Reflexive Polytopes
and the Convergence of Feynman Integrals

Pierre Vanhove

IPhT
Saclay

Amplitudes and Algebraic Geometry
Erwin Schrödinger International Institute
for Mathematics and Physics

based on 2512.10518,
with Leonardo de la Cruz, and Pavel Novichkov
and on 2603.03183

Feynman integrals are ubitquitous to many problems in physics ranging from elementary particle physics to classical gravity computations

Precision physics request for fast and high-precision evaluations

The Feynman integrals are period integrals of mixted motives, and an Hodge theoreric based classification of the motives is an active research program [Doran et al.; Duhr et al.; Weinzierl et al.; Klemm et al.; Wilhelm et al.]

So far the following classes have been found to contribute to physical observables

- Multiple polylogarithms [Brown; Goncharov; . . .]
- Elliptic integrals [Bloch, Vanhove; Broadhurst; Weinzierl; Duhr; . . .]
- Periods of K3, CY 3-fold, CY 4-fold [Bloch, Kerr, Vanhove; Broadhurst; Weinzierl; Duhr; Wilhelm; Klemm, Nega, Bönisch, Plefka, . . .]
- Beyond CY geometry have been identified (genus 2 curves at 2-loops) [Doran, et al.; Wilhelm et al.].

The generic form of a Feynman integral in parametric form is

$$I_\Gamma(\underline{z}) = \int_{\mathbb{R}_+^{N-1}} \frac{\prod_{i=1}^{N} x_i^{\nu_i - 1}}{\mathcal{U}_\Gamma(\underline{x})^{n_\mathscr{U}} \mathcal{F}_\Gamma(\underline{x})^{n_\mathscr{F}}}\Big|_{x_N=1} \prod_{i=1}^{N-1} dx_i$$

with $L \in \mathbb{N}$, $(D, \nu_1, \ldots, \nu_N) \in \mathbb{C}^{N+1}$.

$$n_\mathscr{U} = \frac{(L+1)D}{2} - \nu, \quad n_\mathscr{U} + n_\mathscr{F} = \frac{D}{2}, \quad \nu = \sum_{i=1}^{N} \nu_i$$

They are a special case of Euler–Mellin integrals

Definition (Euler–Mellin integrals)

Let $f_1, \ldots, f_q$ *be Laurent polynomials in variables* $x_1, \ldots, x_n$, $f_i(\boldsymbol{x}) := \sum_{\boldsymbol{a} \in \mathrm{supp}(f_i)} c_i(\boldsymbol{a}) \boldsymbol{x}^{\boldsymbol{a}}$. *An Euler–Mellin integral is defined as*

$$I(\boldsymbol{\nu}, \boldsymbol{t}) := \int_{\mathbb{R}_+^n} \frac{x_1^{\nu_1} \cdots x_n^{\nu_n}}{\prod_{i=1}^{q} (f_i(\boldsymbol{x}))^{t_i}} \prod_{j=1}^{n} \frac{dx_j}{x_j}.$$

An homogeneous degree $L+1$ polynomial in the variables $\boldsymbol{x} = (x_1, \dots, x_N)$

$$\mathcal{F}_\Gamma(\underline{x}) = \mathcal{U}_\Gamma(\underline{x}) \times \mathcal{L}(\underline{m}^2; \underline{x}) - \mathcal{V}_\Gamma(\underline{s}, \underline{x})$$

- Homogeneous polynomial of degree L with $u_{a_1,\dots,a_n} \in \{0,1\}$

$$\mathcal{U}_\Gamma(\underline{x}) = \sum_{a_1+\cdots+a_n=L} u_{a_1,\dots,a_n} \prod_{i=1}^{n} x_i^{a_i}; \qquad a_i \in \{0,1\}$$

- the mass hyperplane

$$\mathcal{L}(\underline{m}^2; \underline{x}) := m_1^2 x_1 + \cdots + m_n^2 x_n$$

- Homogeneous polynomial of degree $L+1$

$$\mathcal{V}_\Gamma(\underline{x}) = \sum_{a_1+\cdots+a_n=L+1} s_{a_i,\dots,a_n} \prod_{i=1}^{n} x_i^{a_i}; \qquad a_i \in \{0,1\}$$

the coefficients $s_{a_i,\dots,a_n}$ are linear combination of the product of the external momenta $\underline{s} = \{p_i \cdot p_j\}$ and relation between these coefficients depend on the dimension D of space-time

Newton polytope

Considering a (Laurent) polynomial in n variables

$$f(x_1,\dots,x_n)=\sum_{\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{Z}^n} c_{\mathbf{a}} x_1^{a_1}\cdots x_n^{a_n}, \qquad c_{\mathbf{a}}\in\mathbb{C}\,,$$

we associate a Newton polytope as the convex hull of its support

$$\Delta(f):=\left\{\sum_{i=1}^n \lambda_i a_i \,\middle|\, \sum_{i=1}^n \lambda_i=1, \lambda_i\in\mathbb{R}_+, \mathbf{a}\in\mathrm{supp}(f)\right\}$$

