What is a Sample?

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A longer title

A longer, and more accurate, title is:

How to express what is a sample of a
real-valued random variable?
A (continuous?) multiplicative linear functional?

So the focus is on real-valued random variables (RVs).
It is about how to cast the notion of of RVs and their samples / observations into a mathematical framework.

How to formalise a random variable and its observation?
What can we do with it and how can we represent it?

Overview

1. Random variables — Kolmogorov and alternatives
2. Algebra of random variables (with a bit of history)
3. Representations of (commuting) random variables
4. Vector spaces of random variables, operational calculus
5. Observations and spectrum — spectral theory
6. Generalised random variables and duality
7. A glimpse at quantum computers

Preliminaries

We will use the following convention:

In the real world there are entities which can be observed or measured, called observables.

The result of such a measurement will be called an observation.

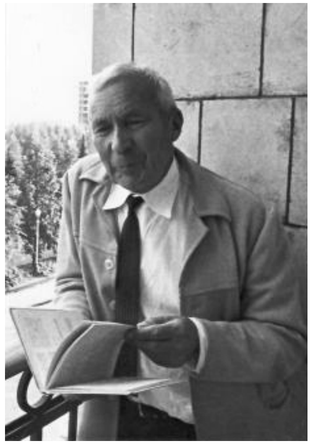
The mathematical model of an observable will be called a random variable (RV).

The mathematical equivalent of an observation will be call a sample of the RV.

Kolmogorov's definition of a random variable

Traditionally, probability theory has the notion of a measure space Ω, σ-algebra $\mathfrak{A}$ of subsets, and a probability measure $\mathbb{P}$, as prime objects. The notion of measure originally arose from measuring lengths, areas, volumes, etc.

Kolmogorov: the first (1930s) rigorous and by now classical definition of a (commutative) RV: A (real valued) RV is a measurable function $r \in \mathrm{L}_0(\Omega)$ on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ into $(\mathbb{R}, \mathfrak{B})$ with the Borel-σ-algebra $\mathfrak{B}$.



Andrej N. Kolmogorov
(1903 – 1987)

Expectation and sample

Given a probability measure $\mathbb{P}$, one may define the expectation $\mathbb{E}\left(\varphi(r)\right) := \int_\Omega \varphi(r(\omega))\,\mathbb{P}(\mathrm{d}\,\omega)$ for any measurable function $\varphi \in \mathrm{L}_0(\mathbb{R})$.

Obviously, $\mathbb{E}\left(\cdot\right)$ is linear, and it is also positive $\mathbb{E}\left(r^2\right) \geq 0$, and normalised $\mathbb{E}\left(\mathbb{1}_\Omega\right) = 1$.

For simplicity, assume that Ω is a compact Hausdorff space, and $\mathbb{P}$ a Radon probability measure. Then all continuous functions $r \in \mathrm{C}(\Omega)$ are RVs, and $\mathbb{E}\left(\cdot\right) : \mathrm{C}(\Omega) \to \mathbb{R}$ is continuous.
Conversely, if $\phi \in \mathrm{C}(\Omega)^*$ is positive and normalised, $\exists\, \mathbb{P}$, a Radon probability measure s.t. $\phi(r) = \int_\Omega r(\omega)\,\mathbb{P}(\mathrm{d}\omega)$.

For $(r \in \mathrm{C}(\Omega), \omega \in \Omega)$, the evaluation $\mathbb{R} \ni r(\omega) = \langle \delta_\omega \mid r \rangle$ is a sample.
$\omega \mapsto (\delta_\omega : \mathrm{C}(\Omega) \to \mathbb{R})$ is linear, continuous, positive; $\langle \delta_\omega \mid \mathbb{1}_\Omega \rangle = 1$.

Some problems

What if $r \notin \mathrm{C}(\Omega)$, but only $r \in \mathrm{L}_\infty(\Omega)$, or even $r \in \mathrm{L}_0(\Omega)$?
Then what does $r(\omega)$ mean? Although Hahn-Banach asserts existence of an extension of δ_ω for $\mathrm{L}_\infty(\Omega)$, there are many possible ones.

Kolmogorov's definition was also found to be too narrow for quantum theory, as observables may not commute.

Thus an alternative point of view arose, which has RVs and expectation as prime objects; or rather, in physics lingo, observables and states.

Incidentally, a similar view was implicitly present at the beginnings of probability theory.

The Bernoullis' view of a random variable

RVs — as implicitly used by the Bernoullis —
can be, with rules like numbers

- a) added to each other,
- b) multiplied by numbers,
- c) multiplied by themselves,
- d) 'averaged'.
- e) and constants 'are' RVs.

Jakob Bernoulli
(1655 – 1705)

Nikolaus (II) Bernoulli
(1695 – 1726)
nephew of Jakob

Daniel Bernoulli
(1700 – 1782)
brother of Nikolaus II

Mathematically (in modern lingo) this means:

1. a) and b) ⇒ RVs form a vector space.
2. c) ⇒ RVs are an associative, distributive algebra.
3. d) ⇒ existence of a state / expectation, a positive linear functional.
4. e) ⇒ existence of a unit / identity element.

Expectation and observations / samples

More of the Bernoullis' implicit rules in modern language:

- Constant RV a always is observed with same value $\alpha \in \mathbb{R}$, i.e. there is a unit constant e, s.t. $a = \alpha e$. And e is multiplicative unit: $a = a \cdot e$.
- RVs can be ordered: a RV is positive $a \geq 0 = 0e$ iff $a = b \cdot b = b^2$. $a \geq c$ iff $(a - c) \geq 0$. RVs form a lattice with $\sup$ and $\inf$.
- The 'average' $\phi \in \mathcal{A}^*$ is called a state in modern physics lingo, defines expected value $\mathbb{E}(a) := \phi(a)$ with $\phi(e) = 1$ and $\phi(b^2) \geq 0$.
- From observations $\omega(a) = a$ and $\omega(b) = b$, one has observations $\omega(\alpha a + b) = \alpha a + b$ and $\omega(a \cdot b) = ab$. Also $\omega(e) = 1$ and $\omega(b^2) \geq 0$.
- This means that an observation $\omega \in \mathcal{A}^*$ is a multiplicative state, an algebra homomorphism $\omega : \mathcal{A} \to \mathbb{R}$ (a character / pure state).

A mathematical formalisation should capture these properties.

Where are we going?

Starting from a probability algebra $\mathcal{A}$, we look at representations $\mathcal{A} \to \mathsf{L}(\mathcal{K})$ and topologies on $\mathcal{A}$, and characterise states $S(\mathcal{A})$ and the spectrum (characters) $\hat{\mathcal{A}}$ on the unit ball of the dual $\mathcal{A}^*$, as all possible samples of all of $\mathcal{A}$; connecting this with $\sigma(a) = \{z \mid a - ze$ not invertible in $\mathcal{A}\}$, the spectrum of a single $a \in \mathcal{A}$ (all possible samples of a). We also want to be able to compute functions $\varphi(a)$ of RVs $a \in \mathcal{A}$.

As classical RVs commute, and as in quantum physics simultaneous observations are only possible on commuting observables—for non-commutative $\mathcal{A}$ typically $\hat{\mathcal{A}} = \emptyset$ — we concentrate on representations of commutative algebras.

The main connection will be the representation on the multiplication algebra $\mathrm{L}_\infty(\Omega)$ and the multiplicative version of the spectral theorem.

