Resurgent Asymptotics of Hopf Algebraic Dyson-Schwinger Equations

Gerald Dunne

University of Connecticut

Higher Structures Emerging from Renormalisation

Schrödinger Institute, Vienna, October 14, 2020

M. Borinsky & GD, 2005.04265; M. Borinsky, GD, M. Meynig, 2020 to appear

O. Costin & GD, 1904.11593, 2003.07451, 2009.01962, ...

[DOE Division of High Energy Physics]

Motivation

- Kreimer-Connes:

[perturbative] QFT renormalisation $\longleftrightarrow$ Hopf algebra stucture

$\Rightarrow$ enables perturbative computations to very high order

- Écalle: resurgent asymptotics

[perturbative] series $\longrightarrow$ [perturbative + nonperturbative] transseries

$\Rightarrow$ nonperturbative physics encoded in perturbative physics

IDEA: use resurgent trans-series to decode nonperturbative properties of QFT from their perturbative Hopf algebra structure

Trans-series

- an interesting observation by Hardy:

No function has yet presented itself in analysis, the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms

G. H. Hardy, *Orders of Infinity*, 1910

- deep result: “this is all we need” (J. Écalle, 1980s)
- also as a closed logic system: Dahn and Göring (1980s)

Resurgent Trans-Series

- Écalle: *resurgent functions* closed under all operations:

(Borel transform) + (analytic continuation) + (Laplace transform)

- basic trans-series expansion in QM & QFT applications:

$$f(x) \sim \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, x^p}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{x}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{x}\right]\right)^l}_{\text{logarithm powers}}$$

- *transmonomial elements*: x, $e^{-\frac{1}{x}}$, $\ln(x)$, familiar in QFT
- new: analytic continuation encoded in trans-series
- new: trans-series coefficients $c_{k,l,p}$ are highly correlated
- new: exponentially improved asymptotics
- explored in ODEs, PDEs, difference eqs., QM, matrix models, QFT, string theory, ...

“Resurgence”

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle

fluctuations about different singularities are quantitatively related

- resurgence is well established in matrix models and QM
- renormalisation makes resurgence in quantum field theory extremely interesting and also difficult
- recent progress for regularised QFTs and lattice QFT
- here: invoke Hopf algebra structure of perturbative QFT

Nonlinear ODEs from Dyson-Schwinger Equations

Combinatoric explosion of renormalization tamed by Hopf algebra: 30-loop Padé-Borel resummation

D.J. Broadhurst [1], D. Kreimer [2]

Erwin Schrödinger Institute, A-1090 Wien, Austria

Physics Letters B 475 (2000) 63–70

Exact solutions of Dyson–Schwinger equations for iterated one-loop integrals and propagator-coupling duality

D.J. Broadhurst [a,1], D. Kreimer [b,2]

Nuclear Physics B 600 (2001) 403–422

An Étude in non-linear Dyson–Schwinger Equations*

Dirk Kreimer[a†]
Karen Yeats[b]

Nuclear Physics B (Proc. Suppl.) 160 (2006) 116–121

Nonlinear ODEs from Dyson-Schwinger Equations

- Broadhurst/Kreimer 1999/2000; Kreimer/Yeats 2006:

for certain QFTs the renormalisation group equations can be reduced to coupled nonlinear ODEs for the anomalous dimension in terms of the renormalised coupling

- resurgence is deeply understood for (nonlinear) ODEs (Écalle, Costin, Kruskal, Ramis, Sauzin, Fauvet, ...)

- so this is a natural place to start

- some paradigmatic cases: Wess-Zumino model (Bellon, Schaposnik, Clavier, 2008, 2016, 2018); 4 dim. Yukawa (Borinsky, GD, 2020); 6 dim. ϕ^3 theory (Bellon & Russo, 2020), (Borinsky, GD, Meynig, 2020)

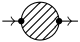
- also related: Maiezza, Vasquez (2019, 2020)

- future goal: gauge theories

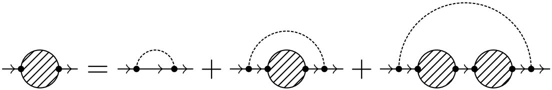
4 dimensional massless Yukawa theory

Broadhurst/Kreimer, 1999

- renormalised fermion self-energy

$$\Sigma(q) :=$$  $$= \not{q}\,\Sigma(q^2)$$

- Dyson-Schwinger equation

 $+ \cdots -$ subtractions

- anomalous dimension $\gamma(\alpha)$ ($\alpha \equiv$ renormalised coupling):

$$\gamma(\alpha) = \frac{d}{d\ln q^2} \ln\left(1 - \Sigma(q^2)\right)\Bigg|_{q^2=\mu^2}$$