The support

$$\mathrm{supp}(f):=\{\mathbf{a}:=(a_1,\dots,a_n)\in\mathbb{Z}^n | c_{\mathbf{a}}\neq 0\}\,.$$

Convergent Euler–Mellin integrals

Definition (Non-vanishing polynomial)

If F is a face of the Newton polytope $\Delta(f)$ of f, then the truncated polynomial with support F is given by $f_F := \sum_{\boldsymbol{a}\in F\cap \mathrm{supp}(f)} c(\boldsymbol{a})\, \boldsymbol{x}^{\boldsymbol{a}}$. The polynomial f is said to be completely non-vanishing on a set X if for each face F of $\Delta(f)$, the truncated polynomial f_F has no zeros on X. In particular, the polynomial f itself does not vanish on X.

Theorem (Berkesch, Forsgård, and Passare)

If each of the polynomials $f_1, \ldots, f_q$ is completely non-vanishing on the positive orthant $\mathbb{R}_+^n$, then the Euler–Mellin integral $I(\boldsymbol{\nu}, \boldsymbol{t})$ converges and defines an analytic function in the tube domain

$$\left\{ (\boldsymbol{\nu}, \boldsymbol{t}) \in \mathbb{C}^{n+q} \;\middle|\; \mathfrak{Re}(\boldsymbol{t}) \in \mathbb{R}_+^q,\ \mathfrak{Re}(\boldsymbol{\nu}) \in \mathrm{int}\left(\sum_{i=1}^{q} \mathfrak{Re}(t_i) \Delta(f_i) \right) \right\}.$$

Finite Feynman integrals

We apply this theorem to the case for Feynman integrals in affine patch

$$I_\Gamma(n_{\mathscr{F}}, n_{\mathscr{U}}, n) = \int_{\mathbb{R}+^{n-1}} \frac{\prod_{i=1}^{n} x_i^{\nu_i}}{\mathcal{U}_\Gamma(\underline{x})^{n_{\mathscr{U}}} \mathcal{F}_\Gamma(\underline{x})^{n_{\mathscr{F}}}}\Bigg|_{x_n=1} \prod_{i=1}^{n} \frac{dx_i}{x_i}$$

$$n_{\mathscr{U}} = -\nu + \frac{L+1}{2}D, \quad n_{\mathscr{F}} = -n_{\mathscr{U}} + \frac{D}{2}, \quad \nu = \nu_1 + \cdots + \nu_N \quad \nu_i \geqslant 1$$

The graph polytope

$$\Delta_\Gamma(L, D; \boldsymbol{\nu}) = n_{\mathscr{U}} \Delta(\mathscr{U}_\Gamma) + n_{\mathscr{F}} \Delta(\mathscr{F}_\Gamma), \qquad n_{\mathscr{U}}, n_{\mathscr{F}} \geqslant 0$$

- We need to compute the number interior points as function of $(n_{\mathscr{U}}, n_{\mathscr{F}})$
- The coordinates of the interior points are $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_N)$ leading to finite integral

Polytope of Feynman graph

$$\Delta_\Gamma(L,D;\boldsymbol{\nu}) = n_{\mathscr{U}}\Delta(\mathscr{U}_\Gamma) + n_{\mathscr{F}}\Delta(\mathscr{F}_\Gamma), \qquad n_{\mathscr{U}}, n_{\mathscr{F}} \geqslant 0$$

- For integer values of $n_{\mathscr{U}}$ and $n_{\mathscr{F}}$ this is a lattice polytope, and $\Delta_\Gamma(L,D;\boldsymbol{\nu}) = \Delta(\mathscr{U}_\Gamma^{n_{\mathscr{U}}}\mathscr{F}_\Gamma^{n_{\mathscr{F}}})$
- The polytope of the mass hyperplane is the standard simplex

$$\Delta(\mathscr{L}_\Gamma) = \Delta(m_1^2 x_1 + \cdots + m_n^2 x_N) = \mathrm{Conv}\{e_1, \ldots, e_N\} = \Delta_{\mathrm{HS}}^{(N,1)}$$

- When all the internal masses are non vanishing

$$\Delta(\mathscr{F}_\Gamma) = \Delta(\mathscr{U}_\Gamma) + \Delta(\mathscr{L}_\Gamma) \Longrightarrow \Delta_\Gamma(L,D;\boldsymbol{\nu}) = \frac{D}{2}\Delta(\mathscr{U}_\Gamma) + n_{\mathscr{F}}\Delta_{\mathrm{HS}}^{(N,1)}$$

  The details of the Second Symanzik is not needed.
  The above is not true when some internal mass vanish

- $\Delta(\mathscr{U}_\Gamma)$ and $\Delta_\Gamma(L,D;\boldsymbol{\nu})$ are generalized permutahedron

