Duality-invariant conformal higher-spin models

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Based on:

- SMK & E. S. N. Raptakis, *Duality-invariant (super)conformal higher-spin models*, arXiv:2107.02001 (to appear in PRD)
- SMK, *Superconformal duality-invariant models and $\mathcal{N} = 4$ SYM effective action*, arXiv:2106.07173 (to appear in JHEP)

What this talk is about

- This talk is about U(1) duality invariance in Fradkin-Tseytlin models for conformal higher-spin fields, their nonlinear and generalised versions.
- Duality is understood in this talk as a continuous symmetry of EoMs.
- There exist different approaches to duality, such as a manifest symmetry of the action, see e.g. Bunster & Henneaux (2011, 2012).
- Many colleagues in this audience have worked on different aspects of duality. I apologise for not being able to discuss their work. (A separate talk would be required to review various approaches to duality in field theory.)
- However, I'd like to mention the oldest (and truly influential) work: E. S. Fradkin & A. A. Tseytlin, "Quantum equivalence of dual field theories," Annals Phys. **162** (1985) 31.
- ... and the most recent one: Z. Avetisyan, O. Evnin & K. Mkrtchyan, "Democratic Lagrangians for nonlinear electrodynamics," [arXiv:2108.01103 [hep-th]].

Outline

1. Electromagnetic duality
2. U(1) duality in nonlinear electrodynamics
3. Conformal duality-invariant electrodynamics
4. Conformal geometry in $D > 3$ dimensions
5. Duality-invariant conformal higher-spin models
6. U(1) duality for complex conformal higher-spin fields

Electromagnetic duality: Maxwell's theory

- Maxwell's electrodynamics is the simplest and oldest example of a duality-invariant theory in four spacetime dimensions.

$$L_{\rm Maxwell}(F) = -\frac{1}{4}F^{ab}F_{ab} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) , \qquad F_{ab} = \partial_a A_b - \partial_b A_a$$

- The Bianchi identity and the equation of motion read

$$\partial^b \widetilde{F}_{ab} = 0 , \qquad \partial^b F_{ab} = 0$$

where $\widetilde{F}_{ab} := \frac{1}{2}\,\varepsilon_{abcd}\,F^{cd}$ is the Hodge dual of F.

- Since these differential equations have the same functional form, one may consider so-called duality rotations

$$F + {\rm i}\widetilde{F} \to {\rm e}^{{\rm i}\varphi}\big(F + {\rm i}\widetilde{F}\big) \quad \Longleftrightarrow \quad \vec{E} + {\rm i}\vec{B} \to {\rm e}^{{\rm i}\varphi}\big(\vec{E} + {\rm i}\vec{B}\big) , \quad \varphi \in \mathbb{R}$$

- Lagrangian $L_{\rm Maxwell}(F)$ changes, but the energy-momentum tensor

$$T^{ab} = \frac{1}{2}\big(F + {\rm i}\widetilde{F}\big)^{ac}\big(F - {\rm i}\widetilde{F}\big)^{bd}\,\eta_{cd} = F^{ac}F^{bd}\eta_{cd} - \frac{1}{4}\eta^{ab}F^{cd}F_{cd}$$

is invariant under U(1) duality transformations.

Electromagnetic duality: Born-Infeld theory

- In 1934, Born & Infeld put forward a particular model for nonlinear electrodynamics

$$L_{\rm BI}(F) = \frac{1}{g^2}\left\{ 1 - \sqrt{-\det(\eta_{ab} + gF_{ab})}\right\} = -\frac{1}{4}F^{ab}F_{ab} + \mathcal{O}(F^4)$$

  as a new fundamental theory of the electromagnetic field (with g the coupling constant).

- In 1935, Schrödinger showed that the Born-Infeld theory possessed invariance under generalised U(1) duality rotations.
  More precisely, he reformulated the Born-Infeld theory in such a way that there was manifest U(1) duality invariance.

- Although the great expectations of Born and Infeld never came true, the Born-Infeld action has re-appeared in the spotlight since the 1980's as a low-energy effective action in string theory.
  Fradkin & Tseytlin (1985)

Electromagnetic duality: Nonlinear electrodynamics

- Patterns of duality invariance in extended supergravity

  Ferrara, Scherk & Zumino (1977)
  Cremmer & Julia (1979)

- General theory of duality invariance in four dimensions

  Gaillard & Zumino (1981)
  Gibbons & Rasheed (1995)
  Gaillard & Zumino (1997)

- General theory of duality invariance in higher dimensions

  Gibbons & Rasheed (1995)
  Araki &Tanii (1999)
  Aschieri, Brace, Morariu & Zumino (2000)

- General theory of duality invariance for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric nonlinear electrodynamics

  SMK & Theisen (2000)

  Partial SUSY breaking often implies U(1) duality invariance.

- Remarkable reformulation of duality-invariant nonlinear electrodynamics (manifest duality-invariant self-interaction).

  Ivanov & Zupnik (2001,2002)

U(1) duality in nonlinear electrodynamics

- Nonlinear electrodynamics

$$L(F_{ab}) = -\frac{1}{4} F^{ab} F_{ab} + \mathcal{O}(F^4)$$

- Using the definition

$$\widetilde{G}_{ab}(F) := \frac{1}{2}\,\varepsilon_{abcd}\, G^{cd}(F) = 2\,\frac{\partial L(F)}{\partial F^{ab}}\;, \qquad G(F) = \widetilde{F} + \mathcal{O}(F^3),$$

  the Bianchi identity (BI) and the equation of motion (EoM) read

$$\partial^b \widetilde{F}_{ab} = 0\;, \qquad \partial^b \widetilde{G}_{ab} = 0\;.$$

- The same functional form of BI and EOM gives us a rationale to introduce a duality transformation

$$\begin{pmatrix} G'(F') \\ F' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} G(F) \\ F \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$$

  For $G'(F')$ one should require

$$\widetilde{G}'_{ab}(F') \;=\; 2\,\frac{\partial L'(F')}{\partial F'^{ab}}$$

  Transformed Lagrangian, $L'(F)$, always exists.

