ESI Workshop 2024

Carrollian Physics and Holography

The Carroll particles in Two Times

Federica Muscolino

Email: federica.muscolino@gmail.com

About this project

"Looking for Carroll particles in two time spacetime"

Alexander Kamenshchik and Federica Muscolino

Phys.Rev.D 109 (2024) 2, 025005

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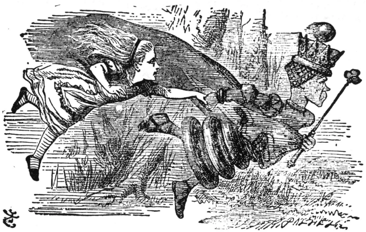
We explore the possibility of describing the Carrollian dynamics in the framework of the Two Time physics.

Itzhak Bars,
"Two-time physics", (1998).

The plan for today

During this presentation, we are going to touch the following points:



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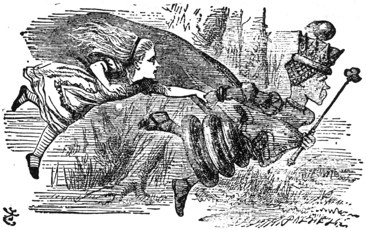
- What is the two time physics?



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- What is the two time physics?
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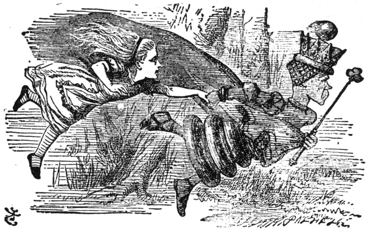

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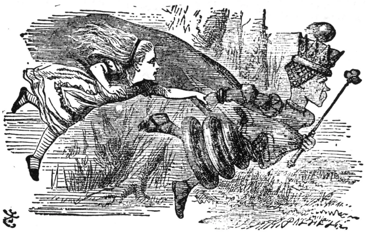
During this presentation, we are going to touch the following points:

- What is the two time physics?
- How can we describe the Carroll particles in this contest?
- Why is this interesting?



The two time physics
What is it and what is not

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- Gauging of $Sp(2,\mathbb{R})$

$$(X^M, P_M)$$
$$\downarrow$$
$$\begin{pmatrix} X'^M \\ P'_M \end{pmatrix} = A \begin{pmatrix} X^M \\ P_M \end{pmatrix}$$

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- It is a gauge theory of the phase space
- Gauging of $Sp(2,\mathbb{R})$
- Forces the introduction of a new coordinate
- The one time spacetime is obtained through a gauge fixing

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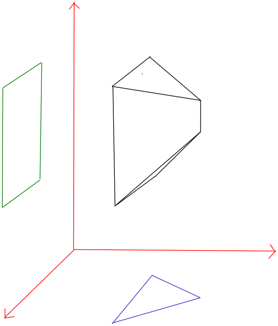

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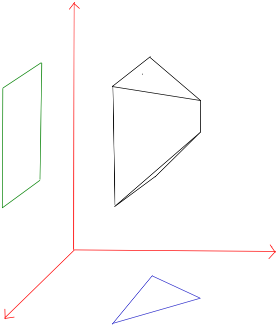

Dual theories

$\downarrow$

They are *separated* by an $Sp(2,\mathbb{R})$ transformation.

The two time physics
The *global* $Sp(2, \mathbb{R})$

The phase-space coordinates

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The *global* $Sp(2,\mathbb{R})$

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The covariant derivative

$$\partial_\tau X_i^M \to D_\tau X_i^M = \partial_\tau X_i^M - \epsilon_{ij}A^{jk}(\tau)X_k^M,$$
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$$Q_{11} = X\cdot X, \quad Q_{12} = X\cdot P, \quad Q_{22} = P\cdot P.$$

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The additional time is necessary in order to have a **non trivial dynamics**.