A bit of algebra

Definition: associative, distributive algebra $\mathcal{A}$ over a field ($\mathbb{K}$-algebra) is a $\mathbb{K}$-vector space ($\mathbb{K}$ here $\mathbb{R}$ or $\mathbb{C}$) with bi-linear associative multiplication $\mathcal{A} \ni a, b \mapsto a \cdot b \in \mathcal{A}$, i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ distributive: $(a + b) \cdot c = a \cdot c + b \cdot c$ and $c \cdot (a + b) = c \cdot a + c \cdot b$
$\mathcal{A}$ is a unital algebra iff $\exists!\, e \in \mathcal{A}$ (identity element): $a \cdot e = e \cdot a = a$.
$\mathcal{A}$ is Abelian or commutative iff $[a, b] := a \cdot b - b \cdot a = 0$.
Note that $(\mathcal{A}, [\cdot, \cdot])$ is a Lie algebra.

Definition: (left) regular representation Υ of $\mathcal{A}$ as linear maps $\mathrm{L}(\mathcal{A})$ on $\mathcal{A}$: $\Upsilon : \mathcal{A} \ni a \mapsto L_a \in \mathrm{L}(\mathcal{A})$ $(L_a b := a \cdot b)$ is an algebra homomorphism.

Definition: A probability algebra $\mathcal{A}$ is an associative (usually complex) unital algebra (with unit element e) and a positive linear functional ϕ — called state or expectation — such that $\mathbb{E}_\phi(e) := \phi(e) = 1$.
'Samples' are algebra homomrph. $\omega_\iota : \mathcal{A} \to \mathbb{K}$ (characters / pure states).

Refinements

Observe that $\mathrm{C}(\Omega)$, $\mathrm{L}_\infty(\Omega)$, and $\mathrm{L}_0(\Omega)$ are algebras, that integral is a state, and $\delta_\omega : \mathrm{C}(\Omega) \to \mathbb{R}$ is an algebra homomorphism.

In quantum physics — although observations are real numbers — one uses complex quantities, and the complex structure is essential for superposition.

Paraphrased from J. Hadamard / P. Painlevé:
“The shortest path between two truths of the real domain quite often passes through the complex domain.”

Therefore we take $\mathbb{K} = \mathbb{C}$, and any real algebra $\mathcal{A}_\mathbb{R}$ may be embedded in a complex one $\mathcal{A} = \mathcal{A}_\mathbb{R} \oplus \mathrm{i}\mathcal{A}_\mathbb{R}$. Additionally we assume an anti-linear involution $(a^*)^* = a$ ($\mathcal{A}$ is a $*$-algebra), such that for $a, b \in \mathcal{A}$, $z \in \mathbb{C}$:

$$(a \cdot b)^* = b^* \cdot a^*, \quad (za)^* = \bar{z}a^*, \text{ and } \phi(a^*) = \overline{\phi(a)}.$$

Example probability algebras

1) $\mathrm{C}(\Omega) \subset \mathrm{L}_\infty(\Omega) \subset \mathrm{L}_{\infty-}(\Omega) := \bigcap_{p\geq 1} \mathrm{L}_p(\Omega)$, all commutative.

2) Complex $n \times n$-matrices with matrix multiplication: $\mathcal{A} = \mathbb{M}(n, \mathbb{C})$ with unit $e \equiv \boldsymbol{I}$, $\phi(\boldsymbol{A}) := \mathrm{tr}(\boldsymbol{A})/n$, and $\boldsymbol{A}^* := \bar{\boldsymbol{A}}^\top$.

A commutative sub-algebra: diagonal matrices.
If $\mathbb{C}^n \ni \boldsymbol{v} \neq 0$ is an eigenvector of a normal $\boldsymbol{A}$ ($\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A}^*\boldsymbol{A}$), then on the commutative sub-algebra $\mathbb{C}[\boldsymbol{A}, \boldsymbol{A}^*]$ the Rayleigh-quotient $\omega_{\boldsymbol{v}}(\boldsymbol{A}) = \boldsymbol{v}^*\boldsymbol{A}\boldsymbol{v}/\boldsymbol{v}^*\boldsymbol{v}$ is a character / pure state.

3) Probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ with simple (step) functions $\mathbb{1}_{\mathcal{E}}$ for events $\mathcal{E} \in \mathfrak{A}$ gives a commutative probability algebra ($e \equiv \mathbb{1}_\Omega$):
$\mathcal{A} = \mathrm{L}_s(\Omega) = \{r(\omega) = \sum_{k=1}^{K} \xi_k \, \mathbb{1}_{\mathcal{E}_k}(\omega) \mid \xi_k \in \mathbb{C}, \mathcal{E}_k \in \mathfrak{S}\} \subseteq \mathrm{L}_\infty(\Omega)$.
$\mathcal{E}_k$ can be chosen disjoint, involution is complex conjugation, and $\phi(r) := \sum_{k=1}^{K} \xi_k \, \mathbb{P}(\mathcal{E}_k) = \int_\Omega r(\omega) \, \mathbb{P}(\mathrm{d}\,\omega)$. Samples are $\omega_k(r) = \xi_k$.

More on algebras

Let $\mathcal{A}$ be a $*$-algebra, and $a, b \in \mathcal{A}$.

- powers $a^n; n \geq 0$, defines polynomials $p(a)$; and inverse $a^{-1} \cdot a = e$.
- $\mathcal{A}$ is a Banach-algebra iff it is a Banach-space and $\|a \cdot b\| \leq \|a\| \, \|b\|$;
- A Banach-*-algebra $\mathcal{A}$ is a C*-algebra iff $\|a^* \cdot a\| = \|a^*\| \, \|a\| = \|a\|^2$;
- $\mathcal{A}$ is a W*- or von Neumann algebra iff it is a C*-algebra which is the dual ($\mathcal{A} = (\mathcal{A}_*)^*$) of another Banach space $\mathcal{A}_*$. Main ex.: $\mathscr{L}(\mathcal{H})$.
- normal, self-adjoint, and unitary elements as for matrices.
- self-adjoint elements $a \in \mathcal{A}$ such that $a = b^* \cdot b$ are called positive; 0 and e are positive, and positive elements form a convex cone $\mathcal{P}$
- positive elements $p \in \mathcal{A}$ such that $p = p \cdot p$ are called projections.
- spectrum of $a \in \mathcal{A}$: $\sigma(a) = \{z \in \mathbb{C} \mid a - ze$ has no inverse in $\mathcal{A}\}$

Goal is to get new RVs by computing functions $f(a)$ (via W*-algebras).

More on functionals and states

Each linear functional $\beta \in \mathcal{A}^*$ defines a sesqui-linear form b on $\mathcal{A} \times \mathcal{A}$, and in turn a *linear map* $B : \mathcal{A} \to \mathcal{A}^*$:

$$\beta(c^* \cdot a) =: b(a, c) =: \langle Ba, c^* \rangle_{(\mathcal{A}^*, \mathcal{A})}\,; \quad \forall a, c \in \mathcal{A}.$$

In case B is Hermitian (self-adjoint) or positive, the same is attached to b and $\beta \in \mathcal{A}^*$.

A Hermitian, strictly positive definite state $\phi \in \mathcal{A}^*$ (a faithful state) defines an inner product $\langle \cdot \mid \cdot \rangle_2$ on $\mathcal{A}$:

$\langle a \mid c \rangle_2 := \phi(c^* \cdot a) = \langle \Phi a, c^* \rangle_{(\mathcal{A}^*, \mathcal{A})}\,, \quad \Phi \in \mathscr{L}(\mathcal{A}, \mathcal{A}^*)$, and Hilbert space completion $\mathcal{H} := \mathrm{L}_2(\mathcal{A}, \phi) := \mathrm{cl}_2\{a \in \mathcal{A} \,:\, \|a\|_2 < \infty\}$.