U(1) duality in nonlinear electrodynamics

The above considerations become nontrivial if the model is required to be duality invariant (self-dual)

$$L'(F) = L(F) \ .$$

The requirement of self-duality implies the following:

- Only U(1) duality transformations can consistently be defined in the nonlinear case.

$$\begin{pmatrix} G'(F') \\ F' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} G(F) \\ F \end{pmatrix}$$

  Maxwell's theory also allows scale duality transformations which, however, are forbidden if the energy-momentum tensor is required to be duality invariant.

- The Lagrangian is a solution of the self-duality equation

$$G^{ab}\,\widetilde{G}_{ab} + F^{ab}\,\widetilde{F}_{ab} = 0 \ , \qquad \widetilde{G}_{ab}(F) = 2\,\frac{\partial L(F)}{\partial F^{ab}}$$

Gibbons & Rasheed (1995) Gaillard & Zumino (1997)

Properties of U(1) duality-invariant models

- Duality invariance of the energy-momentum tensor.
- $\mathsf{SL}(2,\mathbb{R})$ duality invariance in the presence of dilaton and axion.
- Self-duality under Legendre transformation.

Legendre transformation for nonlinear electrodynamics $L(F)$.

- Associate with $L(F)$ an equivalent auxiliary model defined by

$$L(F, F_{\rm D}) = L(F) - \frac{1}{2}\, F \cdot \widetilde{F}_{\rm D}\ , \qquad\qquad F_{\rm D}{}^{ab} = \partial^a A_{\rm D}{}^b - \partial^b A_{\rm D}{}^a\ ,$$

  in which F_{ab} is an unconstrained two-form (auxiliary field).
- Eliminate F_{ab} using its equation of motions $G(F) = F_{\rm D}$ to yield

$$L_{\rm D}(F_{\rm D}) := \Big(L(F) - \frac{1}{2} F \cdot \widetilde{F}_{\rm D}\Big)\Big|_{F=F(F_{\rm D})}\ .$$

- If $L(F)$ solves the self-duality equation $G \cdot \widetilde{G} + F \cdot \widetilde{F} = 0$, then

$$L_{\rm D}(F) = L(F)\ .$$

Self-dual electrodynamics

General structure of self-dual electrodynamics

- Given a model for nonlinear electrodynamics, its Lagrangian $L(F_{ab})$ can be realised as a real function of one complex variable,

$$L(F_{ab}) = L(\omega, \bar{\omega}) \ , \qquad \omega = \alpha + \mathrm{i}\,\beta = F^{\alpha\beta} F_{\alpha\beta} \ ,$$

where $\alpha = \frac{1}{4} F^{ab} F_{ab}$ and $\beta = \frac{1}{4} F^{ab} \widetilde{F}_{ab}$ are the EM invariants.

$$L(\omega, \bar{\omega}) = -\frac{1}{2} \Big(\omega + \bar{\omega} \Big) + \omega \, \bar{\omega} \, \Lambda(\omega, \bar{\omega}) \ .$$

- Self-duality equation (SDE), $G \cdot \widetilde{G} + F \cdot \widetilde{F} = 0$, turns into

$$\mathrm{Im} \Big\{ \frac{\partial(\omega \, \Lambda)}{\partial \omega} - \bar{\omega} \, \Big(\frac{\partial(\omega \, \Lambda)}{\partial \omega} \Big)^2 \Big\} = 0 \ .$$

- Assuming $\Lambda(\omega, \bar{\omega})$ to be real analytic, the general solution of SDE involves a real function of one real argument $f(\omega\bar{\omega})$

$$\Lambda(\omega, \bar{\omega}) \quad = \quad \sum_{n=0}^{\infty} \ \sum_{p+q=n} c_{p,q} \, \omega^p \bar{\omega}^q \ , \qquad c_{p,q} = c_{q,p} \in \mathbb{R}$$

SDE uniquely fixes the level-n coefficients $c_{p,q}$ with $p \neq q$ through those at lower levels, while $c_{r,r}$ remain undetermined.

Duality-invariant theories with higher derivatives

In the case of theories with higher derivatives, the scheme should be generalised in accordance with the rules given in

SMK & Theisen (2000)
Aschieri, Ferrara & Zumino (2008)
Chemissany, Kallosh & Ortin (2012)

- Definition

$$\widetilde{G}^{ab}(F) := \frac{1}{2}\,\varepsilon^{abcd}\,G_{cd}(F) = 2\,\frac{\partial L(F)}{\partial F_{ab}}\ .$$

  is replaced with

$$\widetilde{G}^{ab}[F] = 2\,\frac{\delta S[F]}{\delta F_{ab}}\ .$$

- Self-duality equation $\widetilde{G}^{ab}G_{ab} + \widetilde{F}^{ab}F_{ab} = 0$ is replaced with

$$\int \mathrm{d}^4x\Big(\widetilde{G}^{ab}G_{ab} + \widetilde{F}^{ab}F_{ab}\Big) = 0\ .$$

  This must hold for $S[F]$ being a functional of an unconstrained two-form F_{ab}.