Ehrhart polynomials for Feynman integrals

The polytopes associated to a generic kinematic Feynman integral is the Minkowski sum of the scaling of two lattice polytopes

$$\Delta_\Gamma(L, D; \boldsymbol{\nu}) = \frac{D}{2}\Delta(\mathscr{U}_\Gamma) + n_{\mathscr{F}}\Delta_{\mathrm{HS}}^{(N,1)}$$

A bivariate Ehrhart polynomial $\mathrm{Ehr}_{P,Q}(t_1, t_2)$

$$\mathrm{Ehr}_\Gamma(t_1, t_2) = \#\Big((t_1\Delta(\mathscr{U}_\Gamma) + t_2\Delta_{\mathrm{HS}}^{(N,1)}) \cap \mathbb{Z}^N\Big)\,,$$

counts the lattice points in the Minkowski sum of independently dilated copies of the Newton polytopes

The number interior points is obtained from the reciprocity relation

$$\mathrm{N}_{\rm int}(t_1\Delta(\mathscr{U}_\Gamma) + t_2\Delta_{\mathrm{HS}}^{(N,1)}) = (-1)^N\,\mathrm{Ehr}_\Gamma(-t_1, -t_2)$$

The knowledge of the Ehrhart polynomial gives an efficient way to identify polytope dilations containing a single interior point.

Ehrhart polynomials for Feynman integrals

The determination of the bivariate Ehrhart polynomial is a difficult problem. The Bernstein–McMullen's theorem implies that the Ehrhart polynomial has total degree N with rational coefficients

$$\mathrm{Ehr}_\Gamma(t_1, t_2) = \sum_{\substack{i+j=N\\ i,j\geqslant 0}} c_{ij}\, t_1^i t_2^j,$$

- The leading homogeneous part (degree N) corresponds to the mixed volumes:

$$\mathrm{Vol}(t_1 P + t_2 Q) = \sum_{i=0}^{n} \binom{n}{i} V_i(P, Q)\, t_1^i t_2^{n-i}$$

$$V_0(P, Q) = V(Q),\, V_n(P, Q) = V(P)$$

- Parametrizing $t_1 = \tau_1 t$ and $t_2 = \tau_2 t$ with $\tau_1, \tau_2 \in \mathbb{N}$, one can easily the Ehrhart of t using `polymake` and reconstruct $\mathrm{Ehr}_\Gamma(t_1, t_2)$ by fitting.

Still for two special families we have been able to derive the Ehrhart polynomial

Ehrhart polynomials for One-loop integrals

For the one-loop N-gon Feynman graphs, the first Symanzik is

$$\mathscr{U}_{N-\mathrm{gon}} = x_1 + \cdots + x_N$$

For generic massive internal lines and kinematics we have

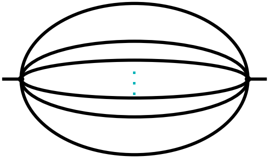
$$\Delta_{N-\mathrm{gon}}(1, D; \boldsymbol{\nu}) = n_{\mathscr{U}} \Delta(\mathscr{U}_{N-\mathrm{gon}}) + n_{\mathscr{F}} \Delta_{\mathrm{HS}}^{(N,1)} = (\nu_1 + \cdots + \nu_N) \Delta_{\mathrm{HS}}^{(N,1)}$$

The number of interior points

$$\mathrm{N}_{\mathrm{int}}(\nu \Delta_{\mathrm{HS}}^{(N,1)}) = (-1)^{N-1} \,\mathrm{Ehr}_{\Delta_{\mathrm{HS}}^{(N,1)}}(-\nu) = \prod_{r=1}^{n-1} \frac{\nu - r}{r}$$

Therefore the polytope $\Delta_{N-\mathrm{gon}}(1, D; \boldsymbol{\nu})$ has only one interior point for $\nu = \nu_1 + \cdots + \nu_N = N$. For $\nu_i \geqslant 1$ for all $i = 1, \ldots, N$, the interior point is $\boldsymbol{\nu} = (1, \ldots, 1)$.

Ehrhart polynomials for sunset integrals



The first Symanzik polynomial is

$$\mathscr{U}_{\ominus}^{N} = x_1 x_2 \cdots x_N \left(\frac{1}{x_1} + \cdots + \frac{1}{x_N} \right)$$

For generic masses and kinematics the sunset graph polytope is

$$\Delta_{\ominus}(N-1, D; \boldsymbol{\nu}) = \frac{D}{2} \Delta(\mathscr{U}_{\ominus}^{N}) + \left(\nu_1 + \cdots + \nu_N - \frac{(N-1)D}{2} \right) \Delta_{\mathrm{HS}}^{(N,1)}$$

The Newton polytope for the first Symanzik polynomial is

$$\Delta(\mathscr{U}_{\ominus}^{N}) = \mathrm{Conv}\{\mathbf{1} - e_i \,|\, i = 1, \ldots, N\} = (1, \ldots, 1) - \Delta_{\mathrm{HS}}^{(N,1)}$$

