Topological field theories with boundaries - some constructions and some applications

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Based on work with
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and Julian Farnsteiner

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Overview

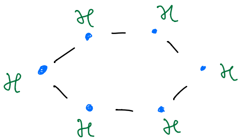
Chapter 1

One-dimensional spin systems

One-dimensional spin systems

$\mathcal{H}$ a finite-dimensional vector space $\mathcal{H}$, with basis $|j\rangle_{j=1,\ldots d}$

Dimension of vector space $\mathcal{H}^{\otimes N}$ for spin chain of length N grows exponentially.



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Dimension of vector space $\mathcal{H}^{\otimes N}$ for spin chain of length N grows exponentially.
Parametrize certain translationally invariant states on $\mathcal{H}^{\otimes N}$:

- Auxillary vector space $\mathcal{V}$ with $\dim_{\mathbb{C}} \mathcal{V} = D$ and basis $|m\rangle_{m=1,\ldots D}$
- Matrix $A^j_{m,n}$ with $m, n = 1, \ldots, D$ and $j = 1, \ldots, d$:

$$\psi(A) := \sum_{j_1, j_2, \ldots, j_N}^{d} \mathrm{Tr}(A^{j_1} A^{j_2} \cdots A^{j_N}) |j_1\rangle \otimes |j_2\rangle \otimes |j_N\rangle \ \in \ \mathcal{H}^{\otimes N}$$

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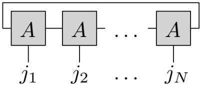
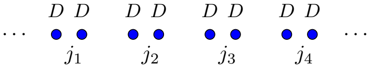
Graphically

$\psi(A) =$ 

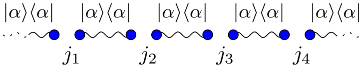
No dynamics specified, just a subspace of states ($\rightarrow$ quantum code)

A different view: PEPS

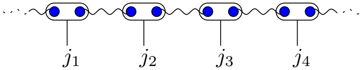
Place at each site $\mathcal{V} \otimes \mathcal{V}$.



Maximally entangle all the pairs of qudits on neighbouring sites by projecting onto the maximally entangled state $|\alpha\rangle := \sum_{m=1}^{D} |m\rangle \otimes |m\rangle$



One-dimensional PEPS tensor is a map $f : \mathcal{V} \otimes \mathcal{V} \to \mathcal{H}$:



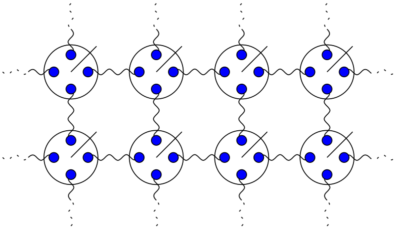
Hence the name Projected entangled pair state

Chapter 2

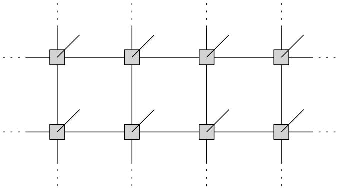
Two-dimensional spin systems

PEPS in two-dimensions

The same prescription works in two dimensions, e.g. for the square lattice



leading to the following structure of the PEPS tensors

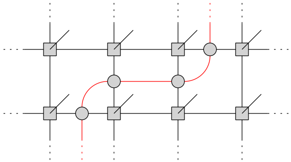


The physical vector space $\mathcal{H}$ of the spin system is sticking out of the plane.

Towards tractable systems: MPO symmetries

This is a two-dimensional system.