Formulation with manifestly U(1) invariant interaction

- Self-duality equation $G \cdot \widetilde{G} + F \cdot \widetilde{F} = 0$ is a nonlinear equation on the Lagrangian $L(F)$, and U(1) duality-invariant deformations of $L(F)$ are difficult to control.
- In 2001, Ivanov & Zupnik proposed a reformulation of nonlinear electrodynamics with the property that U(1) duality invariance becomes equivalent to manifest U(1) invariance of the interaction.
- Twisted self-duality constraint put forward by Bossard & Nicolai (2011) and by Carrasco, Kallosh & Roiban (2012) proves to be a variant of the Ivanov-Zupnik formulation.

Formulation with manifestly U(1) invariant interaction

- The Ivanov-Zupnik formulation involves an auxiliary (unconstrained) antisymmetric tensor $V_{ab} = -V_{ba}$, which is equivalently described by a symmetric rank-2 spinor $V_{\alpha\beta} = V_{\beta\alpha}$ and its conjugate $\bar{V}_{\dot{\alpha}\dot{\beta}}$, where $\alpha, \beta = 1, 2$.
- Inspired by the structure of $\mathcal{N} = 3$ supersymmetric Born-Infeld action in $\mathcal{N} = 3$ harmonic superspace, Ivanov & Zupnik replaced $L(F_{ab})$ with a new Lagrangian

$$L(F_{ab}, V_{ab}) = \frac{1}{4} F^{ab} F_{ab} + \frac{1}{2} V^{ab} V_{ab} - V^{ab} F_{ab} + L_{\rm int}(V_{ab}) \ .$$

  The original Lagrangian $L(F_{ab})$ is obtained from $L(F_{ab}, V_{ab})$ by integrating out the auxiliary variables.
- In terms of $L(F_{ab}, V_{ab})$, the condition of U(1) duality invariance proves to be equivalent to the requirement that the self-interaction

$$L_{\rm int}(V_{ab}) = L_{\rm int}(\nu, \bar{\nu}) \ , \qquad \nu := V^{\alpha\beta} V_{\alpha\beta}$$

  is invariant under linear U(1) transformations $\nu \to {\rm e}^{{\rm i}\varphi} \nu$, with $\varphi \in \mathbb{R}$,

$$L_{\rm int}(\nu, \bar{\nu}) = L_{\rm int}({\rm e}^{{\rm i}\varphi} \nu, {\rm e}^{-{\rm i}\varphi} \bar{\nu}) \quad \Longrightarrow \quad L_{\rm int}(\nu, \bar{\nu}) = h(\nu \bar{\nu}) \ ,$$

  with h a real function of one real variable.

Duality invariance and (super)conformal symmetry

- Perturbative scheme to construct $\mathcal{N}=2$ superconformal U(1) duality-invariant actions for the $\mathcal{N}=2$ vector multiplet (e.g., low-energy effective action for $\mathcal{N}=4$ SU(N) super-Yang-Mills theory on its Coulomb branch)

  SMK & Theisen (2000)

$$
\begin{aligned}
\mathcal{S} &= \frac{1}{8}\int \mathrm{d}^4x\mathrm{d}^4\theta\,\mathcal{W}^2 + \frac{1}{8}\int \mathrm{d}^4x\mathrm{d}^4\bar\theta\,\bar{\mathcal{W}}^2 + \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar\theta\,\mathcal{L}\,, \\
\mathcal{L} &= c\,\ln\mathcal{W}\,\ln\bar{\mathcal{W}} + \frac{1}{4}\,c^2\left(\ln\mathcal{W}\,\nabla\ln\mathcal{W} \;+\; \mathrm{c.c.}\right) \\
&\quad + \frac{1}{4}\,c^3\,d\,(\nabla\ln\mathcal{W})\,\bar\nabla\ln\bar{\mathcal{W}} - \frac{1}{8}\,c^3\left(\ln\mathcal{W}\,(\nabla\ln\mathcal{W})^2 \;+\; \mathrm{c.c.}\right) \\
&\quad + \frac{1}{16}\,c^4\Big((1-4d)\,(\nabla\ln\mathcal{W})^2\,\bar\nabla\ln\bar{\mathcal{W}} + (2d-1)\,(\nabla\ln\mathcal{W})\,\bar\nabla\nabla\ln\mathcal{W} \\
&\qquad\qquad + \frac{5}{3}\,\ln\mathcal{W}\,(\nabla\ln\mathcal{W})^3 \;+\; \mathrm{c.c.}\Big) \;+\; O(\nabla^4)\,.
\end{aligned}
$$

  c the anomaly coefficient; $\nabla := \frac{1}{\bar{\mathcal{W}}^2}\,D^4$ and $\bar\nabla := \frac{1}{\mathcal{W}^2}\,\bar D^4$.
  $\mathcal{W}$ chiral field strength of the $\mathcal{N}=2$ vector multiplet, $\bar D_i^{\dot\alpha}\mathcal{W}=0$, $D^{\alpha i}D_\alpha{}^j\mathcal{W} = \bar D_{\dot\alpha}{}^i\bar D^{\dot\alpha j}\bar{\mathcal{W}}$.

- In 2000, we did not look at the simpler $\mathcal{N}=0$ and $\mathcal{N}=1$ cases.
- Twenty years later, other people have studied the $\mathcal{N}=0$ case.

Conformal duality-invariant electrodynamics

- ModMax theory

$$L_{\rm conf}(\omega,\bar\omega) = -\frac{1}{2}\cosh\gamma\Big(\omega+\bar\omega\Big) + \sinh\gamma\sqrt{\omega\bar\omega}\ ,$$

with γ a positive parameter.