Ehrhart polynomials for sunset integrals

The bivariate Ehrhart polynomial of the multiloop sunset polytope

$$\mathrm{Ehr}_{\ominus}(n_{\mathscr{F}}, D/2, N) = P(n_{\mathscr{F}}, D/2, N) + \sum_{r=0}^{N-1} c(r,N) P(n_{\mathscr{F}}, D/2, r)\,,$$

with P being a polynomial defined as

$$P(t_1,t_2,n) := \sum_{\substack{0\leqslant i,j\leqslant n-1\\ i+j\neq n-1}} (-1)^{-i-j+n-1} \binom{i+j}{i}\binom{i+t_1}{i}\binom{j+t_2}{j}\,,$$

and the coefficients $c(r,n)$ are given by

$$c(r,n) = (-1)^n \mathrm{coeff}_{x^{n-1}}\left(\frac{2x+1}{(1+x)^2}\left(\frac{x}{x+1}\right)^{n-1-r}\right)\,,$$

where $\mathrm{coeff}_{x^r}(f(x))$ means the coefficient of x^r in the series expansion of $f(x)$ around $x=0$.

Ehrhart polynomials for sunset integrals

Using the reciprocity formula for the Ehrhart polynomial we identify the polytopes with a single interior point for generic number of edges:

- $D = 2$ with $\boldsymbol{\nu} = (1, \dots, 1)$: in this case $n_{\mathscr{U}} = 0$ and $n_{\mathscr{F}} = 1$, and the polytope is

$$\Delta_{\ominus}(N-1, 2; (1, \dots, 1)) = \Delta(\mathscr{U}_{\ominus}^{N}) + \Delta_{\mathrm{HS}}^{(N,1)}$$

- $D = 2N$ with $\boldsymbol{\nu} = (N-1, \dots, N-1)$: in this case $n_{\mathscr{U}} = N$ and $n_{\mathscr{F}} = 0$, and the polytope reads

$$\Delta_{\ominus}(N-1, 2N; (N-1, \dots, N-1)) = N\Delta(\mathscr{U}_{\ominus}^{N}) = N\left(\mathbf{1} - \Delta_{\mathrm{HS}}^{(N,1)}\right)$$

Fano and reflexive polytopes

We focus on the polytopes with a single interior point

Definition (Fano polytope)

A lattice polytope is (canonical) Fano if the only lattice point that lies strictly in its interior is the origin.

Let N be the dual lattice to M with respect to the scalar product $\langle \bullet | \bullet \rangle : M \times N \to \mathbb{Z}$. Using this scalar product we can define the dual (or polar) polytope

$$\nabla := \{\boldsymbol{b} = (b_1, \dots, b_n) \in N \,|\, \langle a, b \rangle \geqslant -1, \, \forall \boldsymbol{a} \in \Delta\} \subset N \cong \mathbb{Z}^n \, .$$

Definition (Reflexive polytope)

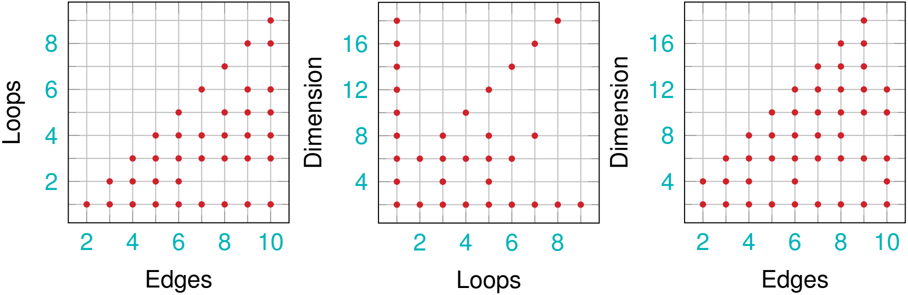
A lattice polytope Δ is reflexive if its polar polytope ∇ is a lattice polytope and has a single interior point. The polytopes (Δ, ∇) are said to be mirror pairs.

Fano and reflexive polytopes

$N = \dim + 1$	graph topologies	Fano	reflexive	non-reflexive Fano
2	1	1	1	0
3	2	2	2	0
4	3	3	3	0
5	6	4	4	1
6	13	8	7	4
7	28	11	6	6
8	70	23	11	16
9	193	36	14	24
10	565	104	26	88

Reflexive polytopes

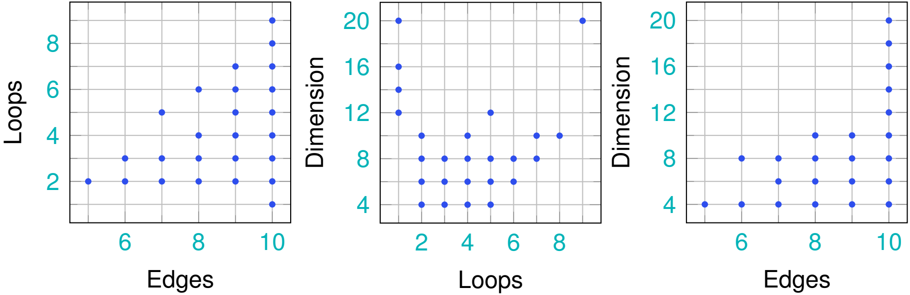
Distribution of reflexive graph polytopes from Feynman integrals with massive internal lines up to 10 edges.