(Possibly factor out $\{a \mid \phi(a^* \cdot a) = \|a\|_2 = 0\}$)

Representation: this is a *-algebra homomorphism $\mathcal{A} \to \mathsf{L}(\mathcal{K})$.

A 1D-representation $\mathcal{A} \to \mathscr{L}(\mathbb{C}) \cong \mathbb{C}$ is called a character.

Regular (left) representation: $\Upsilon : \mathcal{A} \ni a \mapsto L_a \in \mathsf{L}(\mathcal{A}),\; L_a b := a \cdot b$.

Topologies, states, and spectrum

- First topologies on $\mathcal{A}$: one from $\mathrm{L}_2(\mathcal{A}, \phi)$. Others from regular representation $\Upsilon : \mathcal{A} \to \mathsf{L}(\mathcal{A})$. Concentrate on bounded subalgebra

$$\mathcal{A}_b = \{a \in \mathcal{A} \mid \sup(\|L_a b\|_2 \,/\, \|b\|_2) = \|L_a\| < \infty\}, \quad \|a\|_\infty := \|L_a\|;$$

  so $\Upsilon(\mathcal{A}_b) \subseteq \mathscr{L}(\mathcal{H})$. Set $\mathcal{A}_\infty = \mathrm{cl}_\infty \mathcal{A}_b$, a C*-algebra.
- States $S(\mathcal{A})$: positive Hermitean functionals $\phi \in \mathcal{A}^*$ with: $\phi(e) = 1, \quad \phi(a^* \cdot a) \geq 0 \;\Rightarrow\; a = a^* : \; \phi(a) \in \mathbb{R}$. ϕ is faithful iff $\phi(a^* \cdot a) = 0 \Leftrightarrow a = 0$. On $\mathcal{A}_\infty$ thus $\phi(e) = 1 \Rightarrow \|\phi\|_* = 1$, and $\emptyset \neq S(\mathcal{A}_\infty) \subset \mathcal{A}_\infty^*$ is w*-compact and convex.
- Extreme points $\mathrm{ext}(S(\mathcal{A}))$: On a complex commutative *-algebra, ϕ extreme point of $S(\mathcal{A}) \Leftrightarrow \phi$ is a character / pure state ($\phi \in \hat{\mathcal{A}}$).
- Spectrum $\hat{\mathcal{A}} \subset S(\mathcal{A})$ (Character / 1D-representation / pure state): On $\mathcal{A}_\infty$ one has $\emptyset \neq \hat{\mathcal{A}}_\infty = \mathrm{ext}(S(\mathcal{A}_\infty)) \subseteq S(\mathcal{A}_\infty)$ is w*-compact; and $\ker \omega \subset \mathcal{A}_\infty$ is maximal ideal. Krein-Milman: $S(\mathcal{A}_\infty) = \overline{\mathrm{co}}^* \hat{\mathcal{A}}_\infty$.

More about the spectrum

Spectrum of

- $u^{-1} = u^*$ unitary $\Rightarrow \sigma(u) \subseteq \mathbb{T}_1 := \{z \in \mathbb{C} : |z| = 1\}$,
- $a = a^*$ self-adjoint $\Rightarrow \sigma(a) \subseteq \mathbb{R}$,
- $a = b^* b$ positive $\Rightarrow \sigma(a) \subseteq \mathbb{R}_+ := [0, \infty[$,
- $p = p^2$ projection $\Rightarrow \sigma(p) \subseteq \{0, 1\}$ — $(\sigma(\alpha e) = \alpha)$.

View $a \in \mathcal{A}$ as random variables —only self-adjoint ones are observable— and $\mathbb{E}_\phi(a) = \phi(a)$ as expectation (when the "knowledge" is in state ϕ).

Projections p can be seen as events, as $0 \leq \mathbb{P}(p) := \mathbb{E}_\phi(p) \leq 1$.

Split $a = \bar{a} + \tilde{a}$ with $\bar{a} = \phi(a)e$ and $\phi(\tilde{a}) = 0$: $\mathcal{A} = \bar{\mathcal{A}} \oplus \tilde{\mathcal{A}}$, $\tilde{\mathcal{A}} = \ker \phi$.

$a_1, a_2 \in \mathcal{A}$ are uncorrelated iff $\langle \tilde{a}_1 \mid \tilde{a}_2 \rangle_2 = 0$, and are independent iff $\left\langle \widetilde{\mathbb{C}[a_1]} \mid \widetilde{\mathbb{C}[a_2]} \right\rangle_2 = 0$ and $[a_1, a_2] = 0$ ($\Rightarrow \phi(a_1 \cdot a_2) = \phi(a_1)\phi(a_1)$).

Observations / samples are pure states, so $\{\omega(a) \mid \omega \in \hat{\mathcal{A}}\} = \sigma(a)$.

Multiplication algebra

For simplicity, assume Ω is a compact Hausdorff space, $\mathbb{P}$ a Radon probability measure as before.

Multiplication algebra $\mathrm{L}_\infty(\Omega)$: Take $\mathcal{R} = \mathrm{L}_2(\Omega)$ and define for $k \in \mathrm{L}_\infty(\Omega)$ a continuous linear map $M_k \in \mathscr{L}(\mathcal{R})$:

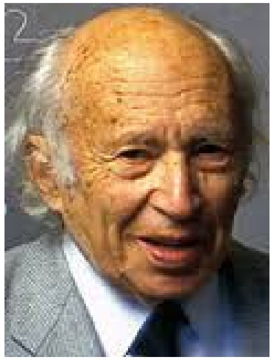
$$M_k : \mathcal{R} \ni f \mapsto M_k(f) := kf \in \mathcal{R} \quad \Rightarrow \|M_k\|_{\mathscr{L}} = \|k\|_\infty .$$

One has $M_k^* = M_{\bar{k}}$. The injective C*-algebra morphism $\mu : \mathrm{L}_\infty(\Omega) \ni k \mapsto M_k \in \mathscr{L}(\mathcal{R})$ is the multiplicator representation, and $\mu(\mathrm{L}_\infty(\Omega)) = \mathcal{M} = \{M_k \mid k \in \mathrm{L}_\infty(\Omega)\}$ is called the multiplication algebra of $\mathcal{R}$, isometrically isomorphic to $\mathrm{L}_\infty(\Omega)$.

The C*-algebra $\mu(\mathrm{L}_\infty(\Omega, \mathbb{R}))$ is maximal Abelian self-adjoint — MASA.

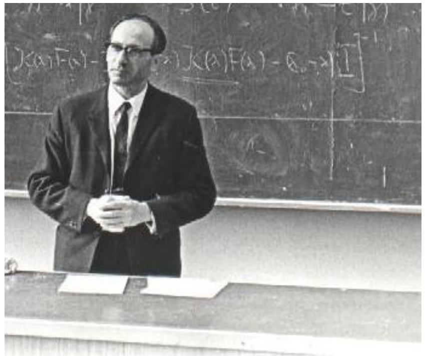
The C*-sub-algebra $\mu(\mathrm{C}(\Omega)) \subset \mathcal{M}$ is called the uniform sub-algebra.