- Topological symmetries should explain ground state degeneracies, if the system is placed on non-trivial topologies.
- In a two-dimensional system, topological symmetries are encoded by one-dimensional defects. (For RCFT: Fuchs, Fröhlich, Runkel, CS, 2004)



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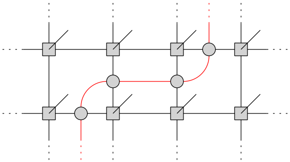
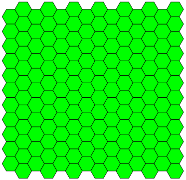


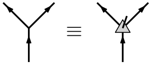
Specialize to trivalent vertices, e.g. honeycomb lattice



Ingredients: Vector spaces $\mathcal{H}, \mathcal{V}, \mathcal{W}$

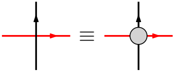
Tensors:

PEPS



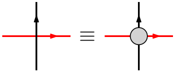
$\mathcal{V}^{\otimes 3} \otimes \mathcal{H} \to \mathbb{C}$

MPO



$\mathcal{V}^{\otimes 2} \otimes \mathcal{W}^{\otimes 2} \to \mathbb{C}$

Fusion category of MPO symmetries

MPO tensor:  :

1. Every $v \in \mathcal{V} \otimes \mathcal{V}$ gives an endomorphism $\mathcal{B}(v) : \mathcal{W} \to \mathcal{W}$.
   Assume that the subalgebra $\mathcal{B}_W := \langle \mathcal{B}(v) \rangle \subset \mathrm{End}(\mathcal{W})$ is semisimple.
   Decompose $\mathcal{W}$ into a direct sum of orthogonal invariant subspaces:
   $\mathcal{W} := \bigoplus_{a \in I_\mathcal{C}} \mathcal{W}_a$ labeled by isoclasses of simple $\mathcal{B}_W$-modules:
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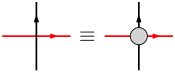
MPO tensor: :

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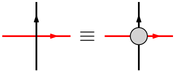
MPO tensor:  :

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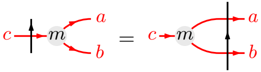
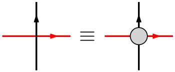
Topological symmetry defects can fuse: fusion tensors. Locality of fusion



implies compatibility with decomposition of $\mathcal{W}$ into subspaces.

Fusion category of MPO symmetries

MPO tensor: :

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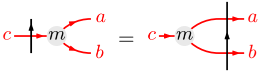

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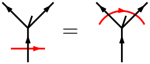
MPO symmetries should be topological symmetries,



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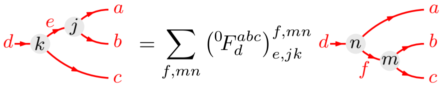
6j-symbols and pentagon identities

Consistency of couplings → 6j symbols



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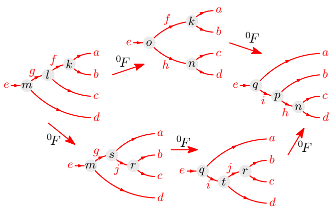

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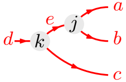
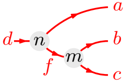


$$= \sum_{f,mn} \left({}^0F^{abc}_d\right)^{f,mn}_{e,jk}$$



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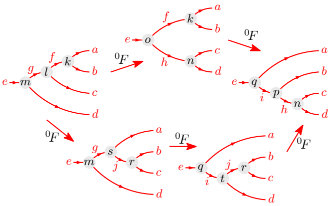

Upshot:

- Invariant subspaces of $\mathcal{W}$ (red labels $a, b, \ldots$) are objects of a (spherical) fusion category $\mathcal{C}$

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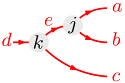

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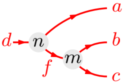

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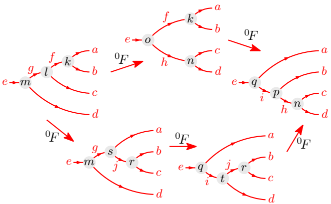
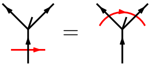


Upshot:

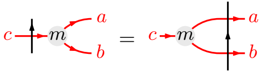
- Invariant subspaces of $\mathcal{W}$ (red labels $a, b, \ldots$) are objects of a (spherical) fusion category $\mathcal{C}$
- Invariant subpaces of $\mathcal{V}$ (labels $\alpha, \beta, \ldots$) are objects of a (spherical) fusion category $\mathcal{D}$

Zipper and pulling through

Compatibility of MPO and PEPS:



and



Idea:
Identities are mixed pentagons. Thus look for a context with mixed pentagons.