At each point there is at least one reflexive polytope

Non reflexive Fano polytopes

Distribution of non-reflexive Fano graph polytopes from Feynman integrals with massive internal lines up to 10 edges.



At each point there is at least one non-reflexive Fano polytope

Polytopes from one-loop graphs

The polytope associated to the one-loop integral is reflexive in $N \leqslant D \leqslant 2N$ dimensions.

$$I_{N-\text{gon}}(1, D; (1, \ldots, 1)) = \int_{\mathbb{R}_+^{N-1}} \frac{1}{\mathscr{U}_{N-\text{gon}}^{D-N} \mathscr{F}_{N-\text{gon}}^{N-\frac{D}{2}}}\Bigg|_{x_N=1} dx_1 \cdots dx_{N-1}$$

$$\begin{aligned}
\mathscr{U}_{N-\text{gon}} &:= x_1 + \cdots + x_N, \\
\mathscr{L}_{N-\text{gon}} &:= m_1^2 x_1 + \cdots + m_N^2 x_N, \\
\mathscr{V}_{N-\text{gon}} &:= \sum_{1 \leqslant i < j \leqslant N} (p_i + \cdots + p_{j-1})^2 x_i x_j, \\
\mathscr{F}_{N-\text{gon}} &:= \mathscr{U}_{N-\text{gon}} \mathscr{L}_{N-\text{gon}} - \mathscr{V}_{N-\text{gon}},
\end{aligned}$$

This is in general multivalued function of the physical parameters.

Reflexive one-loop polytopes

Reflexive polytopes associated to the massive one-loop N-gon graph are given by the scaled standard simplex $N\Delta_{\rm HS}^{(N,1)}$.

The corresponding polar polytope has the simple expression

$$\nabla_{N-\rm gon}(1,D;(1,\dots,1)) = \mathrm{Conv}\Big\{e_1, e_2, \dots, e_{N-1}, -\sum_{i=1}^{N-1} e_i\Big\},$$

where e_i are unit vectors or $\mathbb{R}^{N-1}$.

Therefore the one-loop N-gon encodes the projective space $\mathbb{P}^{N-1}$, which is the associated toric variety of the polar polytope

One-loop Hodge numbers

The Hodge numbers all vanish except for the numbers $h_{N,0} = h_{1,1}(N) = 1$ for $N > 4$ and $h_{1,N-3}(N)$ can be extracted from the Hirzebruch generating formula for a degree N hypersurface X in $\mathbb{P}^N$

$$H(N) = \sum_{p,q} (h_{p,q}(N) - \delta_{pq}) x^p y^q = \frac{(1+y)^{N-1} - (1+x)^{N-1}}{(1+x)^N y - (1+y)^N x}\,.$$

$N = dim + 1$	4	5	6	7	8	9	10
$h_{1,N-3}$	20	101	426	1667	6371	24229	92278

The varieties defined by the singular locus of the one-loop N-gon graphs are not smooth, although they share the same polar polytope as the smooth Calabi–Yau $(N-2)$-folds with the given Hodge numbers.

Reflexive polytope for graphs with $N=3$ edges

With $N=3$ edges we have two types of graph leads to a reflexive polytopes

- The one-loop triangle (with massive and massless internal lines)

  $$\mathscr{U}_{\rm triangle}=x_1+x_2+x_3, \qquad \mathscr{V}_{\rm triangle}=x_1x_2\,p_3^2+x_1x_3\,p_2^2+x_2x_3\,p_1^2.$$

  In $D=4$ dimensions the finite integral is

  $$I^0_{\rm triangle}(1,4;(1,1,1))=\int_{\mathbb{R}^2_+}\left.\frac{1}{\mathscr{U}_{\rm triangle}\mathscr{V}_{\rm triangle}}\right|_{x_3=1}dx_1\,dx_2\,.$$

- The two-loop sunset integral

  $$\mathscr{F}^2_\ominus=(x_1x_2+x_1x_3+x_2x_3)(m_1^2x_1+m_2^2x_2+m_3^2x_3)-p^2x_1x_2x_3$$

  and integral

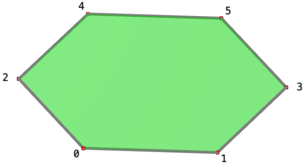
  $$I_\ominus(2,2;(1,1,1))=\int_{\mathbb{R}^2_+}\left.\frac{1}{\mathscr{F}^2_\ominus}\right|_{x_3=1}dx_1\,dx_2$$