Bandos, Lechner, Sorokin & Townsend arXiv:2007.09092

Kosyakov arXiv:2007.13878

- Derivation of ModMax using the Ivanov-Zupnik approach

SMK arXiv:2106.07173

Unique conformal duality-invariant model corresponds to

$$L_{\rm int,conf}(\nu,\bar\nu) = \kappa\sqrt{\nu\bar\nu}\ ,$$

with κ a coupling constant. Integrating out the auxiliary variables $V_{\alpha\beta}$ and $\bar V_{\dot\alpha\dot\beta}$ leads to $L_{\rm conf}(\omega,\bar\omega)$ with

$$\sinh\gamma = \frac{\kappa}{1-(\kappa/2)^2}\ .$$

Superconformal duality-invariant electrodynamics

$\mathcal{N} = 1$ supersymmetric ModMax theory

Bandos, Lechner, Sorokin & Townsend arXiv:2106.07547

SMK arXiv:2106.07173

$$
\begin{aligned}
S[W, \bar{W}] &= \frac{1}{4}\cosh\gamma \int \mathrm{d}^4x\mathrm{d}^2\theta\, \mathcal{E}\, W^2 + \mathrm{c.c.} \\
&+ \frac{1}{4}\sinh\gamma \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, E\, \frac{W^2\,\bar{W}^2}{\sqrt{u\bar{u}}}\ ,
\end{aligned}
$$

where $u := \frac{1}{8}(\mathcal{D}^2 - 4\bar{R})W^2$, $W^2 = W^\alpha W_\alpha$, and

$$
W_\alpha = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\mathcal{D}_\alpha V\ , \qquad \bar{\mathcal{D}}_{\dot\beta} W_\alpha = 0
$$

is the chiral field strength of the vector multiplet.

Conformal geometry in $D > 3$ dimensions

Kaku, Townsend & van Nieuwenhuizen (1977)

- The conformal algebra in $D > 2$ dimensions, $\mathfrak{so}(D, 2)$, is spanned by the generators of translation (P_a), Lorentz (M_{ab}), special conformal (K_a) and dilatation ($\mathbb{D}$). The non-vanishing commutation relations are:

$$[M_{ab}, M_{cd}] = 2\eta_{c[a}M_{b]d} - 2\eta_{d[a}M_{b]c} \ ,$$
$$[M_{ab}, P_c] = 2\eta_{c[a}P_{b]} \ , \qquad [M_{ab}, K_c] = 2\eta_{c[a}K_{b]} \ ,$$
$$[K_a, P_b] = 2\eta_{ab}\mathbb{D} + 2M_{ab} \ , \qquad [\mathbb{D}, P_a] = P_a \ , \quad [\mathbb{D}, K_a] = -K_a \ .$$

- Conformal covariant derivatives ∇_a

$$\nabla_a = e_a{}^m\partial_m - \frac{1}{2}\omega_a{}^{bc}M_{bc} - \mathfrak{b}_a\mathbb{D} - \mathfrak{f}_a{}^bK_b \ .$$

- For $D > 3$ the algebra of conformal covariant derivatives is

$$[\nabla_a, \nabla_b] = -\frac{1}{2}C_{abcd}M^{cd} - \frac{1}{2(D-3)}\nabla^dC_{abcd}K^c \ .$$

  It is determined by a single primary tensor field, the Weyl tensor C_{abcd}.

- Primary field Φ of dimension Δ is characterised by the condition:

$$K_a\Phi = 0 \ , \qquad \mathbb{D}\Phi = \Delta\Phi \ .$$

- Gauge condition $\mathfrak{b}_a = 0 \implies$ tractor calculus.

Conformal geometry in four dimensions

- In the $D = 4$ case, the two-component spinor formalism is indispensable

$$h_a \to h_{\alpha\dot\alpha} = (\sigma^b)_{\alpha\dot\alpha} h_b \quad \Longleftrightarrow \quad h_a = -\frac{1}{2}(\tilde\sigma_a)^{\dot\beta\beta} h_{\beta\dot\beta}$$

- Given a symmetric and traceless field $h_{a(s)} := h_{a_1 \dots a_s}$, it is equivalently described by a symmetric spinor field $h_{\alpha(s)\dot\alpha(s)} := h_{(\alpha_1\dots\alpha_s)(\dot\alpha_1\dots\dot\alpha_s)}$ defined by

$$h_{a(s)} \to h_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} = (\sigma^{a_1})_{\alpha_1\dot\alpha_1} \cdots (\sigma^{a_s})_{\alpha_s\dot\alpha_s} h_{a_1\dots a_s} = h_{\alpha(s)\dot\alpha(s)}$$

- The algebra of conformal covariant derivatives

$$\begin{aligned} [\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] \quad &= \quad -(\varepsilon_{\dot\alpha\dot\beta} C_{\alpha\beta\gamma\delta} M^{\gamma\delta} + \varepsilon_{\alpha\beta} \bar C_{\dot\alpha\dot\beta\dot\gamma\dot\delta} \bar M^{\dot\gamma\dot\delta}) \\ &\quad -\frac{1}{4}(\varepsilon_{\dot\alpha\dot\beta} \nabla^{\delta\dot\gamma} C_{\alpha\beta\delta}{}^{\gamma} + \varepsilon_{\alpha\beta} \nabla^{\gamma\dot\delta} \bar C_{\dot\alpha\dot\beta\dot\delta}{}^{\dot\gamma}) K_{\gamma\dot\gamma} \ . \end{aligned}$$

  Here $C_{\alpha\beta\gamma\delta}$ and $\bar C_{\dot\alpha\dot\beta\dot\gamma\dot\delta}$ are the self-dual and anti self-dual parts of the Weyl tensor C_{abcd}, and are primary.