GNS-construction



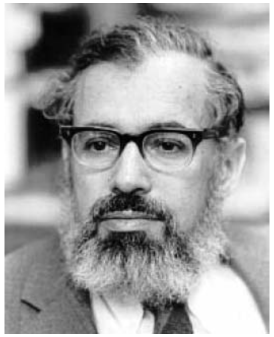
Izrail' M. Gel'fand

(1913 – 2009)



Mark A. Najmark

(1909 – 1978)



Irving E. Segal

(1918 – 1998)

GNS — Gel'fand-Najmark-Segal

Starting from a probability algebra $\mathcal{A}$ with faithful state $\phi \in S(\mathcal{A})$,

- construct $\|\cdot\|_2$, Hilbert space $\mathcal{H} = \mathrm{L}_2(\mathcal{A}, \phi)$, and W*A $\mathscr{L}(\mathcal{H})$,
- construct bounded $\mathcal{A}_b$ s.t. $\Upsilon(\mathcal{A}_b) \subseteq \mathscr{L}(\mathcal{H})$, $\|\cdot\|_\infty$, C*A $\mathcal{A}_\infty = \mathrm{cl}_\infty \mathcal{A}_b$,

Regular C*-representation Υ: C*-algebra $\mathcal{A}_\infty \ni a \mapsto L_a \in \mathscr{L}(\mathcal{H})$ with $L_a b := a \cdot b$, then $\mathbb{E}_\phi(a) = \phi(a) = \phi(e^* \cdot a \cdot e) = \langle L_a e \mid e \rangle_2 = \langle a \mid e \rangle_2$, (vector state), and $\mathcal{A}_\infty$ repr. as a C*-sub-algebra $\Upsilon(\mathcal{A}_\infty) \subseteq \mathscr{L}(\mathcal{H})$.

Gel’fand representation of Abelian algebras

As classical RVs commute, and as in quantum physics simultaneous observations are only possible on commuting observables, assume that $\mathcal{B}$ is a complex Abelian (commutative) C*-algebra, possibly produced as uniform closure of bounded RVs via the GNS-construction.
Gel’fand representation: spectrum $\Omega := \hat{\mathcal{B}}$ is a w*-compact Hausdorff subspace of $\mathcal{B}^*$, each $a \in \mathcal{B}$ is in $\mathrm{C}(\Omega;\mathbb{C}) = \mathrm{C}(\Omega)$. The representation

$$\gamma : \mathcal{B} \to \mathrm{C}(\Omega), \qquad \gamma(a) : \Omega \ni \omega \mapsto \langle \delta_\omega, a \rangle_{(\mathcal{B}^*,\mathcal{B})} =: a(\omega) \in \mathbb{C};$$

is an isometric isomorphism of C*-algebras (via Stone-Weierstrass).
$\mu \circ \gamma : \mathcal{B} \to \mathcal{M}$ on $\mathcal{H}$ is an iso-representation in the uniform algebra.
For $a \in \mathcal{B}$, spectrum $\sigma(a) = \operatorname{im} \gamma(a) = \gamma(a)(\Omega) \subset \mathbb{C}$ (range of $\gamma(a)$).
For bounded, normal a, and $\mathcal{B} = \mathrm{C}^*\mathbb{C}[a, a^*] := \mathrm{cl}_\infty \mathbb{C}[a, a^*]$, there is an iso-repr. $\nu : \mathrm{C}(\hat{\mathcal{B}}) \to \mathrm{C}(\sigma(a))$ s.t. $\nu(a)$ is $z \mapsto z$ and $\nu(a^*)$ is $z \mapsto \bar{z}$.
For $\mathrm{C}(\Omega)$ the spectrum $\widehat{\mathrm{C}(\Omega)} \cong \Omega$ are all $\delta_\omega \in \mathrm{C}(\Omega)^*$, $\omega \in \Omega$.

Continuous spectral calculus and L_p-spaces

If $\varphi \in \mathrm{C}(\sigma(a))$, then define $\varphi(a) := \gamma^{-1} \circ \varphi \circ \gamma(a) \in \mathcal{B}$
— the continuous operational calculus, and $\sigma(\varphi(a)) = \varphi(\sigma(a))$ —
If φ is real-valued, $\varphi(a)$ is self-adjoint, if $|\varphi| = 1$, $\varphi(a)$ is unitary.

This allows (also for non-Abelian) C*-algebras $\mathcal{A}$ to define
continuous functions of normal $a \in \mathcal{A}$ by computing it
on the Abelian C*-sub-algebra $\mathcal{B} = \mathrm{C}^*\mathbb{C}[a, a^*]$;
and as $c = (a^* a) \geq 0$ is positive and self-adjoint, hence normal,
for any $a \in \mathcal{A}$, the absolute value is $|a| = c^{1/2} \in \mathcal{B} \subseteq \mathcal{A}$.

For any C*-probability algebra $\mathcal{A}$ one may now define for $0 < p < \infty$:
$\|a\|_p = (\phi(|a|^p))^{1/p} = (\mathbb{E}_\phi(|a|^p))^{1/p}$ and $\mathrm{L}_p(\mathcal{A}, \phi) = \mathrm{cl}_p\, \mathcal{A}$,
and $\mathrm{L}_{\infty-}(\mathcal{A}, \phi) := \bigcap_{p \geq 1} \mathrm{L}_p(\mathcal{A}, \phi)$, the largest algebra in
$\mathrm{L}_0(\mathcal{A}, \phi) := \mathrm{cl}_d\, \mathcal{A}$ with metric $d(a, b) = \mathbb{E}_\phi\left({|a-b|}/{(1+|a-b|)}\right)$.

L_∞-space and weak closure

$\mathrm{C}(\Omega) \subset \mathrm{L}_\infty(\Omega)$ is a C*-algebra, hence a proper closed subspace in the $\|\cdot\|_\infty$-norm in the W*-algebra $\mathrm{L}_\infty(\Omega)$. So for $\mathrm{L}_\infty(\mathcal{A}, \phi)$ a bit more care is needed, as a C*-probability algebra $\mathcal{A}$ with norm $\|\cdot\|_\infty$ is a Banach space and hence complete. For any $\mathcal{C} \subseteq \mathscr{L}(\mathcal{H})$ the commutant

$$\mathcal{C}' := \{A \in \mathscr{L}(\mathcal{H}) \mid [A, C] = AC - CA = 0 \quad \forall C \in \mathcal{C}\} \subset \mathscr{L}(\mathcal{H})$$

is a C*-sub-algebra, and the double commutant $\mathcal{C}'' \supseteq \mathcal{C}$ is a W*-sub-algebra, and also $\mathcal{C}'' = \mathrm{W}^*\mathcal{C} := \mathrm{cl}_{\tau_w} \mathcal{C}$, the closure / completion in the weak operator topology τ_w defined by the semi-norms $|C|_{f,g} = |\langle Cf \mid g\rangle|$ for $C \in \mathscr{L}(\mathcal{H})$ and $f, g \in \mathcal{H}$.

Thus, for $\mathcal{A}$ define topology τ_w with semi-norms $|a|_{f,g} = |\phi(g^* \cdot a \cdot f)|$ $(a, f, g \in \mathcal{A})$ and the completion $\mathrm{L}_\infty(\mathcal{A}, \phi) := \mathrm{cl}_{\tau_w} \mathcal{A}$ — a W*-algebra.

Borel spectral calculus

If $\mathcal{B}$ is a commutative C*-probability algebra, the state / expectation defines a continuous, positive, and normalised functional $\mathbb{E}_\Omega := \phi \circ \gamma^{-1} \in \mathrm{C}(\Omega)^*$, hence $\exists\, \mathbb{P}_\phi$ — Radon probability measure. Hence one may define all the spaces $\mathrm{L}_p(\Omega)$ $(0 \leq p \leq \infty)$ as completions.