- renormalisation group $\Rightarrow$ non-linear ODE

$$2\gamma = -\alpha - \gamma^2 + 2\alpha\,\gamma \frac{d}{d\alpha}\gamma$$

Perturbative Solution (rescale: $\gamma(\alpha) := 2\, C\left(-\frac{\alpha}{4}\right)$)

$$\left[C(x)\left(2\,x\,\frac{d}{dx}-1\right)-1\right]C(x)=-x$$

- perturbative solution: $C(x)=\sum_{n=1}^{\infty} C_n x^n$ (OEIS: A000699)

$$C_n = [1, 1, 4, 27, 248, 2830, 38232, 593859, 10401712, 202601898, \dots]$$

- combinatorics: generating function for "connected chord diagrams"

- large order asymptotics

$$C_n \sim e^{-1}\frac{2^{n+\frac{1}{2}}\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{2\pi}}\left(1-\frac{\frac{5}{2}}{2\left(n-\frac{1}{2}\right)}-\frac{\frac{43}{8}}{2^2\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)}-\dots\right)$$

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- missing boundary condition parameter ?

Écalle: formal series → trans-series : $\quad C(x)=\sum_{k=0}^{\infty}\sigma^k\, C^{(k)}(x)$

Trans-series Solution

M. Borinsky & GD, 2005.04265

- expand $C(x) = C^{(0)}(x) + \sigma\, C^{(1)}(c) + \sigma^2\, C^{(2)}(x) + \dots$
- $C^{(0)}(x) =$ previous formal perturbative series solution
- linear inhomogeneous equations for $C^{(k)}(x)$ for $k \geq 1$

$$C^{(1)}(x) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{x}}{C^{(0)}(x)} \exp\left[-\frac{\left(C^{(0)}(x)+1\right)^2}{2x}\right]$$

$$\sim \frac{e^{-1/(2x)}}{\sqrt{x}} \frac{e^{-1}}{\sqrt{2\pi}} \left[1 - \frac{5}{2}x - \frac{43}{8}x^2 - \frac{579}{16}x^3 - \dots\right]$$

- resurgence: $C^{(1)}(x)$ expressed in terms of $C^{(0)}(x)$

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- resurgence: $C^{(1)}(x)$ expressed in terms of $C^{(0)}(x)$
- characteristic signature of resurgent structure:

$$C_n^{(0)} \sim e^{-1} \frac{2^{n+\frac{1}{2}}\, \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{2\pi}} \left(1 - \frac{\frac{5}{2}}{2\left(n-\frac{1}{2}\right)} - \frac{\frac{43}{8}}{2^2\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)} - \ldots\right)$$

- combinatorics of $C_n^{(1)}$: Mahmoud & Yeats, 2020

Resurgent structure

- large order asymptotics of $C_n^{(1)}$ coefficients

$$C_n^{(1)} \sim -2e^{-2}\,\frac{2^{n+\frac{3}{2}}\,\Gamma\left(n+\frac{3}{2}\right)}{2\pi}\left(1-\frac{5}{2\left(n+\frac{1}{2}\right)}-\frac{\frac{11}{2}}{2^2\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}-\dots\right)$$

- next nonperturbative solution ($\xi(x)\equiv\frac{1}{\sqrt{x}}\,e^{-1/(2x)}$):

$$C^{(2)}(x)\sim\xi(x)^2\,\frac{e^{-2}}{2\pi}\left[\frac{1}{x}-5-\frac{11}{2}x-\frac{97}{2}x^2-\dots\right]$$

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- continues to all orders $\Rightarrow$ all-orders summation

$$C(x)=\left[\exp\left(\sigma\,\xi(x)f(x,y)\,\frac{\partial}{\partial y}\right)\cdot y\right]_{y=C^{(0)}(x)}$$

generating function : $\quad f(x,y)\equiv\frac{1}{\sqrt{2\pi}}\frac{x}{y}\exp\left[-\frac{1}{2x}\,y(y+2)\right]$

- also follows from Borinsky's alien derivative on the ring of formal power series

Resurgence in the 4 dimensional massless Yukawa Model

- trans-series: the (asymptotic) perturbative solution to the nonlinear ODE for the anomalous dimension can be extended to a trans-series which resums all nonperturbative orders

- non-perturbative terms $C^{(k)}(x)$ $(k \geq 1)$ $\longleftrightarrow$ singularities of the Borel transform of the perturbative series

- resurgence: all non-perturbative terms are expressed explicitly in terms of the original formal series $C^{(0)}(x)$

fluctuations about different singularities are quantitatively related

Resurgence in the 6 dim. Scalar ϕ^3 Theory Broadhurst/Kreimer, 1999

- physically more interesting model

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu \phi)^2 + \frac{g}{3!}\,\phi^3 \qquad , \qquad \alpha := \frac{g^2}{(4\pi)^3}$$

