued process X is a semimartingale if it can be decomposed as $X = X_0 + M + A$ with M a càdlàg local martingale, A a càdlàg adapted process of locally bounded variation. The continuous martingale part of X is denoted by X^c. The space of semimartingales is denoted by $\mathscr{S}(\mathbb{R})$.

Definition (FHT (2020+))

A tensor-valued semimartingale is a series $\mathbf{X}$ where X^w is a semimartingale for all w.

Definition (see e.g. Protter's book)

Square bracket:

$$[X, Y] := XY - \int X_- \, \mathrm{d}Y - \int Y_- \, \mathrm{d}X$$

Angle bracket:

$$\langle X^c, Y^c \rangle := [X, Y] - \sum_{s \leq \cdot} \Delta X_s \Delta Y_s$$

Definition (FHT (2020+))

Outer bracket:

$$[\![\mathbf{X}, \mathbf{Y}]\!] := \sum_{u,v} [X^u, Y^v] u \otimes v$$

Inner bracket:

$$\langle \mathbf{X}^c, \mathbf{Y}^c \rangle := \sum_{w} \left(\sum_{uv=w} \langle X^{u,c}, Y^{v,c} \rangle \right) w$$

Generalized signatures

Definition (Friz–Shekhar (2017), FHT (2020+))

Let $\mathbf{X} \in \mathscr{S}(\mathfrak{t})$. Its generalized signature $\mathrm{Sig}(\mathbf{X})_{s,t}$ is the unique solution to the Marcus equation

$$\mathbf{S}_{s,t} = 1 + \int_{(s,t]} \mathbf{S}_{s,u-}\,\mathrm{d}\mathbf{X}_u + \frac{1}{2}\int_s^t \mathbf{S}_{u-}\,\mathrm{d}\langle \mathbf{X}^c\rangle_u + \sum_{s<u\le t} \mathbf{S}_{s,u-}(\exp(\Delta\mathbf{X}_u) - 1 - \Delta\mathbf{X}_u) =: 1 + \int_s^t \mathbf{S}_{s,u-} \circ \mathrm{d}\mathbf{X}_u.$$

Proposition (Chen (1953), Lyons (1998), FHT (2020+))

Let $\mathbf{X} \in \mathscr{S}(\mathfrak{t})$ *and* $s, u, t \in [0, T]$.

$$\mathrm{Sig}(\mathbf{X})_{s,u}\,\mathrm{Sig}(\mathbf{X})_{u,t} = \mathrm{Sig}(\mathbf{X})_{s,t}.$$

Definition (Hambly–Lyons (2010))

$$\boldsymbol{\mu}_t(T) := \mathbb{E}_t\,\mathrm{Sig}(\mathbf{X})_{t,T} \in T_1$$
$$\boldsymbol{\kappa}_t(T) := \log \boldsymbol{\mu}_t(T) \in \mathfrak{t}$$

Theorem (Bonnier–Oberhauser (2019))

$$\mu^w_{s,t} = \sum_{a \in \mathcal{OP}(w)} \frac{1}{|a|!}\kappa^a$$

Theorem (FHT (2020+))

Under suitable integrability conditions,

$$\boldsymbol{\mu}_t(T) \in \mathscr{S}(T_1), \quad \boldsymbol{\kappa}_t(T) \in \mathscr{S}(\mathfrak{t}).$$

Main result

Theorem (FHT (2020+))

For a sufficiently integrable $\mathfrak{t}$-valued semimartingale $\mathbf{X}$, $\boldsymbol{\kappa}_t(T)$ is the unique solution to the functional equation