- Important commutation relation

$$[K_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] \quad = \quad 4(\varepsilon_{\dot\alpha\dot\beta} M_{\alpha\beta} + \varepsilon_{\alpha\beta} \bar M_{\dot\alpha\dot\beta} - \varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} \mathbb{D}) \ .$$

- The Lorentz generators act on vectors and Weyl spinors as follows:

$$M_{ab} V_c = 2\eta_{c[a} V_{b]} \ , \qquad M_{\alpha\beta} \psi_\gamma = \varepsilon_{\gamma(\alpha} \psi_{\beta)} \ , \qquad \bar M_{\dot\alpha\dot\beta} \bar\psi_{\dot\gamma} = \varepsilon_{\dot\gamma(\dot\alpha} \bar\psi_{\dot\beta)} \ .$$

Conformal higher-spin fields in curved space

- Given an integer $s \geq 1$, consider a real spin-s field $h_{\alpha(s)\dot\alpha(s)} := h_{\alpha_1 \dots \alpha_s \dot\alpha_1 \dots \dot\alpha_s} = h_{(\alpha_1 \dots \alpha_s)(\dot\alpha_1 \dots \dot\alpha_s)}$ in curved spacetime. Its conformal properties are fixed by demanding

$$K_b h_{\alpha(s)\dot\alpha(s)} = 0 \ , \qquad \mathbb{D} h_{\alpha(s)\dot\alpha(s)} = (2-s) h_{\alpha(s)\dot\alpha(s)} \ .$$

- Associated with $h_{\alpha(s)\dot\alpha(s)}$ is its descendant

$$\mathcal{C}_{\alpha(2s)} = \nabla_{(\alpha_1}{}^{\dot\beta_1} \dots \nabla_{\alpha_s}{}^{\dot\beta_s} h_{\alpha_{s+1} \dots \alpha_{2s})\dot\beta(s)}$$

with nice conformal properties:

$$K_b \mathcal{C}_{\alpha(2s)} = 0 \ , \qquad \mathbb{D} \mathcal{C}_{\alpha(2s)} = 2 \mathcal{C}_{\alpha(2s)} \ .$$

- Since $\mathcal{C}_{\alpha(2s)}$ is primary and of dimensions $+2$, the functional

$$S^{(s)}_{\rm FTL}[\mathcal{C}, \bar{\mathcal{C}}] = \frac{(-1)^s}{2} \int \mathrm{d}^4 x\, e \left\{ \mathcal{C}^{\alpha(2s)} \mathcal{C}_{\alpha(2s)} + \text{c.c.} \right\}$$

is locally conformally invariant.

Conformal higher-spin fields in curved space

- The conformal properties of $h_{\alpha(s)\dot\alpha(s)}$ are consistent with gauge transformations of the form

$$\delta_\zeta h_{\alpha(s)\dot\alpha(s)} = \nabla_{(\alpha_1(\dot\alpha_1}\zeta_{\alpha_2\dots\alpha_s)\dot\alpha_2\dots\dot\alpha_s)}~,$$

  where the gauge parameter $\zeta_{\alpha(s-1)\dot\alpha(s-1)}$ is also primary.
- However, for a generic background, the gauge transformations leave the field strength $\mathcal{C}_{\alpha(2s)}$ invariant only when $s=1$, $\delta_\zeta \mathcal{C}_{\alpha(2)}=0$.
- For $s\geq 2$ gauge invariance holds only if the background is conformally flat,

$$C_{\alpha(4)}=0 \quad \Longrightarrow \quad \delta_\zeta\mathcal{C}_{\alpha(2s)}=0~.$$

- In what follows, the background spacetime is assumed to be conformally flat, $C_{\alpha(4)}=0$.

Bianchi identity (BI) and equation of motion (EoM)

- Let $S^{(s)}[\mathcal{C},\bar{\mathcal{C}}]$ be a gauge-invariant action functional describing the propagation of a conformal spin-s field $h_{\alpha(s)\dot{\alpha}(s)}$.
- The field strengths $\mathcal{C}_{\alpha(2s)}$ and $\bar{\mathcal{C}}_{\dot{\alpha}(2s)}$ obey BI

$$\nabla^{\beta_1}{}_{(\dot{\alpha}_1} \dots \nabla^{\beta_s}{}_{\dot{\alpha}_s)}\mathcal{C}_{\alpha(s)\beta(s)} = \nabla_{(\alpha_1}{}^{\dot{\beta}_1} \dots \nabla_{\alpha_s)}{}^{\dot{\beta}_s}\bar{\mathcal{C}}_{\dot{\alpha}(s)\dot{\beta}(s)} \ .$$

- Extending $S^{(s)}[\mathcal{C},\bar{\mathcal{C}}]$ to be a functional of an unconstrained field $\mathcal{C}_{\alpha(2s)}$ and its conjugate, we introduce primary dimension-2 field

$$\mathrm{i}\mathcal{M}_{\alpha(2s)} := \frac{\delta S^{(s)}[\mathcal{C},\bar{\mathcal{C}}]}{\delta \mathcal{C}^{\alpha(2s)}} \ , \qquad K_b\mathcal{M}_{\alpha(2s)} = 0 \ , \quad \mathbb{D}\mathcal{M}_{\alpha(2s)} = 2\mathcal{M}_{\alpha(2s)} \ ,$$

where the functional derivative is defined by

$$\delta S^{(s)}[\mathcal{C},\bar{\mathcal{C}}] = \int \mathrm{d}^4x\, e\, \delta\mathcal{C}^{\alpha(2s)} \frac{\delta S^{(s)}[\mathcal{C},\bar{\mathcal{C}}]}{\delta\mathcal{C}^{\alpha(2s)}} + \text{c.c.}$$

- Varying $S^{(s)}[\mathcal{C},\bar{\mathcal{C}}]$ with respect to $h_{\alpha(s)\dot{\alpha}(s)}$ yields EoM

$$\nabla^{\beta_1}{}_{(\dot{\alpha}_1} \dots \nabla^{\beta_s}{}_{\dot{\alpha}_s)}\mathcal{M}_{\alpha(s)\beta(s)} = \nabla_{(\alpha_1}{}^{\dot{\beta}_1} \dots \nabla_{\alpha_s)}{}^{\dot{\beta}_s}\bar{\mathcal{M}}_{\dot{\alpha}(s)\dot{\beta}(s)} \ .$$

- The BI and EoM have the same functional form.