Massless triangle and two-loop sunset

Their Newton polytope is identical

$$\Delta(\mathscr{U}_{\text{triangle}}) + \Delta(\mathscr{V}_{\text{triangle}}) = \Delta(\mathscr{F}^2_{\ominus})$$

is the two-dimensional reflexive hexagon



The associated toric Fano surface is the del Pezzo surface dP_6 obtained as the blow-up of $\mathbb{P}^2$ at three non-collinear points with anticanonical hypersurface

$$a_0 z_0^2 z_1^2 z_2 z_3 + a_1 z_0 z_1^2 z_3^2 z_4 + a_2 z_0^2 z_1 z_2^2 z_5 + a_3 z_1 z_3^2 z_4^2 z_5 + a_4 z_0 z_2^2 z_4 z_5^2 + a_5 z_2 z_3 z_4^2 z_5^2 + a_6 z_0 z_1 z_2 z_3 z_4 z_5 = 0$$

Massless triangle and dP_6

We have a smooth elliptic curve

$$\frac{a_0 x_1}{x_2} + \frac{a_1 x_1}{x_3} + \frac{a_2 x_3}{x_2} + \frac{a_3 x_2}{x_3} + \frac{a_4 x_3}{x_1} + \frac{a_5 x_2}{x_1} + a_6 = 0$$

setting

$$a_0 = a_2 = m_1^2, \quad a_1 = a_3 = m_2^2, \quad a_4 = a_5 = m_3^2, \quad a_6 = m_1^2 + m_2^2 + m_3^2 - p^2$$

gives the graph polynomial of the two-loop sunset

$$\{\mathscr{F}_{\ominus}^2 = (x_1 x_2 + x_1 x_3 + x_2 x_3)(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3) - p^2 x_1 x_2 x_3 = 0\}$$

But the one for the triangle the locus of the integral is singular

$$\{(x_1 + x_2 + x_3)(x_1 x_2 p_3^2 + x_1 x_3 p_2^2 + x_2 x_3 p_1^2) = 0\}$$

obtained by setting

$$a_0 = a_2 = m_1^2, \quad a_1 = a_3 = m_2^2, \quad a_4 = a_5 = m_3^2, \quad a_6 = m_1^2 + m_2^2 + m_3^2$$

Massless triangle and dP_6

The two-loop sunset Feynman integral

$$I_{\ominus}(2,2;(1,1,1)) = \int_{\mathbb{R}^2_+} \frac{1}{\mathscr{F}_{\ominus}}\bigg|_{x_3=1} dx_1\, dx_2$$

is interpreted as a regulator period, and the limiting mixed Hodge structure realizes a version of local mirror symmetry for dP_6 [Bloch, Kerr, Vanhove]

On the other hand, the hypersurface $\{\mathscr{U}_{\rm triangle}\mathscr{F}_{\rm triangle} = 0\}$ is a degeneration of the smooth sunset elliptic curve.

The two-loop sunset is an elliptic dilogarithm, and the massless triangle integral evaluates to a single-valued dilogarithm, obtained by taking the degenerating limit $p^2 \to 0$.

Massive triangle and dP_3

The massive triangle in $D = 4$ has the integral

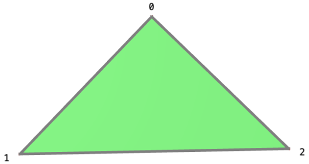
$$I^0_{\text{triangle}}(1,4;(1,1,1)) = \int_{\mathbb{R}^2_+} \frac{1}{\mathscr{U}_{\text{triangle}}\mathscr{F}_{\text{triangle}}}\Big|_{x_3=1} dx_1\, dx_2\,.$$

with

$$\mathscr{F}_{\text{triangle}} = (x_1+x_2+x_3)\left(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3\right) - \left(x_1 x_2\, p_3^2 + x_1 x_3\, p_2^2 + x_2 x_3\, p_1^2\right)$$

The polytope is the scaled standard simplex which is reflexive

$$\Delta_{\text{triangle}}(1,4;(1,1,1)) = 3\Delta^{(3,1)}_{\text{HS}}\,.$$



Massive triangle and dP_3

The toric Fano variety $\mathbb{P}_{\Delta_{\rm triangle}}$ is the projective plane $\mathbb{P}^2$ with anticanonical hypersurface

$$a_0 z_0^3 + a_1 z_1^3 + a_2 z_2^3 + a_3 z_0 z_1 z_2 = 0\,.$$

which are smooth elliptic curves

Again the singular locus of the integral $\mathscr{U}_{\rm triangle}\mathscr{F}_{\rm triangle}$ is singular degeneration of the elliptic curve

$$\mathscr{U}_{\rm triangle}\mathscr{F}_{\rm triangle} = z_0\left(a_0 z_0^2 + a_3 z_1 z_2\right).$$