$$\boldsymbol{\kappa}_t = \mathbb{E}_t\Big\{\int_{(t,T]} H(\operatorname{ad}\boldsymbol{\kappa}_{u-})(\mathrm{d}\mathbf{X}_u) + \frac{1}{2}\int_t^T H(\operatorname{ad}\boldsymbol{\kappa}_{u-})(\mathrm{d}\langle\mathbf{X}^c\rangle_u)$$
$$+ \frac{1}{2}\int_t^T H(\operatorname{ad}\boldsymbol{\kappa}_{u-})Q(\operatorname{ad}\boldsymbol{\kappa}_{u-})(\mathrm{d}[\![\boldsymbol{\kappa}]\!]^c_u) + \int_t^T H(\operatorname{ad}\boldsymbol{\kappa}_{u-})\circ(\operatorname{id}\odot G(\operatorname{ad}\boldsymbol{\kappa}_{u-}))(\mathrm{d}[\![\mathbf{X},\boldsymbol{\kappa}]\!]^c_u)$$
$$+ \sum_{t<u\le T}\big(H(\operatorname{ad}\boldsymbol{\kappa}_{u-})(\exp(\Delta\mathbf{X}_u)\exp(\boldsymbol{\kappa}_u)\exp(-\boldsymbol{\kappa}_{u-}) - 1 - \Delta\mathbf{X}_u) - \Delta\boldsymbol{\kappa}_u\big)\Big\}$$

$$G(z) := \frac{\mathrm{e}^z - 1}{z} = \sum_{k=0}^{\infty}\frac{z^n}{(n+1)!}, \qquad Q(z,\tilde{z}) := \sum_{n,m=0}^{\infty}\frac{z^n\tilde{z}^m}{(n+1)!m!(n+m+2)}, \qquad U\odot V(\mathbf{X}\otimes\mathbf{Y}) = U(\mathbf{X})V(\mathbf{Y}).$$

Main result: Recursion

Corollary (FHT (2020+))

The graded components $\boldsymbol{\kappa}_t^{(n)}(T)$ satisfy the recursion

$$\boldsymbol{\kappa}_t^{(1)} = \mathbb{E}_t[\mathbf{X}_{t,T}^{(1)}]$$

$$\boldsymbol{\kappa}_t^{(n)} = \mathbb{E}_t[\mathbf{X}_{t,T}^{(n)}] + \sum_{k=1}^{n} (\mathbf{X}^{(k)} \diamond \mathbf{X}^{(n-k)})_t(T) + \sum_{\mathcal{I} \vdash n} \mathbb{E}_t[\Omega(\mathcal{I}) + \mathbb{Q}(\mathcal{I}) + \mathbb{C}(\mathcal{I}) + \mathbb{J}(\mathcal{I})]$$

where

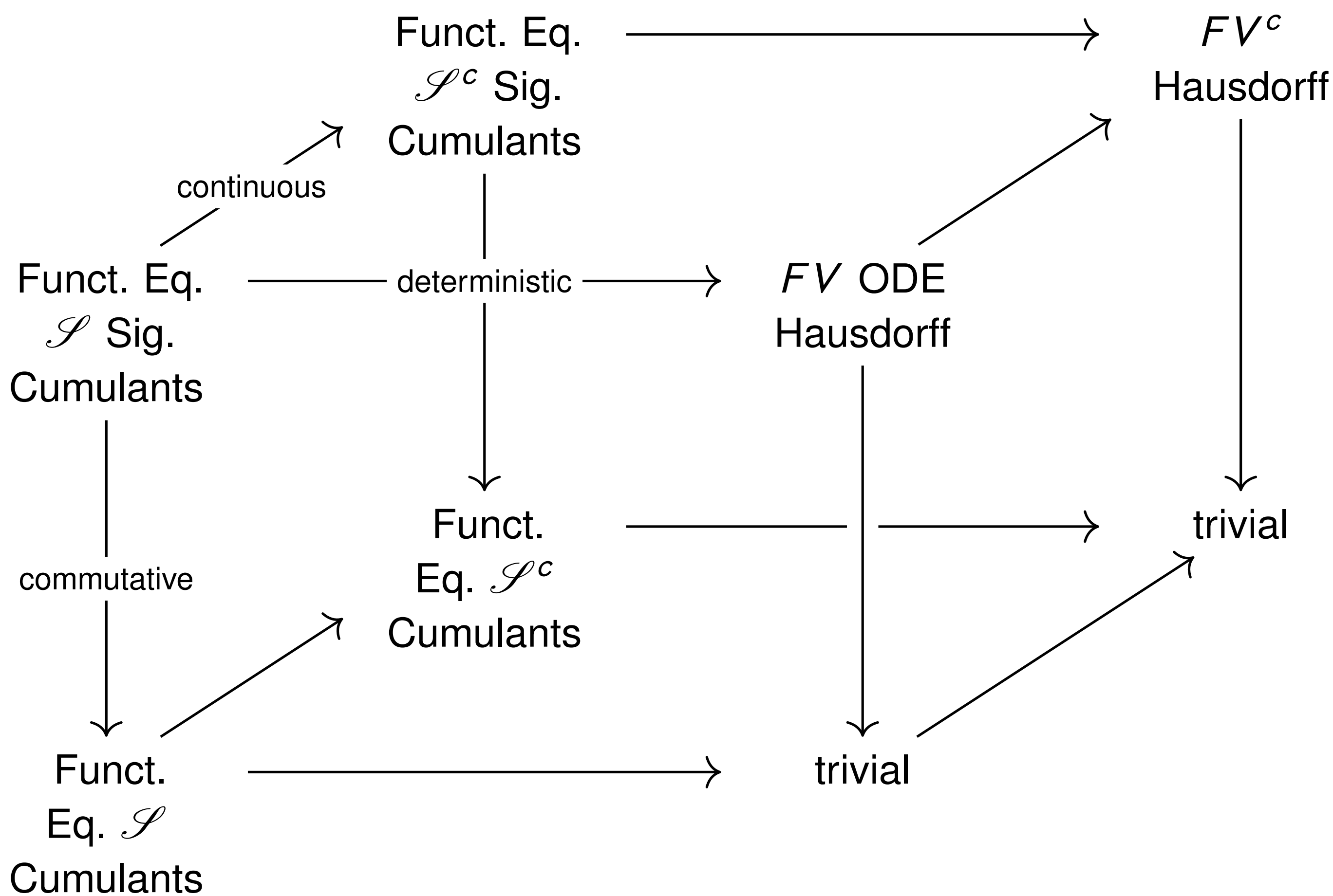
$$\Omega(\mathcal{I}) = \frac{1}{k!} \int_{(t,T]} \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_2)} \cdots \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_k)} (\mathrm{d}\mathbf{X}^{(i_1)})$$