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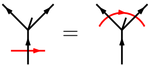

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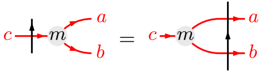

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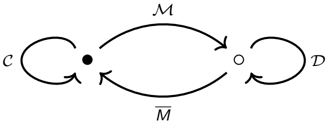
Bicategory with two objects



Remarks

- Familar situation in local rational CFT and subfactor theory.
- $\mathcal{C}, \mathcal{D}$ are monoidal categories, $\mathcal{M}$ a $\mathcal{C}$-$\mathcal{D}$-bimodule.
- Minimality requirement: $\mathcal{M}$ is an invertible bimodule.
  Then $\mathcal{D} \cong \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ and $\mathcal{C} \cong \mathrm{Fun}_{\mathcal{D}}(\mathcal{M}, \mathcal{M})$.

Setup of the general spin model

Two object bicategory $\rightarrow \mathcal{C}, \mathcal{D}$ monoidal category, $\mathcal{M}$ a $\mathcal{C}$-$\mathcal{D}$-bimodule.

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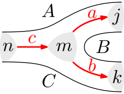
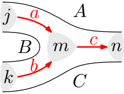
Two object bicategory $\rightarrow \mathcal{C}, \mathcal{D}$ monoidal category, $\mathcal{M}$ a $\mathcal{C}$-$\mathcal{D}$-bimodule.
Leads to the following vector spaces:

- Physical vector space: $\mathcal{H} := \oplus_{\alpha,\beta,\gamma \in I_\mathcal{D}} \mathrm{Hom}_\mathcal{D}(\alpha \otimes \beta, \gamma)$
- Auxilliary vectors space $\mathcal{V} := \oplus_{A,B \in I_\mathcal{M}} \oplus_{\alpha \in I_\mathcal{D}} \mathrm{Hom}_\mathcal{M}(A.\alpha, B)$
- Vector spaces for MPO symmetries $\mathcal{V} := \oplus_{A,B \in I_\mathcal{M}} \oplus_{a \in I_\mathcal{C}} \mathrm{Hom}_\mathcal{M}(a.A, B)$



$:= \left(\frac{d_a d_b}{d_c}\right)^{\frac{1}{4}} \frac{\left({}^1F^{abC}_A\right)^{B,kj}_{c,mn}}{\sqrt{d_B}},$



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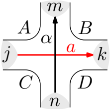

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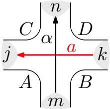

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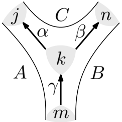

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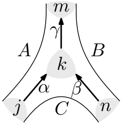

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Summary about PEPS

Summary:

Surface Σ with trivalent vertices, e.g. hexagonal lattice Δ:

- Vector space for spin model: $\mathcal{H}_\Sigma := \bigotimes_{v \in \Delta_0} \mathcal{H}$ given in terms of Hom spaces of a spherical fusion category $\mathcal{D}$,

$$\mathcal{H} := \oplus_{\alpha,\beta,\gamma \in I_\mathcal{D}} \mathrm{Hom}_\mathcal{D}(\alpha \otimes \beta, \gamma)$$

- A PEPS given in terms of mixed 6j-symbols for a module category $\mathcal{M}/\mathcal{C}$.
- State in subspace $\mathcal{H}^0_\Sigma \subset \mathcal{H}_\Sigma$, obtained by contracting the PEPS tensor
- Any (indecomposable, pivotal) module category over $\mathcal{D}$ gives a PEPS.
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Lessons:

- Given a spin model in terms of $\mathcal{D}$, the MPO symmetries are not unique.
- Different PEPS for different module categories $\mathcal{M}$ give different "coordinates" for the system that allow to see different symmetries.
- Dual descriptions are related by categorical Morita equivalence
- Hamiltonian $\to$ Lewin-Wen

Chapter 3

PEPS and state-sum TFT with boundaries

Why TFT?