The massive triangle integral is a degeneration of the smooth elliptic curve associated to dP_3

Pentagon graph and the mirror Quintic 3-fold

The pentagon has graph polynomials

$$\mathscr{U}_{\text{pentagon}} = x_1 + \cdots + x_5,$$
$$\mathscr{F}_{\text{pentagon}} = \mathscr{U}_{\text{pentagon}}\left(m_1^2 x_1 + \cdots + m_5^2 x_5\right) - \sum_{1\leqslant i<j\leqslant 5} (p_i + \cdots + p_{j-1})^2 x_i x_j\,.$$

its polytope is the unique reflexive simplex in dimension four given by the scaled standard 4-simplex $5\Delta_{\text{HS}}^{(5,1)}$

The associated toric variety is $\mathbb{P}^4$, and the anticanonical linear system consists of smooth quintic hypersurfaces

$$a_0 z_0^5 + \cdots + a_4 z_4^5 + a_5 z_0 \cdots z_4 = 0\,.$$

The Hodge number $h_{1,2} = 101$

Pentagon graph and the mirror Quintic 3-fold

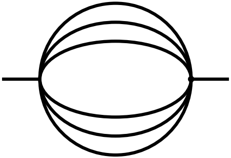
The graph hypersurfaces for the pentagon are given by in $D = 6, 8, 10$

$$\mathscr{U}_{\text{pentagon}}\mathscr{F}^2_{\text{pentagon}} = 0, \quad \mathscr{U}^3_{\text{pentagon}}\mathscr{F}_{\text{pentagon}} = 0, \quad \mathscr{U}^5_{\text{pentagon}} = 0$$

These pentagon graph hypersurfaces can be viewed as toric degeneration of smooth quintic threefold and hence fits naturally into the Batyrev construction.

This pentagon integrals are hypergeometric functions which compute a period of a singular Calabi–Yau threefold that is birational to a toric degeneration of the quintic.

Mirror symmetry and sunset integrals



We have seen that the multiloop sunset integral in $D=2$ leads to a reflexive polytope so it is tempting to connect this to mirror symmetry.

We restric to the all equal mass case but the same discussion applies to general mass case.

Consider the multiloop sunset integral in two dimensions for all equal masses in the Euclidean regime $-p^2>0$

$$I_{\ominus}^{(L)}(p^2)=\int_{x_i\geqslant 0} e^{-(x_1+\cdots+x_{L+1})-\frac{(-p^2)}{x_1^{-1}+\cdots+x_{L+1}^{-1}}}\,\frac{1}{x_1^{-1}+\cdots+x_{L+1}^{-1}}\,\frac{dx_1\cdots dx_{L+1}}{x_1\cdots x_{L+1}}$$

Exact asymptotic expansion

Using Mellin-Barnes techniques one can give an exact expression for the multiloop sunset [Bonisch, Duhr, Fischbach, Klemm, Nega, Safari; Vanhove]

$$I_{\ominus}^{(L)}(p^2) = \pi_{\ominus}^{(L)}(p^2)\Big(-(L+1)\big(-R^{(L)}(p^2)\big)^L + \sum_{l=1}^{\infty} e^{lR^{(L)}(p^2)} \sum_{r=0}^{L-1} d_r^{(L)}(l)\,\big(R^{(L)}(p^2)\big)^r\Big) + \sum_{r=0}^{L-1} \alpha_r^{(L)}\,\mathrm{Frob}_r^{(L)}(p^2)$$

The holomorphic period near $t=\infty$ reads

$$\pi_{\ominus}^{(L)}(p^2) = \int_{|x_i|=1} \frac{dx_1\cdots dx_{L+1}}{\mathscr{F}_{\ominus}^{N}}\Big|_{x_{L+1}=1} = \sum_{\substack{(r_1,\dots,r_{L+1})\in\mathbb{N}^{L+1}\\ n=r_1+\cdots+r_{L+1}}} \left(\frac{n!}{r_1!\cdots r_{L+1}!}\right)^2 \left(\frac{1}{p^2}\right)^{n+1}$$

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The flat coordinate $dR^{(L)}(p^2)/dp^2 = \pi_{\ominus}^{(L)}(p^2)$

$$R(p^2) = i\pi - \log(p^2) + \sum_{\substack{(r_1,\dots,r_{L+1})\in\mathbb{N}^{L+1}\\ n=r_1+\cdots+r_{L+1}>0}} \left(\frac{n!}{r_1!\cdots r_{L+1}!}\right)^2 \left(\frac{1}{p^2}\right)^{n+1} \frac{1}{n}$$

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A Frobenius basis of solutions for $0 \leqslant r \leqslant L-1$

$$\mathrm{Frob}_r^{(L)}(p^2) =:= \frac{1}{r!}\pi^{(L)}(p^2)\left(R^{(L)}(p^2)\right)^{r} + \sum_{n\geqslant 1} c_n e^{nR^{(L)}(p^2)}$$

where c_n are rational numbers.