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The symmetries of the action

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$$S = \int d\tau \ \left[\eta^{MN} \partial_\tau X_M P_N - \frac{1}{2} A^{ij}(\tau) Q_{ij} \right]$$

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The $SO(2,d)$ generators:

$$L^{MN} = X^M P^N - X^N P^M = \epsilon^{ij} X_i^M X_j^N \quad \longrightarrow \quad \text{Invariant under } Sp(2,\mathbb{R})$$

When the gauge is fixed, the $SO(2,d)$ remains a symmetry of the action.

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The gauge fixing

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$$\downarrow$$
$$\begin{cases} \dot{x}^i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial x^i} \end{cases}$$

The two time physics
An example: the massless relativistic particle

$$X_i^+ = \frac{1}{2}\left(X_i^{0'} + X_i^{1'}\right)$$
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	$+$	$-$	μ
X^M	X^+	X^-	X^μ
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Equations for the μ component: $\begin{cases} \dot{X}^M = A^{12}X^M + A^{22}P^M \\ \dot{P}^M = -A^{12}P^M - A^{11}X^M \end{cases}$

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$P \cdot P = p^2$

The two time physics

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The action reduces to the one expected for the relativistic massless particle in the first order formalism

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The $SO(2,d)$ generators:

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \qquad L^{+\mu} = p^\mu, \qquad L^{+-} = x \cdot p$$

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L^{MN} generate the **conformal symmetry** of the relativistic particle.

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Another example: the non-relativistic particle

We can define a parametrization also for the **non-relativistic** particle

	$+$	$-$	0	i
X^M	t	$\frac{\mathbf{x}\cdot\mathbf{p}-tH}{m}$	$\pm\left\lvert\mathbf{x}-\frac{t}{m}\mathbf{p}\right\rvert$	x^i
P^M	m	H	0	p^i

$$A^{11} = A^{12} = 0$$
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The two time physics

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and the **equations of motion** become

$$\dot{t} = \lambda, \quad \dot{\mathbf{x}} = \lambda \frac{\mathbf{p}}{m}, \quad \dot{E} = 0, \quad \dot{\mathbf{p}} = 0.$$

The two time physics

Another example: the non-relativistic particle

The $SO(2,d)$ generators:

$$L^{ij} = x^i p^j - x^j p^i, \qquad L^{0i} = \pm\left|\mathbf{x} - \frac{t}{m}\mathbf{p}\right|\ p^i, \qquad L^{+i} = tp^i - mx^i$$

$$L^{-i} = \frac{\mathbf{x}\cdot\mathbf{p} - tH}{m}p^i - Hx^i, \qquad L^{+-} = -\mathbf{p}\cdot\mathbf{x},$$

$$L^{-0} = \mp\left|\mathbf{x} - \frac{t}{m}\mathbf{p}\right| H, \qquad L^{+0} = \mp\left|\mathbf{x} - \frac{t}{m}\mathbf{p}\right| m.$$