U(1) duality invariance

- The functional form of EoM mirrors that of BI. Consequently, we can introduce $\mathrm{SO}(2) \cong \mathrm{U}(1)$ duality transformations:

$$\delta_\lambda \mathcal{C}_{\alpha(2s)} = \lambda \mathcal{M}_{\alpha(2s)} \ , \quad \delta_\lambda \mathcal{M}_{\alpha(2s)} = -\lambda \mathcal{C}_{\alpha(2s)} \ ,$$

  where λ is a constant, real parameter.
- Two equivalent expressions for the variation of $S^{(s)}[\mathcal{C}, \bar{\mathcal{C}}]$

$$\delta_\lambda S^{(s)}[\mathcal{C}, \bar{\mathcal{C}}] = \frac{\mathrm{i}\lambda}{4} \int \mathrm{d}^4x\, e \left\{ \mathcal{C}^2 - \mathcal{M}^2 \right\} + \text{c.c.} = -\frac{\mathrm{i}\lambda}{2} \int \mathrm{d}^4x\, e\, \mathcal{M}^2 + \text{c.c.}$$

- Self-duality equation

$$\mathrm{Im} \int \mathrm{d}^4x\, e \left\{ \mathcal{C}^{\alpha(2s)} \mathcal{C}_{\alpha(2s)} + \mathcal{M}^{\alpha(2s)} \mathcal{M}_{\alpha(2s)} \right\} = 0$$

  It must hold for unconstrained fields $\mathcal{C}_{\alpha(2s)}$ and $\bar{\mathcal{C}}_{\dot{\alpha}(2s)}$.
- $s = 1$: Gibbons-Rasheed-Gaillard-Zumino self-duality equation.

Simplest solutions of the self-duality equation

- Fradkin-Tseytlin-Linetsky conformal spin-s action

$$S^{(s)}_{\rm FTL}[\mathcal{C},\bar{\mathcal{C}}] = \frac{(-1)^s}{2}\int {\rm d}^4x\, e\,\Big\{\mathcal{C}^{\alpha(2s)}\mathcal{C}_{\alpha(2s)} + {\rm c.c.}\Big\}$$

$\mathbb{M}^4$: Fradkin & Tseytlin (1985); Fradkin & Linetsky (1989)
Conformally flat backgrounds: SMK & Ponds (2019)

- Higher-spin ModMax theory

$$S^{(s)}_{\rm ModMax}[\mathcal{C},\bar{\mathcal{C}}] = \frac{(-1)^s\cosh\gamma}{2}\int {\rm d}^4x\, e\,\Big\{\mathcal{C}^2 + \bar{\mathcal{C}}^2\Big\} + \sinh\gamma\int {\rm d}^4x\, e\,\sqrt{\mathcal{C}^2\bar{\mathcal{C}}^2}\,,$$

with $\mathcal{C}^2 = \mathcal{C}^{\alpha(2s)}\mathcal{C}_{\alpha(2s)}$.

SMK & Raptakis (2021)

This nonlinear theory is conformal and U(1) duality-invariant. It is a one-parameter ($\gamma \in \mathbb{R}$) extension of $S^{(s)}_{\rm FTL}[\mathcal{C},\bar{\mathcal{C}}]$.
For $s = 1$ the model coincides with ModMax electrodynamics.

Auxiliary field formulation

- To generate duality-invariant higher-spin models, a formulation with auxiliary variables is desirable.
- Consider the following action functional

$$S^{(s)}[\mathcal{C},\bar{\mathcal{C}},\eta,\bar{\eta}] = (-1)^s \int \mathrm{d}^4x\, e \left\{2\eta\mathcal{C} - \eta^2 - \frac{1}{2}\mathcal{C}^2\right\} + \text{c.c.} + \mathcal{S}^{(s)}_{\text{int}}[\eta,\bar{\eta}]$$

  Here $\eta_{\alpha(2s)}$ is an unconstrained primary dimension-2 field,

$$K_b\eta_{\alpha(2s)} = 0 \ , \qquad \mathbb{D}\eta_{\alpha(2s)} = 2\eta_{\alpha(2s)} \ .$$

- Equation of motion for $\eta^{\alpha(2s)}$

$$\eta_{\alpha(2s)} = \mathcal{C}_{\alpha(2s)} + \frac{(-1)^s}{2}\frac{\delta\mathcal{S}^{(s)}_{\text{int}}[\eta,\bar{\eta}]}{\delta\eta^{\alpha(2s)}}$$

  allows one to express $\eta_{\alpha(2s)}$ as a functional of $\mathcal{C}_{\alpha(2s)}$ and $\bar{\mathcal{C}}_{\dot{\alpha}(2s)}$.
- U(1) duality invariance is equivalent to the requirement that $\mathcal{S}^{(s)}_{\text{int}}[\eta,\bar{\eta}]$ is invariant under rigid U(1) phase transformations

$$\mathcal{S}^{(s)}_{\text{int}}[\mathrm{e}^{\mathrm{i}\varphi}\eta, \mathrm{e}^{-i\varphi}\bar{\eta}] = \mathcal{S}^{(s)}_{\text{int}}[\eta,\bar{\eta}] \ , \quad \varphi \in \mathbb{R} \ .$$