What is the spectrum — and hence possible sample — of $\varphi \in \mathrm{L}_\infty(\Omega)$?

$$\sigma(\varphi) = \{z \in \mathbb{C} \,:\, (\varphi - z\mathbb{1}_\Omega)^{-1} \notin \mathrm{L}_\infty(\Omega)\} =$$

$$\{z \in \mathbb{C} \,:\, \varphi_* \mathbb{P}(U) = \mathbb{P}(\varphi^{-1}(U)) > 0 \quad \forall \text{ neighbourhoods } U \subset \mathbb{C} \text{ of } z\},$$

the essential range or spectrum of $\varphi \in \mathrm{L}_\infty(\Omega)$ —may be extended to $\mathrm{L}_p(\Omega)$ $(0 \leq p < \infty)$.

For $0 \leq p \leq \infty$ and $\varphi \in \mathrm{L}_p(\sigma(a))$ define $\varphi(a) \in \mathrm{L}_p(\mathcal{B}, \phi)$ as the limit of a net $\gamma^{-1} \circ \nu^{-1}(\tilde{\varphi}_\iota)$, where $\tilde{\varphi}_\iota \in \mathrm{C}(\sigma(a))$ is a net converging to φ in appropriate topology — the bounded Borel operational calculus.

Uncertainty relation

For s.a. RV $a \in \mathcal{A}_\infty$ define uncertainty $\varsigma(a)_\phi$ ($\tilde{a} = a - \bar{a} = a - \phi(a)e$) by

$$\varsigma(a)^2_\phi := \mathbb{E}_\phi(\tilde{a}^2) = \phi(\tilde{a} \cdot \tilde{a}) = \phi(\tilde{a}^* \cdot \tilde{a}) = \phi(a^2) - \phi(a)^2 = \langle \tilde{a} \mid \tilde{a} \rangle_2.$$

If ϕ is multiplicative — an extreme state resp. a sample $\phi = \omega \in \hat{\mathcal{A}}_\infty = \mathrm{ext}(S(\mathcal{A}_\infty))$ — one has $\varsigma(a)^2_\omega = \omega(\tilde{a}^2) = \omega(\tilde{a})\omega(\tilde{a}) = 0$: observation or sampling without uncertainty is possible.

For combined uncertainty of two s.a. RVs $a, b \in \mathcal{A}_\infty$ one has:

$$\varsigma(a)^2_\phi \varsigma(b)^2_\phi = \langle \tilde{a} \mid \tilde{a} \rangle_2 \langle \tilde{b} \mid \tilde{b} \rangle_2 \geq \left| \langle \tilde{a} \mid \tilde{b} \rangle_2 \right|^2 \text{ (Cauchy–Schwarz).}$$

For $z = \langle \tilde{a} \mid \tilde{b} \rangle_2 \in \mathbb{C}$ one has $|z|^2 = (\Re z)^2 + (\Im z)^2 \geq (\Im z)^2$ and

$$(\Im z)^2 = (1/2\mathrm{i}(z - \bar{z}))^2 = 1/4(z - \bar{z})^2 = 1/4(\langle \tilde{a} \mid \tilde{b} \rangle_2 - \langle \tilde{b} \mid \tilde{a} \rangle_2)^2$$
$$= 1/4(\phi(\tilde{a} \cdot \tilde{b}) - \phi(\tilde{b} \cdot \tilde{a}))^2 = 1/4(\phi(ab) - \phi(ba))^2 = 1/4\phi([a, b])^2,$$

whence (Robertson) $\varsigma(a)_\phi \varsigma(b)_\phi \geq 1/2 \left| \mathbb{E}_\phi([a, b]) \right|$:

lower limit of uncertainty for non-commuting RVs ($[a, b] \neq 0$); generally observation / sample of non-commuting RVs is not possible ($\hat{\mathcal{A}}_\infty = \emptyset$).

Where have we come to?

Formalised the abstract idea of (possibly non-commuting) observables and average as probability algebra $(\mathcal{A}, \phi)$.

Representations of $(\mathcal{A}, \phi)$:
Key to many developments is the regular representation $\Upsilon : a \mapsto L_a$,
observations / samples are characters—elements of the spectrum $\hat{\mathcal{A}}$.
Kolmogorov's classical definition appears as one particular representation.

Every Abelian C*-probability algebra $\mathcal{B}_\infty = \mathrm{C}^*\mathcal{B}$ 'is like' $\mathrm{C}(\Omega)$,
and also like uniform algebra $\mu(\mathrm{C}(\Omega)) \subseteq \mathscr{L}(\mathcal{H})$.
Its weak closure W*$\mathcal{B}$ 'is like' $\mathrm{L}_\infty(\Omega)$,
and also like multiplication algebra—a MASA— $\mu(\mathrm{L}_\infty(\Omega)) \subseteq \mathscr{L}(\mathcal{H})$.
Samples appear as continuous characters.

Functions $\varphi(a)$ of RVs may be computed via operational calculus.

Which topologies can one use?

For all $0 < p \leq \infty$ the $\mathrm{L}_p(\mathcal{A})$-spaces were defined.
Possibly unbounded $\mathcal{A}$ can be identified with a sub-algebra of $\mathrm{L}_{\infty-}(\mathcal{A})$,
for Abelian $\mathcal{B}$ represented as multiplication by
possibly unbounded function $f \in \mathrm{L}_0(\Omega)$ in
unbounded multiplication algebra $\mu(\mathrm{L}_0(\Omega)) \subseteq \mathsf{L}(\mathcal{H})$.

Hence one has the system of algebras of RVs:

$$\mathcal{A}_b \subseteq \mathcal{A}_\infty \subseteq \mathrm{L}_\infty(\mathcal{A}) \subseteq \mathcal{A} \subseteq \mathrm{L}_{\infty-}(\mathcal{A}) \subset \mathrm{L}_0(\mathcal{A})$$

Similarly, one has the system of Banach-Gelfand triples (with $2 \leq p < \infty$ and $1/p + 1/p^* = 1$); continuous embeddings with different topologies:

$$\mathcal{A}_\infty \hookrightarrow \mathrm{L}_\infty(\mathcal{A}) \hookrightarrow \mathrm{L}_p(\mathcal{A}) \hookrightarrow \mathrm{L}_2(\mathcal{A}) = \mathcal{H} \cong \mathcal{H}^* \hookrightarrow \mathrm{L}_{p^*}(\mathcal{A}) \hookrightarrow \mathrm{L}_1(\mathcal{A})$$

$\mathrm{L}_2(\mathcal{A}) = \mathcal{H} \cong \mathcal{H}^*$ is a ‘natural’ pivot space.

Some questions

Some possible questions:

- Are there other representations of RV-algebras which may be helpful to understand certain aspects, or for specific numerical computations? We have seen regular repr. on $\mathrm{L}(\mathcal{A})$, and for Abelian ones the function algebra repr. (both faithful), and for samples the 1D character repr.
- How to sample RVs $a \in \mathrm{L}_0(\Omega) \setminus \mathrm{C}(\Omega)$?
- Is a good regularity theory possible for solutions of stochastic equations, and functionals (QoIs) thereof? What could be good spaces of RVs? Can this reduce the $\#$ of samples to approximate $\mathbb{E}_\phi$?
- How to define and deal with "wild" RVs, or generalised RVs as "idealised" elements — e.g. white noise, Donsker's delta, etc.

Sampling

Looking at self-adjoint $a = a^*$.

If $\gamma(a) = a \in \mathrm{C}(\Omega)$, where $\Omega = \hat{\mathcal{B}}$, then a sample is just $a(\omega) := \langle \delta_\omega, a \rangle$.
What to do for a RV $a \in \mathrm{L}_0(\Omega) \setminus \mathrm{C}(\Omega)$?