$$\mathbb{Q}(\mathcal{I}) = \frac{1}{k!} \sum_{m=2}^{k} \binom{n-1}{m-1} \int_t^T \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_3)} \cdots \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_m)} \odot \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_{m+1})} \cdots \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_k)} (\mathrm{d}[\![\boldsymbol{\kappa}^{(i_1)}, \boldsymbol{\kappa}^{(i_2)}]\!]_u^c)$$

$$\mathbb{C}(\mathcal{I}) = \frac{1}{(k-1)!} \int_t^T (\mathrm{id} \odot \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_3)} \cdots \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_k)})(\mathrm{d}[\![\mathbf{X}^{(i_1)}, \boldsymbol{\kappa}^{(i_2)}]\!]_u^c)$$

$$\mathbb{J}(\mathcal{I}) = \sum_{t<u\leq T} \left(\sum_{1\leq m\leq j\leq k} (-1)^{k-j} \frac{\Delta \mathbf{X}_u^{(i_1)} \cdots \Delta \mathbf{X}_u^{(i_m)} \boldsymbol{\kappa}_u^{(i_{m+1})} \cdots \boldsymbol{\kappa}_u^{(i_j)} \boldsymbol{\kappa}_{u-}^{(i_{j+1})} \cdots \boldsymbol{\kappa}_{u-}^{(i_k)}}{m!(m-j)!(k-j)!} - \frac{1}{k!} \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_2)} \cdots \operatorname{ad} \boldsymbol{\kappa}_{u-}^{(i_k)} (\Delta \boldsymbol{\kappa}_u^{(i_1)}) \right)$$

Main result: Overwiew



Consequences: Time-inhomogeneous Lévy processes

Suppose $X \in \mathscr{S}(\mathbb{R}^d)$ is an Itô semimartingale with independent increments. Then

$$X_t = \int_0^t b(u)\,\mathrm{d}u + \int_0^t \sigma(u)\,\mathrm{d}B_u + \int_{(0,t]}\int_{|x|\le 1} x(\mu^X - \nu)(\mathrm{d}u,\mathrm{d}x) + \int_{(0,t]}\int_{|x|>1} x\mu^X(\mathrm{d}u,\mathrm{d}x).$$

where $b \in L^1, \sigma \in L^2$, μ^X is an independent inhomogeneous Poisson random measure with intensity measure ν, such that $\nu(\mathrm{d}u,\mathrm{d}x) = K_u(\mathrm{d}x)\,\mathrm{d}u$ and K_u are Lévy measures with

$$\int_{\mathbb{R}^d}(|x|^2 \wedge 1)K_u(\mathrm{d}x) < \infty, \quad \int_0^T \int_{|x|>1} |x|^n K_u(\mathrm{d}x)\mathrm{d}u < \infty.$$