Nonlinear duality-invariant conformal spin-2 model

- Algebraic invariants of the symmetric rank-$(2s)$ spinor $\eta_{\alpha(2s)}$

$$\eta^2 := (-1)^s \eta_{\alpha(s)}{}^{\beta(s)} \eta_{\beta(s)}{}^{\alpha(s)} ~, \qquad \eta^3 := \eta_{\alpha(s)}{}^{\beta(s)} \eta_{\beta(s)}{}^{\gamma(s)} \eta_{\gamma(s)}{}^{\alpha(s)} ~, \qquad \dots$$

  If s is odd, all invariants η^{2n+1} vanish.

- For $s = 2$ there are two independent algebraic invariants, η^2 and η^3.

$$s = 2: \qquad \eta^4 = \frac{1}{2}(\eta^2)^2$$

- Conformal U(1) invariant self-interaction for $s = 2$

$$\mathcal{S}^{(2)}_{\text{int}}[\eta, \bar{\eta}] = \int \mathrm{d}^4 x \, e \left\{ \beta \left(\eta^2 \bar{\eta}^2\right)^{\frac{1}{2}} + \kappa \left(\eta^3 \bar{\eta}^3\right)^{\frac{1}{3}} \right\} ~,$$

  where β and κ are real coupling constants.

- Elimination of the auxiliary variables gives

$$\begin{aligned} S^{(2)}[\mathcal{C}, \bar{\mathcal{C}}] &= \int \mathrm{d}^4 x \, e \left\{ \frac{1}{2}\left(1 + \frac{1}{2}\beta^2\right)(\mathcal{C}^2 + \bar{\mathcal{C}}^2) + \beta(\mathcal{C}^2 \bar{\mathcal{C}}^2)^{\frac{1}{2}} + \kappa(\mathcal{C}^3 \bar{\mathcal{C}}^3)^{\frac{1}{3}} \right. \\ &\quad + \frac{1}{2}\beta\kappa \frac{(\mathcal{C}^3)^2 \bar{\mathcal{C}}^2 + (\bar{\mathcal{C}}^3)^2 \mathcal{C}^2}{(\mathcal{C}^3 \bar{\mathcal{C}}^3)^{\frac{2}{3}} (\mathcal{C}^2 \bar{\mathcal{C}}^2)^{\frac{1}{2}}} + \frac{1}{12}\kappa^2 \frac{(\mathcal{C}^2)^2 + (\bar{\mathcal{C}}^2)^2}{(\mathcal{C}^3 \bar{\mathcal{C}}^3)^{\frac{1}{3}}} \\ &\quad \left. - \frac{1}{24}\kappa^2 \frac{(\mathcal{C}^3)^2 (\bar{\mathcal{C}}^2)^2 + (\bar{\mathcal{C}}^3)^2 (\mathcal{C}^2)^2}{(\mathcal{C}^3 \bar{\mathcal{C}}^3)^{\frac{4}{3}}} + \dots \right\} . \end{aligned}$$

U(1) duality for complex conformal higher-spin fields

- So far our attention was restricted to conformal higher-spin (CHS) models described by real gauge prepotentials $h_{\alpha(s)\dot\alpha(s)}$.
- Supersymmetric duality-invariant CHS theories also involve fermionic gauge prepotentials $\psi_{\alpha(s+1)\dot\alpha(s)}$, and thus there should exist a way to define duality transformations for fermions.
- More generally, one may consider a complex CHS gauge prepotential $\phi_{\alpha(m)\dot\alpha(n)}$, with $m, n \geq 1$ and $m \neq n$, defined modulo gauge transformations

$$\delta_\ell \phi_{\alpha(m)\dot\alpha(n)} = \nabla_{(\alpha_1(\dot\alpha_1}\ell_{\alpha_2\dots\alpha_m)\dot\alpha_2\dots\dot\alpha_n)} .$$

- Conformal properties

$$K_b \phi_{\alpha(m)\dot\alpha(n)} = 0 , \qquad \mathbb{D}\phi_{\alpha(m)\dot\alpha(n)} = (2 - \frac{1}{2}(m+n))\phi_{\alpha(m)\dot\alpha(n)}$$

$\mathbb{M}^4$: Vasiliev (2009)

SMK, Manvelyan & Theisen (2017)

Conformally flat backgrounds: SMK & Ponds (2019)

U(1) duality for complex conformal higher-spin fields

- Introduce field strengths

$$\hat{\mathcal{C}}_{\alpha(m+n)} = \nabla_{(\alpha_1}{}^{\dot\beta_1} \dots \nabla_{\alpha_n}{}^{\dot\beta_n} \phi_{\alpha_{n+1}\dots\alpha_{m+n})\dot\beta(n)} ,$$
$$\check{\mathcal{C}}_{\alpha(m+n)} = \nabla_{(\alpha_1}{}^{\dot\beta_1} \dots \nabla_{\alpha_m}{}^{\dot\beta_m} \bar\phi_{\alpha_{m+1}\dots\alpha_{m+n})\dot\beta(m)} .$$

- They are primary fields in generic backgrounds,

$$K_b \hat{\mathcal{C}}_{\alpha(m+n)} = 0 , \qquad \mathbb{D}\hat{\mathcal{C}}_{\alpha(m+n)} = \Big(2 + \frac{1}{2}(n-m)\Big)\hat{\mathcal{C}}_{\alpha(m+n)} ;$$
$$K_b \check{\mathcal{C}}_{\alpha(m+n)} = 0 , \qquad \mathbb{D}\check{\mathcal{C}}_{\alpha(m+n)} = \Big(2 + \frac{1}{2}(m-n)\Big)\check{\mathcal{C}}_{\alpha(m+n)} .$$