If $a \in \mathrm{L}_0(\Omega)$, it suffices to look at $a_n = \mathrm{sign}(a) \cdot \operatorname{ess\,inf}(|a|, n) \in \mathrm{L}_\infty(\Omega)$,
whence for any $n \in \mathbb{N}$: $\sigma(a_n) \in [-n, n]$ and $\forall \lambda \in \sigma(a_n)$
$\exists \omega_\lambda \in \widehat{\mathrm{L}_\infty(\Omega)} \subset (\mathrm{L}_\infty(\Omega))^*$ with $\langle \omega_\lambda, a_n \rangle = \lambda$.

Elements which are in $(\mathrm{L}_\infty(\Omega))^*$ are difficult to access directly.
Therefore via regular representation $\Upsilon : a \mapsto L_a$ on $\mathcal{H} = \mathrm{L}_2(\Omega)$.

Any $a \in \mathrm{L}_0(\mathcal{A})$ defines a possibly unbounded,
densely defined, self-adjoint operator $L_a =: A \in \mathrm{L}(\mathcal{H})$ in $\mathcal{H}$.

Sampling therefore is equivalent to computation of spectral values of A.

Three Spectral Theorems for Matrices

Let $\boldsymbol{A} \in \mathbb{M}(n, \mathbb{C})$ be normal $([\boldsymbol{A}, \boldsymbol{A}^*] = \mathbf{0},\ \boldsymbol{A}^* = \bar{\boldsymbol{A}}^\mathsf{T}) \Rightarrow$
$\exists \lambda_1, \ldots, \lambda_n \in \sigma(\boldsymbol{A}) \subset \mathbb{C}$ and orthonormal eigenvectors $\boldsymbol{e}_k \in \mathbb{C}^n$.
Three versions, and operational calculus for any $f : \sigma(\boldsymbol{A}) \to \mathbb{C}$:

1. Multiplicative: Set $\boldsymbol{V} = [\boldsymbol{e_1, \ldots, e_n}]$ (unitary) and $\boldsymbol{\Lambda}_A = \operatorname{diag}(\lambda_k)$.
   Then $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}_A\boldsymbol{V}^*$. For any function f: $f(\boldsymbol{A}) = \boldsymbol{V} f(\boldsymbol{\Lambda}_A)\boldsymbol{V}^*$,
2. Projections: Set $\boldsymbol{E}_0 = 0, \boldsymbol{E}_k = \sum_{\lambda_m \le \lambda_k} \boldsymbol{e}_m \boldsymbol{e}_m^* = \sum_{\lambda_m \le \lambda_k} \boldsymbol{e}_m \otimes \bar{\boldsymbol{e}}_m$,
   and $\Delta \boldsymbol{E}_k := \boldsymbol{E}_k - \boldsymbol{E}_{k-1}$. Then for all $\boldsymbol{v} \in \mathbb{C}^n$:
   $\boldsymbol{A}\boldsymbol{v} = \sum_{\lambda_1 \le \lambda_k \le \lambda_n} \lambda_k \, \Delta \boldsymbol{E}_k \boldsymbol{v}$; and $f(\boldsymbol{A})\boldsymbol{v} = \sum_{\lambda_1 \le \lambda_k \le \lambda_n} f(\lambda_k) \, \Delta \boldsymbol{E}_k \boldsymbol{v}$.
3. Eigenvectors: For all $\boldsymbol{v} \in \mathbb{C}^n$:
   $\boldsymbol{A}\boldsymbol{v} = \sum_{\lambda_k} \lambda_k \, \langle \boldsymbol{v}, \boldsymbol{e}_k \rangle_{\mathbb{C}^n} \, \boldsymbol{e}_k$ and $f(\boldsymbol{A})\boldsymbol{v} = \sum_{\lambda_k} f(\lambda_k) \, \langle \boldsymbol{v}, \boldsymbol{e}_k \rangle_{\mathbb{C}^n} \, \boldsymbol{e}_k$.

$f(\boldsymbol{A})$ is normal, and for real-valued f, $f(\boldsymbol{A})$ is Hermitean.

Spectral analysis overview

Multiplicative spectral theorem:

for a self-adjoint (s.a.) operator $A \in \mathscr{L}(\mathcal{H}), A = A^\dagger$, we may take $\mathcal{B} = \mathrm{C}^*\mathbb{C}[A]$ and apply the Gel'fand representation; then $\sigma(A) \subset \mathbb{R}$. There is a unitary $U \in \mathscr{L}(\mathrm{L}_2(\sigma(A), \mathcal{H}))$, s.t. $A = UM_\lambda U^*$ with multiplication operator M_λ in the uniform algebra $\mathrm{C}(\sigma(A))$, and $f(A) = UM_{f(\lambda)}U^*$ in the MASA $\mathrm{W}^*\mathcal{B} = \mathcal{B}''$, equiv. to $\mathrm{L}_\infty(\sigma(A))$.

Projection measure spectral theorem:

take for $\lambda \in \sigma(A)$ the projections $p_\lambda = \mathbb{1}_{]-\infty,\lambda]} \in \mathrm{L}_\infty(\sigma(A))$, define projections $E_\lambda = p_\lambda(A) \in \mathrm{W}^*\mathcal{B}$ to obtain the usual

$A = \int_{\sigma(A)} \lambda \, \mathrm{d}E_\lambda$ and $f(A) = \int_{\sigma(A)} f(\lambda) \, \mathrm{d}E_\lambda$.

For a s.a. RV a one has the cumulative distribution function

$F_a(\lambda) = \phi(p_\lambda(a)) = \mathbb{E}\left(p_\lambda(a)\right) =: \mathbb{P}(\omega(a) \leq \lambda)$, and

$$a = \int_{\sigma(a)} \lambda \, \mathrm{d}p_\lambda(a) \quad \Rightarrow \quad \mathbb{E}\left(a\right) = \int_{\sigma(a)} \lambda \, \mathrm{d}F_a(\lambda).$$

Countably Hilbertian and nuclear spaces

Space $\mathcal{F}$ with sequence of Hilbert norms $\{\|\cdot\|_n\}_0^\infty$ s.t. for $n < m$: $\|x\|_n \leq \|x\|_m$. With $\mathcal{V}_n = \mathrm{cl}_n \mathcal{F}$ and $\mathcal{V}_{-n} = \mathcal{V}_n^*$ one has densely $\mathcal{F} \hookrightarrow \mathcal{V}_m \hookrightarrow \mathcal{V}_n \hookrightarrow \mathcal{H} = \mathcal{V}_0 \cong \mathcal{H}^* \hookrightarrow \mathcal{V}_{-n} \hookrightarrow \mathcal{V}_{-m}$, and

$$\mathcal{F} \hookrightarrow \mathcal{V} := \bigcap_{k=0}^{\infty} \mathcal{V}_k \hookrightarrow \ldots \mathcal{V}_n \cdots \hookrightarrow \mathcal{H} \hookrightarrow \ldots \mathcal{V}_{-n} \cdots \hookrightarrow \mathcal{V}^* := \bigcup_{k=0}^{\infty} \mathcal{V}_{-k}$$

Projective limit $\mathcal{V} = \varprojlim \mathcal{V}_k$ is a complete countably Hilbertian Fréchet space. Its dual is the injective limit $\mathcal{V}^* = \varinjlim \mathcal{V}_{-k}$. All are separable. If $\forall n \, \exists m > n$ s.t. $\mathcal{V}_m \hookrightarrow \mathcal{V}_n$ is nuclear, $\mathcal{V}$ is a nuclear space.