The two time physics

Other "dual" theories

Gauge choice		$+'$	$-'$	$m=(\mu\oplus i),\ \mu=0,1,\cdots$
Relativistic massless particle	$X^M=$	1	$\frac{1}{2}x^2$	x^μ
$p^2=0$	$P^M=$	0	$x\cdot p$	p^μ
$\mathrm{AdS}_{d-n}\times\mathrm{S}^n$	$X^M=$	$\frac{R_0^2}{\lvert\vec{y}\rvert}$	$\frac{1}{2\lvert\vec{y}\rvert}(x^2+\vec{y}^2)$	$\frac{R_0}{\lvert\vec{y}\rvert}x^\mu,\ \frac{R_0}{\lvert\vec{y}\rvert}\vec{y}^i$
$\vec{y}^2(p^2+\vec{k}^2)=0$	$P^M=$	0	$\frac{\lvert\vec{y}\rvert}{R_0^2}(x\cdot p+\vec{y}\cdot\vec{k})$	$\frac{\lvert\vec{y}\rvert}{R_0}p^\mu,\ \frac{\lvert\vec{y}\rvert}{R_0}\vec{k}^i$
Maximally Symmetric Spaces	$X^M=$	$1+\sqrt{1-Kx^2}$	$\frac{x^2/2}{1+\sqrt{1-Kx^2}}$	x^μ
$p^2-\frac{K\,(x\cdot p)^2}{1-Kx^2}=0$	$P^M=$	0	$\frac{\sqrt{1-Kx^2}}{1+\sqrt{1-Kx^2}}x\cdot p$	$p^\mu-\frac{Kx\cdot p\ x^\mu}{1+\sqrt{1-Kx^2}}$
Free function $\alpha(x)$	$X^M=$	$x^2+\alpha(x)$	$\frac{x^2/2}{x^2+\alpha(x)}$	x^μ
$p^2+\frac{4\alpha(x)(x\cdot p)^2}{(x^2-\alpha(x))^2}=0$	$P^M=$	0	$\frac{x\cdot p}{\alpha(x)-x^2}$	$p^\mu-\frac{2x\cdot p}{x^2-\alpha(x)}x^\mu$
Conformally flat $g_{\mu\nu}=e_\mu^m(x)e_\nu^n(x)\eta_{mn}$	$X^M=$	$\pm e^{\sigma(x)}$	$\pm\frac{1}{2}e^{\sigma(x)}q^2(x)$	$\pm e^{\sigma(x)}q^m(x^\mu)$ $e_\mu^m(x)\equiv\pm e^{\sigma(x)}\frac{\partial q^m(x)}{\partial x^\mu}$
$g^{\mu\nu}(x)\,p_\mu p_\nu=0$	$P^M=$	0	$q^m(x)\,e_m^\mu(x)\,p_\mu$	$e_m^\mu(x)\,p_\mu$
Relativistic massive particle	$X^M=$	$\frac{1+a}{2a}$	$\frac{x^2a}{1+a}$	x^μ $a\equiv\sqrt{1+\frac{m^2x^2}{(x\cdot p)^2}}$
$p^2+m^2=0$	$P^M=$	$\frac{-m^2}{2ax\cdot p}$	$a\ x\cdot p$	p^μ
Non-relativistic massive particle	$X^M=$	t	$\frac{\mathbf{r}\cdot\mathbf{p}-tH}{m}$	$X^0=\pm\left\lvert\mathbf{r}-\frac{t}{m}\mathbf{p}\right\rvert,\ X^i=\mathbf{r}^i$
$H-\frac{\mathbf{p}^2}{2m}=0$	$P^M=$	m	H	$P^0=0,\ P^i=\mathbf{p}^i$

Table1: Parametrization of X^M, P^M for $M=(+',-',(m\text{ or }\mu))$

I. Bars, "Dual Field Theories In $(d-1)+1$ Emergent Spacetimes From A Unifying Field Theory In $d+2$ Spacetime"

The Carroll dynamics from the two time physics

The two time physics is able to describe
relativistic and **non-relativistic** particles
Why not the Carroll particles?

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Some consequences:

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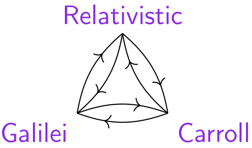
Dualities



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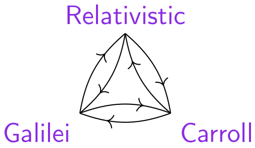

Hidden symmetries

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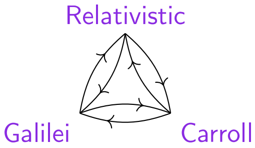

Hidden symmetries

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Different limits of the speed of light can be understood in terms of **different gauge fixing**

The Carroll dynamics from the two time physics
The *rest* particle

We want to reproduce the dynamics of the particle at rest

$$S = \int d\tau \left[-\dot{t}E + \dot{\mathbf{x}} \cdot \mathbf{p} + \lambda_t \left(E - E_0 \right) \right],$$

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$$E, \qquad p_i, \qquad B^i = Ex^i \qquad \text{and} \qquad L^{ij} = x^i p^j - x^j p^i$$