Goal: go beyond trivalent vertices (and lattices)

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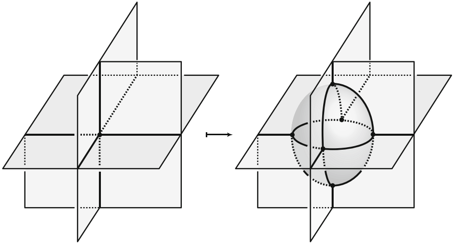
Features of state-sum construction, based on spherical fusion category $\mathcal{D}$:

- Choose as a auxillary datum a skeleton Δ of a 3-manifold.



- Construct for free boundary surface Σ a big vector space $\mathrm{preTFT}_{\mathcal{D}}(\Sigma, \Delta)$ that depends on Δ and a subspace

$$\mathrm{TFT}_{\mathcal{D}}(\Sigma) \subset \mathrm{preTFT}_{\mathcal{D}}(\Sigma, \Delta)$$

  that is independent of Δ.

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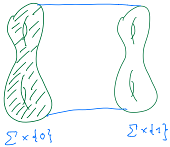
"Holographic" strategy:

Given a closed oriented surface Σ, consider 3-manifold $M_\Sigma := \Sigma \times [0,1]$



- Physical boundary $M \times \{0\}$ (possibly with a network of boundary Wilson lines)
- Gluing boundary $M \times \{1\}$ with

$$\mathrm{preTFT}_{\mathcal{D}}(\Sigma, \Delta) = \mathcal{H}_\Sigma \quad \text{and} \quad \mathrm{TFT}_{\mathcal{D}}(\Sigma) = \mathcal{H}^0_\Sigma$$

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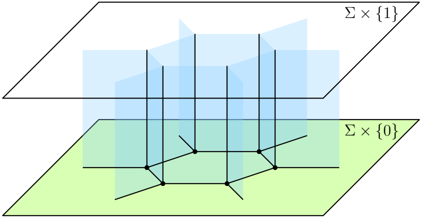
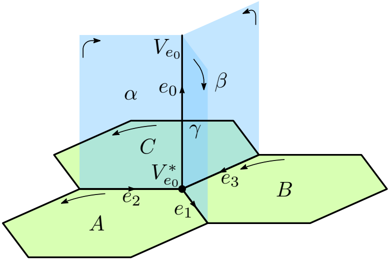
- Then $\mathrm{TFT}_{\mathcal{D}}(\Sigma) : \mathbb{C} \to \mathrm{TFT}_{\mathcal{D}}(\Sigma)$ gives a state described by the PEPS.

Turaev-Viro construction with boundaries





- No vertices on the gluing boundary $M \times \{1\}$
- State sum variables assigned to plaquettes
  - $\alpha \in \mathcal{D}$ to (blue) plaquettes in interior
  - $A \in \mathcal{M}$ to (green) plaquette on the physical boundary
- Vector spaces of invariants to each half-edge

  $$\mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma) =: V_{e_0} \quad \text{and} \quad \mathrm{Hom}_{\mathcal{D}}(\gamma, \alpha \otimes \beta) \cong \mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \beta, \gamma)^* = V^*_{e_0}\,,$$

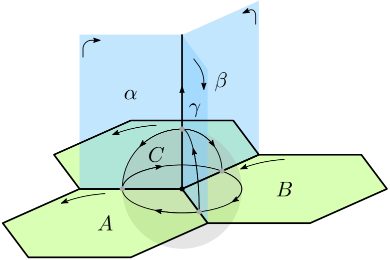
- Two vector spaces for same edge are in duality, hence canonical vector in $V^*_{e_0} \otimes V_{e_0}$

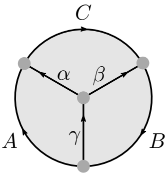
Turaev-Viro construction: evaluation at vertices

Thus, given a skeleton Δ of 3-manifold, get vector space V_Δ with canonical vector $v_\Delta \in V_\Delta$.