If $A : \mathcal{F} \to \mathcal{F} (\hookrightarrow \mathcal{H}$ densely) is ess. s.a., set $\|x\|_n^2 = \sum_{k=0}^n \left\|A^k x\right\|_{\mathcal{H}}^2$. One has $A \in \mathscr{L}(\mathcal{V})$ and $A^k \in \mathscr{L}(\mathcal{V}^{m+k}, \mathcal{V}^{m-k})$. In case $\mathcal{V}$ is not nuclear, it is possible to find nuclear $\mathcal{U} \hookrightarrow \mathcal{V}$ densely, s.t. $A \in \mathscr{L}(\mathcal{U})$.

The nuclear Gelfand triple $\mathcal{U} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{U}^*$ is called a rigged Hilbert space.

Generalised eigenvectors

Generalised eigenvectors / nuclear spectral theorem:

Rigged Hilbert space $\mathcal{U} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{U}^*$ with $A \in \mathscr{L}(\mathcal{U})$ and $\lambda \in \sigma(A)$:

$\exists\, u_\lambda \in \mathcal{U}^*$ (generalised eigenvector) s.t. $\forall v \in \mathcal{U}$:

$$\langle Av, u_\lambda \rangle_{(\mathcal{U},\mathcal{U}^*)} =: \langle v, Au_\lambda \rangle_{(\mathcal{U},\mathcal{U}^*)} = \lambda \langle v, u_\lambda \rangle_{(\mathcal{U},\mathcal{U}^*)} .$$

In case $u_\lambda \in \mathcal{H}$, it is a usual eigenvector: $Au_\lambda = \lambda u_\lambda$.

Then it is possible to define a measure ρ_A on $\sigma(A)$ and a measurable $\lambda \mapsto u_\lambda$, s.t. $\forall v \in \mathcal{U}$:

$$Av = \int_{\sigma(A)} \lambda \langle v, u_\lambda \rangle_{(\mathcal{U},\mathcal{U}^*)} u_\lambda \, \rho_A(\mathrm{d}\lambda)$$

and

$$f(A)v = \int_{\sigma(A)} f(\lambda) \langle v, u_\lambda \rangle_{(\mathcal{U},\mathcal{U}^*)} u_\lambda \, \rho_A(\mathrm{d}\lambda)$$

Spectral subsets, stability

For s.a. $A \in \mathrm{L}(\mathcal{H})$: $\sigma(A) = \{\lambda \in \mathbb{R} \mid (A - \lambda I)$ not invertible in $\mathscr{L}(\mathcal{H})\}$

- $\lambda \in \sigma_p(A)$ — point spectrum, if $\ker(A - \lambda I) \neq \{0\}$,
- $\lambda \in \sigma_c(A) = \sigma(A) \setminus \sigma_p(A)$ — continuous spectrum,
- $\lambda \in \sigma_s(A) \subseteq \sigma_p(A)$ — simple spectrum,
  if λ isolated in $\sigma(A)$ and $\dim(\ker(A - \lambda I)) < \infty$,
- $\lambda \in \sigma_e(A) = \sigma(A) \setminus \sigma_s(A)$ — essential spectrum,

Spectral stability:
Stability under compact s.a. perturbation $C \in \mathscr{L}(\mathcal{H})$:

$$\sigma_e(A) = \sigma_e(A + C).$$

For any s.a. $A \in \mathscr{L}(\mathcal{H})$ there is a compact s.a. perturbation $C \in \mathscr{L}(\mathcal{H})$
(of arbitrarily small norm $\|C\|$) s.t.

$$\sigma_c(A + C) = \emptyset, \text{ i.e. } \sigma(A + C) = \sigma_p(A + C).$$

Sampling the spectrum for $a \in \mathrm{L}_0(\Omega) \setminus \mathrm{C}(\Omega)$

For $\lambda \in \sigma_s(A)$ there is an normalised eigenvector $v_\lambda \in \mathcal{H}$ and $\omega_\lambda(a) = \langle v_\lambda \mid A v_\lambda \rangle$ is a pure state.

For $\lambda \in \sigma_e(A)$ only smoothed approximation, as λ may not be isolated.

But for $\lambda \in \sigma_p(A) \setminus \sigma_s(A)$ there is an normalised eigenvector $v_\lambda \in \mathcal{H}$ and $\omega_\lambda(a) = \langle v_\lambda \mid A v_\lambda \rangle$ is a pure state.

For $\lambda \in \sigma_c(A)$, there is a generalised eigenvector $u_\lambda \in \mathcal{U}^*$, and $v \in \mathcal{U}$ s.t. $\langle v, u_\lambda \rangle_{(\mathcal{U},\mathcal{U}^*)} \neq 0$, and a pure state is given by

$$\omega_\lambda(a) = \frac{\langle A v, u_\lambda \rangle_{(\mathcal{U},\mathcal{U}^*)}}{\langle v, u_\lambda \rangle_{(\mathcal{U},\mathcal{U}^*)}}.$$

Sampling the spectrum II

Alternatively, for $\lambda \in \sigma_e(A)$ there is an orthonormal sequence $g_n \in \mathcal{H}$ s.t. $\omega_\lambda(a) = \lim_{n\to\infty} \|Ag_n\|$ is a pure state.

Other calculations based on resolvent $R_A(z) = (A - zI)^{-1}$.
This corresponds to $r_a(z) = (a - ze)^{-1}$.
For s.a. a one has that $r_a(z) \in \mathrm{L}_\infty(\mathcal{A})$ if $\Im z \neq 0$.

For $\lambda \in \sigma(A)$, $\langle R_A(\lambda + \mathrm{i}\epsilon)g \mid g\rangle$ has a jump at $\epsilon = 0$.
Therefore, with $\langle r_a(z)e \mid e\rangle = \mathbb{E}_\phi(r_a(z))$, look at
$s_\lambda(\epsilon) = \mathbb{E}_\phi(r_a(\lambda + \mathrm{i}\epsilon)) - \mathbb{E}_\phi(r_a(\lambda - \mathrm{i}\epsilon))$.
If $\lim_{\epsilon\to 0} s_\lambda(\epsilon) \neq 0 \;\Rightarrow\; \lambda \in \sigma(a)$.

Hence, for $\lambda \in \sigma_c(A) \Leftrightarrow s_\lambda(\epsilon) \neq 0$ for $\epsilon \ll 1$
is a ϵ-smoothed approximation for a pure state $\omega_\lambda(a)$.

Polynomial chaos

A frequent situation is that one has a Hilbert space of (commuting) Gaussian RVs $\mathcal{G}$, and one considers the algebra $\mathcal{P} := \mathbb{C}[\mathcal{G}]$ with state $\mathbb{E} = \phi$.

Choosing a complete orthonormal system (CONS) $\{\theta_k\}$ in $\mathcal{G}$, one builds well known orthogonal Hermite polynomial chaos in $\mathcal{P}$
$H_{\boldsymbol{\alpha}}(\boldsymbol{\vartheta}) = \prod_j h_j(\alpha_j)(\theta_j)$, s.t. $\langle H_{\boldsymbol{\alpha}} \mid H_{\boldsymbol{\beta}} \rangle_2 = (\boldsymbol{\alpha}!)\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}$
with $\boldsymbol{\vartheta} = (\theta_1, \dots)$ and $\boldsymbol{\alpha} = (\alpha_1, \dots) \in \mathscr{J} = \mathbb{N}_0^{(\mathbb{N})}$ a multi-index, and 1D Hermite polynomials h_j.