Corollary (FHT (2020+), Friz–Shekhar (2017))

The signature cumulants satisfy

$$\boldsymbol{\kappa}_t = \int_t^T H(\operatorname{ad}\boldsymbol{\kappa}_u)(\mathfrak{h}(u))\,\mathrm{d}u,$$

where

$$\mathfrak{h}(u) := b(u) + \frac{1}{2}a(u) + \int_{\mathbb{R}^d}(\exp(x) - 1 - x1_{|x|\le 1})K_u(\mathrm{d}x), \qquad a = \sigma.\sigma^\top.$$

Consequences: Brownian motion

Let B be a standard BM and let $\mathrm{d}X_t = \sigma(t)\,\mathrm{d}B_t$.

Corollary (Fawcett (2002), FHT (2020+))

The signature cumulants of X satisfy the functional equation

$$\boldsymbol{\kappa}_t(T) = \frac{1}{2}\int_t^T H(\operatorname{ad}\boldsymbol{\kappa}_u)(a(u))\,\mathrm{d}u.$$

In particular, if $X = B$, i.e. $\sigma = I = \sum_{i=1}^d ii$ we recover Fawcett's formula

$$\boldsymbol{\kappa}_t(T) = \frac{1}{2}\sum_{i=1}^d (T-t)ii, \qquad \mathbb{E}_t \operatorname{Sig}(B)_{t,T} = \exp\left(\frac{1}{2}\sum_{i=1}^d (T-t)ii\right)$$

Theorem (Lyons–Ni (2015), FHT (2020+))

Let $\Gamma \subset \mathbb{R}^d$ bounded, regular domain and τ_Γ the first exit time of a BM B.
The signature cumulants $\boldsymbol{\kappa}_t = \log \mathbb{E}_t[\operatorname{Sig}(B)_{t\wedge\tau_\Gamma,\tau_\Gamma}]$ up to the first exit time from Γ have the form $\boldsymbol{\kappa}_t = 1_{\{t<\tau_\Gamma\}}\mathbf{F}(B_t)$ where

$$-\Delta\mathbf{F}(x) = \sum_{i=1}^d H(\operatorname{ad}\mathbf{F}(x))\Big(ii + Q(\operatorname{ad}\mathbf{F}(x))(\partial_i\mathbf{F}(x)^2) + 2iG(\operatorname{ad}\mathbf{F}(x))(\partial_i\mathbf{F}(x))\Big)$$

with boundary condition $\mathbf{F}|_{\partial\Gamma} = 0$.

Consequences: Symmetrization

Given $\mathbf{S} \in T((d))$ denote by $\hat{\mathbf{S}} \in S((d))$ its symmetrization.

Theorem (FHT (2020+))

We have,

$$\widehat{\mathrm{Sig}(\mathbf{X})}_{s,t} = \exp\big(\hat{\mathbf{X}}_{t,T}\big)$$

Moreover, if $\mathbf{X} = (0, X, 0, \dots)$,

$$\hat{\boldsymbol{\mu}}_t(T) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}_t[(X_T - X_t)^n].$$

Theorem (Friz–Gatheral–Radoičić (2019), FHT (2020+))

Let $\tilde{\mathbf{X}} \in \mathscr{S}(\hat{\mathfrak{t}})$ *and* $\mathbb{K}_t(T) := \log \mathbb{E}_t \exp(\tilde{\mathbf{X}}_T) = \tilde{\mathbf{X}}_t + \tilde{\boldsymbol{\kappa}}_t(T)$. *Then*

$$\mathbb{K}_t(T) = \frac{1}{2}(\mathbb{K} \diamond \mathbb{K})_t(T) + \sum_{t<u\leq T} \mathbb{E}_t[\exp(\Delta\mathbb{K}_u) - 1 - \Delta\mathbb{K}_u].$$

Consequences: Hausdorff and Magnus

Theorem (Hausdorff (1906), FHT (2020+))