- asymptotically free; $d = 6$ critical dimension; Lipatov instanton; renormalons; $\to$ non-perturbative physics

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- asymptotically free; $d = 6$ critical dimension; Lipatov instanton; renormalons; $\rightarrow$ non-perturbative physics
- Broadhurst/Kreimer: 3rd order ODE (with quartic nonlinearity) for anomalous dimension

$$\left[C\left(2\,x\,\frac{d}{dx} - 1\right) - 1\right]\left[C\left(2\,x\,\frac{d}{dx} - 1\right) - 2\right]\left[C\left(2\,x\,\frac{d}{dx} - 1\right) - 3\right]C = x$$

- perturbative solution: $C(x) = \sum_{n=1}^{\infty} C_n x^n$: (OEIS: A051862)

$|C_n| : \{1, 11, 376, 20241, 1427156, 121639250, 12007003824, \ldots\}$

- no known combinatorial interpretation of C_n

Trans-series Analysis

Borinsky, GD, Meynig, 2020, to appear

- Broadhurst/Kreimer: $C_n \sim (-1)^n \Gamma(n+2)$
- with more data

$$C_n \sim (-1)^n \Gamma\left(n + \frac{23}{12}\right)\left(1 - \frac{\frac{97}{144}}{2\left(n+\frac{11}{12}\right)} - \frac{\frac{53917}{124416}}{2^2\left(n+\frac{11}{12}\right)\left(n-\frac{1}{12}\right)} - \dots\right) + \dots$$

- now there are 3 "missing" b.c. parameters !

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- now there are 3 "missing" b.c. parameters !
- transseries ansatz for terms "beyond all orders"

$$C(x) \sim x^c\, e^{-b/x} \;\rightarrow\; \text{three solutions}$$

$$b=1 \qquad \& \qquad c=-\frac{23}{12}$$

$$b=2 \qquad \& \qquad c=+\frac{1}{6}$$

$$b=3 \qquad \& \qquad c=-\frac{11}{4}$$

$\Rightarrow$ three <u>resonant</u> Borel singularities at $t=-1,-2,-3$

Trans-series Analysis

- full three-term trans-series

$$C(x) \sim C_{\text{pert}}(x) \quad + \quad S_{[1]} \sum_{k=1}^{\infty} \sigma_{[1]}^{k} \left(\frac{e^{-\frac{1}{x}}}{x^{23/12}} \right)^{k} \sum_{n=0}^{\infty} C_{[1],n}^{(k)} x^{n}$$
$$+ \quad S_{[2]} \sum_{k=1}^{\infty} \sigma_{[2]}^{k} \left(\frac{e^{-\frac{2}{x}}}{x^{-1/6}} \right)^{k} \sum_{n=0}^{\infty} C_{[2],n}^{(k)} x^{n}$$
$$+ \quad S_{[3]} \sum_{k=1}^{\infty} \sigma_{[3]}^{k} \left(\frac{e^{-\frac{3}{x}}}{x^{11/4}} \right)^{k} \sum_{n=0}^{\infty} C_{[3],n}^{(k)} x^{n}$$

- compute fluctuation coefficients from ODE: e.g. $C_{[1],n}^{(k=1)}$

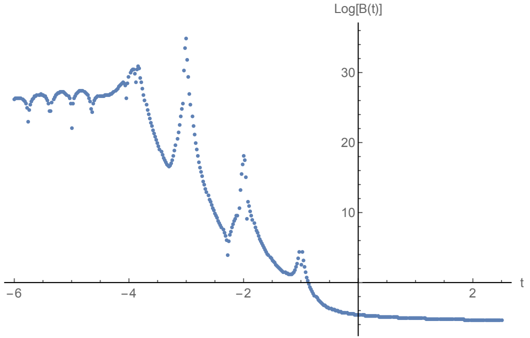
$$C_{[1],n}^{(k=1)} = \left\{ 1, \frac{97}{144}, \frac{53917}{124416}, \dots \right\}$$

- resurgence relation:

$$C_n^{\text{pert}} \sim (-1)^n \Gamma\left(n + \frac{23}{12}\right) \left(1 - \frac{\frac{97}{144}}{2\left(n + \frac{11}{12}\right)} - \frac{\frac{53917}{124416}}{2^2 \left(n + \frac{11}{12}\right)\left(n - \frac{1}{12}\right)} - \dots \right)$$

Borel Analysis

- location and nature of singularities, and associated Stokes constants $S_{[j]}$, can be efficiently extracted numerically
- perturbative series: Borel singularities on negative axis