- They are gauge-invariant in any conformally flat background,

$$C_{\alpha(4)} = 0 \quad \Longrightarrow \quad \delta_\ell \hat{\mathcal{C}}_{\alpha(m+n)} = \delta_\ell \check{\mathcal{C}}_{\alpha(m+n)} = 0 .$$

- Free gauge-invariant CHS action

$$S^{(m,n)}_{\rm free}[\hat{\mathcal{C}}, \check{\mathcal{C}}, \bar{\hat{\mathcal{C}}}, \bar{\check{\mathcal{C}}}] = {\rm i}^{m+n} \int {\rm d}^4x\, e\, \hat{\mathcal{C}}^{\alpha(m+n)} \check{\mathcal{C}}_{\alpha(m+n)} + {\rm c.c.}$$

U(1) duality for complex conformal higher-spin fields

- Bianchi identity

$$\nabla^{\beta_1}{}_{(\dot\alpha_1} \dots \nabla^{\beta_m}{}_{\dot\alpha_m)} \hat{\mathcal{C}}_{\alpha(n)\beta(m)} = \nabla_{(\alpha_1}{}^{\dot\beta_1} \dots \nabla_{\alpha_n)}{}^{\dot\beta_n} \bar{\check{\mathcal{C}}}_{\dot\alpha(m)\dot\beta(n)} \ .$$

- Given a dynamical system with action $S^{(m,n)}[\hat{\mathcal{C}}, \check{\mathcal{C}}, \bar{\hat{\mathcal{C}}}, \bar{\check{\mathcal{C}}}]$, the equation of motion for $\phi_{\alpha(m)\dot\alpha(n)}$ is

$$\nabla^{\beta_1}{}_{(\dot\alpha_1} \dots \nabla^{\beta_m}{}_{\dot\alpha_m)} \hat{\mathcal{M}}_{\alpha(n)\beta(m)} = \nabla_{(\alpha_1}{}^{\dot\beta_1} \dots \nabla_{\alpha_n)}{}^{\dot\beta_n} \overline{\check{\mathcal{M}}}_{\dot\alpha(m)\dot\beta(n)} \ ,$$

where we have defined

$$\mathrm{i}^{m+n+1} \hat{\mathcal{M}}_{\alpha(m+n)} := \frac{\delta S^{(m,n)}[\hat{\mathcal{C}}, \check{\mathcal{C}}, \bar{\hat{\mathcal{C}}}, \bar{\check{\mathcal{C}}}]}{\delta \check{\mathcal{C}}^{\alpha(m+n)}} \ ,$$

$$\mathrm{i}^{m+n+1} \check{\mathcal{M}}_{\alpha(m+n)} := \frac{\delta S^{(m,n)}[\hat{\mathcal{C}}, \check{\mathcal{C}} \ , \bar{\hat{\mathcal{C}}}, \bar{\check{\mathcal{C}}}]}{\delta \hat{\mathcal{C}}^{\alpha(m+n)}} \ ,$$

- Conformal properties of the equations of motion:

$$K_b \hat{\mathcal{M}}_{\alpha(m+n)} = 0 \ , \qquad \mathbb{D} \hat{\mathcal{M}}_{\alpha(m+n)} = \Big(2 + \frac{1}{2}(n-m)\Big) \hat{\mathcal{M}}_{\alpha(m+n)} \ ;$$

$$K_b \check{\mathcal{M}}_{\alpha(m+n)} = 0 \ , \qquad \mathbb{D} \check{\mathcal{M}}_{\alpha(m+n)} = \Big(2 + \frac{1}{2}(m-n)\Big) \check{\mathcal{M}}_{\alpha(m+n)} \ .$$

U(1) duality for complex conformal higher-spin fields

- U(1) duality rotations

$$\delta_\lambda \hat{\mathcal{C}}_{\alpha(m+n)} = \lambda \hat{\mathcal{M}}_{\alpha(m+n)} , \quad \delta_\lambda \check{\mathcal{C}}_{\alpha(m+n)} = \lambda \check{\mathcal{M}}_{\alpha(m+n)} ,$$
$$\delta_\lambda \hat{\mathcal{M}}_{\alpha(m+n)} = -\lambda \hat{\mathcal{C}}_{\alpha(m+n)} , \quad \delta_\lambda \check{\mathcal{M}}_{\alpha(m+n)} = -\lambda \check{\mathcal{C}}_{\alpha(m+n)} .$$

- Self-duality equation

$$\mathrm{i}^{m+n+1} \int \mathrm{d}^4 x \, e \left\{ \hat{\mathcal{C}}^{\alpha(m+n)} \check{\mathcal{C}}_{\alpha(m+n)} + \hat{\mathcal{M}}^{\alpha(m+n)} \check{\mathcal{M}}_{\alpha(m+n)} \right\} + \text{c.c.} = 0$$

- The simplest solution of this equation is the free CHS action

$$S^{(m,n)}_{\text{free}}[\hat{\mathcal{C}}, \check{\mathcal{C}}, \bar{\hat{\mathcal{C}}}, \bar{\check{\mathcal{C}}}] = \mathrm{i}^{m+n} \int \mathrm{d}^4 x \, e \, \hat{\mathcal{C}}^{\alpha(m+n)} \check{\mathcal{C}}_{\alpha(m+n)} + \text{c.c.}$$

Thank you!