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where:

- $d_r^{(L)}(l) \in \mathbb{Q}$ are rational coefficients (easy calculated with `sagemath`)
- $\alpha_0^{(1)} = \alpha_0^{(2)} = \alpha_1^{(2)} = 0$ and for $L \geqslant 3$, $\alpha_r^{(L)} \in \mathbb{Q}[\zeta(3), \zeta(5), \zeta(7), \dots]$ are determined by the generating function

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{\alpha_r^{(n+3)}\lambda^r x^n}{(n+4)!} = \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{2k+1}\frac{2\,x^{2k-2}}{1+\lambda x}\ \exp\left(x^3 \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{2k+1}\right)$$

Sunset exact asymptotic expansion

- The form of the expansion is a expected from Weinberg analysis of the large euclidean kinematics of Feynman integrals

  $$I_\Gamma(\lambda) \simeq \sum_{r=0}^{r_{\max}} (\log(\lambda))^r \sum_{s \geqslant s_0} \frac{g_{r,s}}{\lambda^s}$$

  and the analysis of `[Bergère, de Calan, Malbouisson]` on the convergent case.
- The coefficients of the Frobenius basis are a realisation of the Γ-class conjecture [Bonisch, Duhr, Fischbach, Klemm, Nega, Safari; Iritani]
- The expansion is not just asymptotic but exact and valid above threshold
- This a example of the case where the thimble integral agrees with its the Borel resummation and fits with the results of `[Fantini, Fenyes; Kontsevich, Soibelman]`

The two-loop case

At two loops order $L=2$ we have

$$\frac{I_{\ominus}^{(2)}(p^2)}{\pi_{\ominus}^{(2)}(p^2)} = -3\left(R^{(2)}(p^2)\right)^2 + \sum_{l=1}^{\infty}\left(d_0^{(2)}(l) + d_1^{(2)}(l)\,R^{(2)}(p^2)\right) e^{lR^{(2)}(p^2)}$$

- No Frobenius correction appears at $L=2$
- The coefficients are given by [Bloch, Kerr, Vanhove]

$$\frac{d_0^{(2)}(l)}{-6l} = \frac{d_1^{(2)}(l)}{6l^2} = \mathrm{GW}(2,l) \;=\; \sum_{d|l}\frac{1}{d^3}\,n_{l/d}$$

where n_l are the integer virtual genus-zero Gromov–Witten invariants of the non-compact Hori-Vafa mirror Calabi–Yau threefold associated to the two-loop sunset elliptic curve

$$\left\{(m_1^2x_1+m_2^2x_2+m_3^2x_3)(x_1x_2+x_1x_3+x_2x_3)-p^2x_1x_2x_3=uv\right\}\subset(\mathbb{C}^*)^2\times\mathbb{C}^2$$

The three-loop case

At three loops order $L = 3$ we have

$$\underline{I_{\ominus}^{(3)}(p^2)} = \pi_{\ominus}^{(3)}(p^2)\left(4(R^{(3)}(p^2))^3 + \sum_{l=1}^{\infty} e^{lR^{(3)}(p^2)} \sum_{r=0}^{2} d_r^{(3)}(l)(R^{(3)}(p^2))^r\right) + 16\zeta(3)\mathtt{Frob}^{0\,(3)}(p^2)$$

ℓ	$(-1)^{\ell+1} d_0^{(3)}(\ell)$	$(-1)^{\ell} d_1^{(3)}(\ell)$	$(-1)^{\ell} d_2^{(3)}(\ell)$
1	48	0	24
2	138	48	132
3	$\frac{7144}{9}$	528	800
4	$\frac{63695}{12}$	4652	6018
5	$\frac{15112894}{375}$	41672	$\frac{242424}{5}$
6	$\frac{73656439}{225}$	$\frac{5695028}{15}$	415088
7	$\frac{23997011104}{8575}$	$\frac{17647088}{5}$	$\frac{25875552}{7}$
8	$\frac{116790128543}{4704}$	$\frac{233804429}{7}$	33914049
9	$\frac{134933770848983}{595350}$	$\frac{33700241452}{105}$	$\frac{954894248}{3}$

Conclusion

We have found that the number of reflexive polytopes that arise from Feynman integral is rather sparse

Calabi–Yau varieties naturally arise from Feynman integrals when their Symanzik graph polynomials are interpreted through toric geometry.

When the Symanzik polynomials of a Feynman graph define a reflexive polytope, the graph integral naturally embeds in this mirror pair geometry in Batyrev's sense.

The Feynman integral computes a period of the graph hypersurface, while the reflexive polytope determines dual toric Fano varieties whose anticanonical hypersurfaces form a Batyrev mirror pair.
The integral's analytic behavior corresponds to the dimension and degeneration type of the underlying toric hypersurface.

This connection bridges quantum field theory and mirror symmetry in a interesting way that need to be systematized