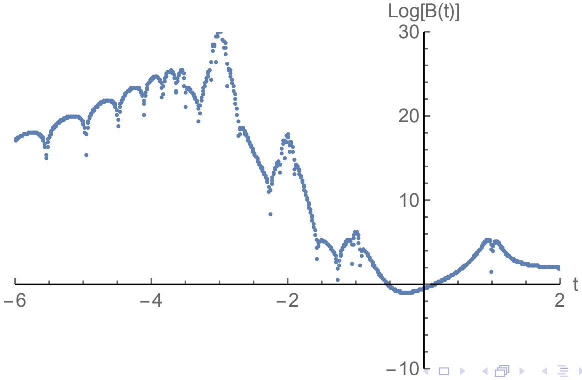


- implies subleading exponentially small corrections

Borel Analysis

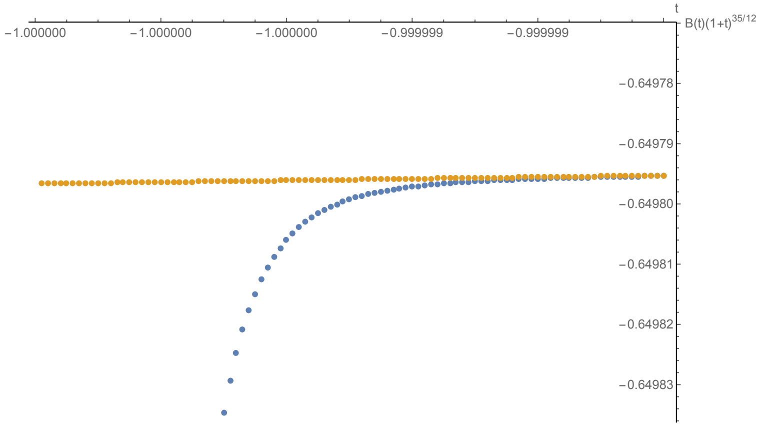
- decoding the full non-perturbative information (e.g. Stokes constants) requires new Borel analysis: Borel-Padé & conformal/uniformizing maps [Costin, GD: 2009.01962]

- 2-instanton fluctuations: Borel singularities on both negative and positive axis



Borel Analysis

- uniformization map in Borel plane enables (optimal) high precision extraction of Stokes constants:

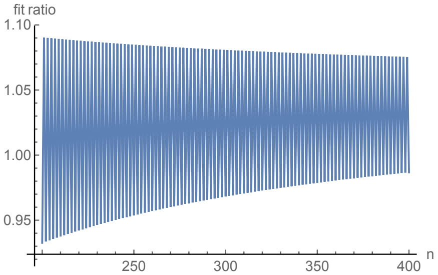


- conformal map [blue]; uniformizing map [gold]

Borel Analysis

- uniformized Borel analysis → large order growth
- fluctuations about $t = -2$ have interference terms

$$C^{(k=1)}_{[2],n} \sim (-1)^n \, \Gamma\left(n + \frac{35}{12}\right) \left[c_1 + \frac{c_2}{\left(n + \frac{23}{12}\right)} + \ldots\right]$$
$$+ \Gamma\left(n + \frac{25}{12}\right) \left[d_1 + \frac{d_2}{\left(n + \frac{13}{12}\right)} + \ldots\right]$$



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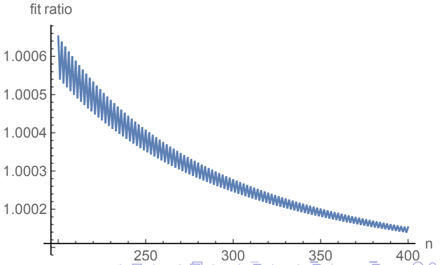

Resurgence in the 6 dimensional Scalar ϕ^3 Theory

- richer non-perturbative structure than Yukawa model
- 3rd order ODE with 4th order non-linearity
- 3 different non-perturbative structures, with different fluctuation powers
- resonance: Borel singularity locations are integer multiples of leading one
- large order/low order resurgence relations
- non-perturbative terms expressed in terms of formal perturbative series

Origin of Non-perturbative Physics in 6 dim scalar ϕ^3 QFT ?

- Lipatov instanton $\Rightarrow$ one Borel singularity, repeated
- Hopf algebra iterative structure $\Rightarrow$ 3 independent (but resonant) Borel branch points, repeated
- "renormalon" bubble-chain diagrams
  $\Rightarrow$ rescaled Lipatov Borel singularity (?)
- dominant effect ?   other effects ?
- diagrammatic interpretation ?

Conclusions

perturbative Hopf algebra renormalisation

resurgent $\Downarrow$ analysis

non-perturbative completion

- does there exist a “natural” Hopf algebraic non-perturbative (trans-series) structure ?
- functional relation & Borinsky’s “alien derivation” ?
- multi-component fields ? (Gracey, 2015; Giombi et al ...)
- relation with instantons and renormalons ?
- other renormalisation schemes ?
- 2d σ models, Chern-Simons, SUSY, QED, QCD, ... ?