Assume $\mathbf{X}$ *is a deterministic càdlàg path of bounded variation. The log-signature* $\Omega_{t,T} := \log \mathrm{Sig}(\mathbf{X})_{t,T}$ *satisfies*

$$\Omega_{t,T} = \int_{(t,T]} H(\operatorname{ad}\Omega_{u-,T})(\mathrm{d}\mathbf{X}_u) + \sum_{t<u\le T} H(\operatorname{ad}\Omega_{u-,T})(\exp(\Delta\mathbf{X}_u) - 1 - \Delta\mathbf{X}_u).$$

Corollary (Magnus (1954), FHT (2020+))

The graded components of Ω *satisfy*

$$\Omega^{(1)}_{t,T} = \mathbf{X}^{(1)}_T - \mathbf{X}^{(1)}_t$$

$$\Omega^{(n+1)}_{t,T} = \sum_{k=1}^{n} \frac{B_k}{k!} \sum_{I\vdash n} \int_{(t,T]} \operatorname{ad}\Omega^{(i_2)}_{t,T} \cdots \operatorname{ad}\Omega^{(i_k)}_{t,T}(\mathrm{d}\mathbf{X}^{(i_1)}_u)$$
$$+ \sum_{2\le m\le k\le n} \frac{B_{k-m}}{(k-m)!m!} \sum_{I\vdash n} \operatorname{ad}\Omega^{(i_{m+1})}_{t,T} \cdots \operatorname{ad}\Omega^{(i_k)}_{t,T}(\Delta\mathbf{X}^{(i_1)}_u \cdots \Delta\mathbf{X}^{(i_m)}_u).$$

Remarks: pre- vs. post-Lie algebras

For $\mathbf{X}, \mathbf{Y} \in \mathscr{S}(\mathfrak{t})$ let

$$\mathbf{X} \succeq \mathbf{Y} := \int \mathbf{X}_- \circ \mathrm{d}\mathbf{Y}, \quad \mathbf{X} \preceq \mathbf{Y} := \int \circ \mathrm{d}\mathbf{X}\, \mathbf{Y}_-.$$

Then $\mathbf{X} * \mathbf{Y} = \mathbf{X} \succeq \mathbf{Y} + \mathbf{X} \preceq \mathbf{Y} = \mathbf{X}\mathbf{Y}$.

The signature is a solution to the *fixed point equation*

$$\mathbf{S} = 1 + \mathbf{S} \succeq \mathbf{X}$$

We write $\mathbf{S} = \mathcal{E}_{\succeq}(\mathbf{X})$.

Theorem (Manchon–Ebrahimi-Fard (2007))

There is a unique element $\Omega_{\triangleright}(\mathbf{X})$ *such that* $\mathbf{S} = \exp(\Omega_{\triangleright}(\mathbf{X}))$, *with* $a \triangleright b = a \succeq b - b \preceq a$.

Define also

$$\mathbf{X} \succ \mathbf{Y} := \int \mathbf{X}_- \, \mathrm{d}\mathbf{Y}, \quad \mathbf{X} \prec \mathbf{Y} := \int \mathrm{d}\mathbf{X}\, \mathbf{Y}_-,$$
$$\mathbf{X} \bullet \mathbf{Y} := [\mathbf{X}, \mathbf{Y}].$$

Then $\mathbf{X} \circledast \mathbf{Y} = \mathbf{X} \succ \mathbf{Y} + \mathbf{X} \prec \mathbf{Y} + \mathbf{X} \bullet \mathbf{Y} = \mathbf{X}\mathbf{Y}$.

The signature is a solution to the *fixed point equation*

$$\mathbf{S} = 1 + \mathbf{S} \succ (\exp_{\bullet}(\mathbf{X}) - 1)$$

We write $\mathbf{S} = \mathcal{E}_{\succ}(\mathbf{X})$.

Conjecture

There is a unique element $\Omega_{\blacktriangleright}(\mathbf{X})$ such that $\mathcal{E}_{\succ}(\mathbf{X}) = \exp(\Omega_{\blacktriangleright}(\mathbf{X}))$:

$$\Omega_{\blacktriangleright}(\mathbf{X}) := \Omega_{\triangleright}(\log_{\bullet}(1 + \mathbf{X}))$$