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$$E, \qquad p_i, \qquad B^i = Ex^i \qquad \text{and} \qquad L^{ij} = x^i p^j - x^j p^i$$

with the following **equations of motion**

$$\dot{t} = \lambda_t, \qquad \dot{x}^i = 0, \qquad \dot{E} = 0 \qquad \text{and} \qquad \dot{p}^i = 0.$$

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	$+$	$-$	0	i
X^M	X^+	X^-	X^0	X^i
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The equations of motions:

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X^M	X^+	X^-	X^0	$x^i + tp^i$
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The constraints:

$$X \cdot X = tX \cdot P = t^2 P \cdot P \qquad\qquad P \cdot P = -2(E - E_0)$$

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The *rest* particle: Gauge fixing

The final gauge choice: $P^+ = E_0, \qquad P^0 = 0$

The Carroll dynamics from the two time physics

The *rest* particle: Gauge fixing

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X^M	$E_0\ t$	$\frac{\mathbf{p}\cdot\mathbf{x}}{E_0} + \frac{t}{E_0}\left(E - E_0 + \frac{\mathbf{p}\cdot\mathbf{p}}{2}\right)$	$\sqrt{\mathbf{x}\cdot\mathbf{x}}$	$x^i + tp^i$
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The action:

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$$\downarrow$$

$$S = \int d\tau \left[\dot{\mathbf{x}}\cdot\mathbf{p} - E_0\right]$$

The Carroll dynamics from the two time physics

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The $SO(2,d)$ generators: $(r = \sqrt{\mathbf{x}\cdot\mathbf{x}})$

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The Carroll dynamics from the two time physics
The *rest* particle: Quantization

The **quantization** is defined by means of the canonical commutation rules:

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L^{MN} and Q_{ij} are functions of the operators x^i and p_i. They become Hermitian operators written in terms of x^i and p_i operators.

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The Carroll dynamics from the two time physics

The *rest* particle: The ordering problem

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The Carroll dynamics from the two time physics
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One can observe the same behavior for the higher Casimirs

$$C_2(Sp(2,\mathbb{R}))|\mathrm{Phys}\rangle = 0 \quad \Rightarrow \quad C_2(SO(2,d))|\mathrm{Phys}\rangle = \left(1 - \frac{d^2}{4}\right)|\mathrm{Phys}\rangle$$

The Carroll dynamics from the two time physics
The *rest* particle: The ordering problem

We can find an ordering that satisfies our conditions:

$$L^{ij} = x^i p^j - x^j p^i, \qquad L^{0i} = \frac{1}{2}\left(r\ p^i + p^i r\right), \qquad L^{+i} = -E_0 x^i$$

$$L^{-i} = -\frac{1}{2E_0} p_j x^i p^j + \frac{1}{2E_0}\left(\mathbf{p}\cdot\mathbf{x} p^i + p^i \mathbf{x}\cdot\mathbf{p}\right) + \frac{x^i}{8E_0 r^2},$$

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The Carroll dynamics from the two time physics

The *rest* particle: The ordering problem

We can find an ordering that satisfies our conditions:

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These generators forms the $SO(2,d)$ algebra, with the following **commutation rules**

$$\left[L^{MN}, L^{RS}\right] = i\eta^{MR} L^{NS} + i\eta^{NS} L^{MR} - i\eta^{MS} L^{NR} - i\eta^{NR} L^{MS}$$

The Carroll dynamics from the two time physics
The *rest* particle and the Hydrogen atom

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The $SO(2, d)$ generators are the same at $\tau = 0$.

I. Bars,
"Conformal Symmetry and Duality between Free Particle, H-atom and Harmonic Oscillator"

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Recap and concluding remarks

- We defined a gauge fixing that is able to reproduce the dynamics of a Carroll particle from the two time physics.
- The $SO(2, d)$ generators show peculiar correspondence with the H Atom.
- We are working on a gauge fixing that describe Carrollian tachyons.
- A systematic characterization and comprehension of the "sub-theories" would be useful.

Thank you

Thank you for listening!