Setting $\mathcal{C}_n = \mathrm{cl}_2(\mathrm{span}\{H_{\boldsymbol{\alpha}} \; : \; |\boldsymbol{\alpha}| = n\})$ for n-th homogeneous chaos, one has $\mathcal{H} = \mathrm{L}_2(\mathcal{P}) = \mathrm{cl}_2(\bigoplus_{n \geq 0} \mathcal{C}_n)$ (chaos decomposition), and for $f \in \mathcal{H}$ with $f = \sum_{\boldsymbol{\alpha} \in \mathscr{J}} f_{\boldsymbol{\alpha}} H_{\boldsymbol{\alpha}}$ one has $\|f\|_2^2 = \sum_{\boldsymbol{\alpha}} (\boldsymbol{\alpha}!) f_{\boldsymbol{\alpha}}^2$.

Spaces of random variables I

For appropriate $f \in \mathcal{H}$ define unbounded s.a. positive operators for $0 \le \varrho \le 1$ and $k \in \mathbb{N}_0$ by

$$A_{\varrho,k} H_{\boldsymbol{\alpha}} = (2\mathbb{N})^{k\boldsymbol{\alpha}} (\boldsymbol{\alpha}!)^{\varrho} H_{\boldsymbol{\alpha}} \quad \text{with } (2\mathbb{N})^{\boldsymbol{\gamma}} = \prod_j (2j)^{\gamma_j} \text{ for } \boldsymbol{\gamma} \in \mathscr{J}$$

and Hilbert norms $\|f\|^2_{\varrho,k} = \langle A_{\varrho,k} f \mid f \rangle_2$. This gives separable Hilbert spaces $(\mathcal{S})_{\varrho,k} = \{f \in \mathcal{H} \;:\; \|f\|_{\varrho,k} < \infty\}$, $(\mathcal{S})_{\varrho,-k} := (\mathcal{S})^*_{\varrho,k}$, and

— Hida-Kondratiev spaces —

$$(\mathcal{S})_{\varrho} := \bigcap_{k=0}^{\infty} (\mathcal{S})_{\varrho,k} \hookrightarrow \dots (\mathcal{S})_{\varrho,m} \dots \hookrightarrow \mathcal{H}$$
$$\hookrightarrow \dots (\mathcal{S})_{\varrho,-n} \dots \hookrightarrow (\mathcal{S})^*_{\varrho} := \bigcup_{k=0}^{\infty} (\mathcal{S})_{\varrho,-k}$$

All $(\mathcal{S})_{\varrho}$ are nuclear spaces. The nuclear test RV space $(\mathcal{S})_0$ is the Hida test space, and the distribution / generalised RV $(\mathcal{S})^*_0$ is the Hida test space, containing white noise. The pair $(\mathcal{S})_1, (\mathcal{S})^*_1$ are the Kondratiev test and distribution / generalised RV spaces. One has

$$(\mathcal{S})_1 \hookrightarrow (\mathcal{S})_{\varrho} \hookrightarrow (\mathcal{S})_0 \hookrightarrow \mathcal{H} \hookrightarrow (\mathcal{S})^*_0 \hookrightarrow (\mathcal{S})^*_{\varrho} \hookrightarrow (\mathcal{S})^*_1$$

Spaces of random variables II

An important s.a. positive operator is the Ornstein-Uhlenbeck or number operator Λ: for $H_{\boldsymbol{\alpha}} \in \mathcal{C}_n$ $(|\boldsymbol{\alpha}| = n)$ one sets $\Lambda H_{\boldsymbol{\alpha}} = n H_{\boldsymbol{\alpha}}$.
Define the norms $\|f\|_{k,p} := \left\|(I+\Lambda)^{k/2} f\right\|_p$ and the Sobolev-Malliavin spaces $\mathbb{D}_p^k = \{f \in (\mathcal{S})_0 \;:\; \|f\|_{k,p} < \infty\}$.

A $f \in \mathbb{D}_p^k$ is k-times Malliavin differentiable in $\mathrm{L}_p(\mathcal{P})$.

For $p = 2$ the $\mathbb{D}_2^k$ are Hilbert spaces, $\mathbb{D}_2^{-k} := (\mathbb{D}_2^k)^*$, and one has

$$\mathbb{D}_2^\infty := \bigcap_{k=0}^{\infty} \mathbb{D}_2^k \hookrightarrow \dots \mathbb{D}_2^m \dots \hookrightarrow \mathcal{H} \hookrightarrow \dots \mathbb{D}_2^{-m} \dots \hookrightarrow \mathbb{D}_2^* := \bigcup_{k=0}^{\infty} \mathbb{D}_2^{-k}$$

If $f \in \mathbb{D}_2^1$, it can have a density.

If $f \in \mathbb{D}^\infty := \bigcap_{p \geq 1} \bigcap_{k=0}^{\infty} \mathbb{D}_p^k$, it can have a smooth density.

A glimpse at quantum computing I

A qubit is a vector in $\mathcal{Q}_1 := \mathbb{C}^2$, the usual basis is $|0\rangle, |1\rangle$.
n qubits are represented as a vector in $\mathcal{Q}_n = \mathcal{Q}_1^{\otimes n} = (\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^{2^n}$.

A quantum computation is in most quantum computers a unitary transformation $\boldsymbol{U} \in \mathscr{L}(\mathcal{Q}_n)$, transforming an input $\boldsymbol{g} \in \mathcal{Q}_n$ to an output $\boldsymbol{a} = \boldsymbol{Q}\boldsymbol{g}$.

At the end of the computation, the output state $\boldsymbol{a}$ has to be measured, to determine which qubit a_j is in which state ($|0\rangle$ or $|1\rangle$).

The possible measurements / observations $\boldsymbol{M}_\ell$ are s.-a. and not without error, so they are non-commuting RVs in the algebra $\mathcal{A} = \mathscr{L}(\mathcal{Q}_n)$.

In each run / computation, only a commuting set $\{\boldsymbol{H}_k\}_{k=1}^K$ can be observed / sampled.

A glimpse at quantum computing II

The expectation is given by a state $\mathbb{E}\left(\boldsymbol{M}\right) = \phi(\boldsymbol{M}) = {}^1\!/\!_n\,\mathrm{tr}(\boldsymbol{R}\boldsymbol{M})$, where $\boldsymbol{R}$ is a positive definite s.a. matrix with $\phi(\boldsymbol{R}) = 1$, the density matrix.

The observations / samples are $\omega(\boldsymbol{H}_k + \boldsymbol{E}) = \frac{\langle(\boldsymbol{H}_k + \boldsymbol{E})\boldsymbol{a}|\boldsymbol{a}\rangle}{\langle\boldsymbol{a}|\boldsymbol{a}\rangle}$, where $\boldsymbol{E} \in \mathbb{C}[\{\boldsymbol{H}_k\}]$ is the error in the measurement.

So instead of $\omega(\boldsymbol{H}_k)$, one observes $\omega(\boldsymbol{H}_k) + \omega(\boldsymbol{E})$, where hopefully $\mathbb{E}\left(\boldsymbol{E}\right) := \phi(\boldsymbol{E}) = 0$.

One then has to perform many (MC) samples ω_ι, so that the statistical error for each qubit becomes small enough.

Conclusion

- Probability algebras give 'probability without measures'.
- RVs and expectation operator become central objects.
- Algebras (and probability measures — integration) have close connection with spectral theory and functional calculus.
- Sampling is theoretically the same as determination of a spectral value.
- Duality theory allows "wild" random variables, and Gelfand triples give stochastic test RVs and stochastic "distributions".
- Quantum computing is a simple non-commutative example.