Next ingredient:

Evaluation at all vertices, using graphical calculus on $\mathbb{S}^2$





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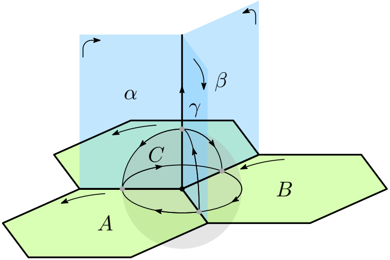
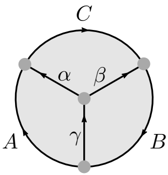
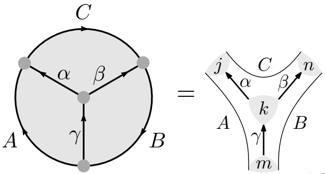




For hexagonal lattices, we get tetrahedra on $\mathbb{S}^2$ and thus 6j-symbols:



Result of evaluation

The evaluations at all vertices $v \in \Delta_0$ compose to a map

$$\mathrm{ev}_\Delta := \otimes_{v\in\Delta_0}\mathrm{ev}_v : \; V_\Delta \to \otimes_{\text{dangling edges}} V_e = \mathcal{H}_\Sigma$$

Then

$$\mathrm{TFT}_{\mathcal{D}}(M_\Sigma)(1) = \mathrm{ev}_\Delta(v_\Delta) = \mathrm{PEPS}_{\mathcal{M},\mathcal{D}}$$

Upshot:
A holographic understanding of PEPS that is independent of lattices.

Remark

Can be generalized by including boundary Wilson lines on the free boundary =MPO symmetries in the tensor network language.

Chapter 4

Equivariant Frobenius Schur indicators and state-sum TFT

Equivariant Frobenius Schur indicators and boundaries

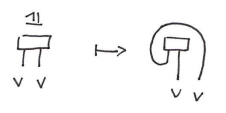
Recap

V a finite-dimensional irreducible $\mathbb{C}[G]$-module.

$$\nu_2(V) := \frac{1}{|G|} \sum_{g \in G} \chi_V(g^2) \in \{0, \pm 1\}$$

$\nu = \pm 1 \Leftrightarrow$ non-deg. invariant bilinear form on V symmetric or antisymmetric.

$\nu_2(V)$ is the trace of the endomorphism on the one-dimensional vector space $\mathrm{Hom}(V \otimes V, 1)$:



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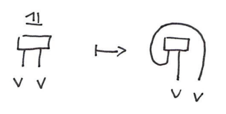
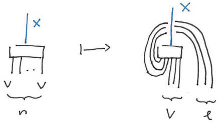
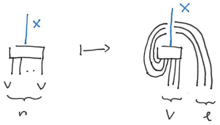


Generalization for pivotal categories: $V \in \mathcal{C}$ and $X \in \mathcal{Z}(\mathcal{C})$:
[Kashina, Sommerhäuser, Zhu; Ng, Schauenburg]



Generalized Frobenius Schur indicator:
$\nu_{V,X,(n,l)} := \mathrm{tr} \xi_{V,X,(n,l)}$.
Equivariance under $\mathrm{SL}(2,\mathbb{Z})$.

Application to the equivariant Frobenius-Schur indicators



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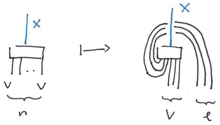

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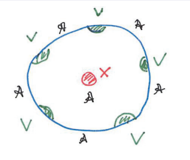
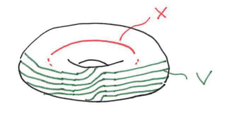
Theorem (Farnsteiner, 2020)





$\rightsquigarrow \mathrm{Hom}_{\mathcal{C}}(V^{\otimes n}, X)$ *Solid torus with Wilson line* $\rightsquigarrow \nu_{V,X,(n,l)}$
$\mathrm{SL}(2,\mathbb{Z})$*-equivariance becomes geometric and follows from TFT axioms.*