# A homotopy theory for higher Lie theory



## Background and motivation

Lie Theory: Lie I, II, III

Where it fails: obstructions of homotopical nature

## HLT: Introducing main characters and plotlines

- dg (NQ) mflds, simplicial (super)manifolds
- contractible submersions and horn-contractibility
- HLT as an $\infty$-adjunction

## Constructing the homotopy theories:

fibrations, weak equivalences, iCFO's

## Where we stand, what is known

# LIE THEORY

$$\mathrm{LGP} \xrightarrow{\ l\ } \mathrm{LALG} \qquad (\text{f.d. real})$$

$$G \longmapsto \mathfrak{g} = l(G), \quad \text{a functor}$$

**Lie I** Given a $G$, ~~...~~ with $l(G) = \mathfrak{g}$, $\exists!$ (up to iso) ~~G~~ 1-connected ~~st.~~ $\hat{G}$, s.t. $l(\hat{G}) \simeq \mathfrak{g}$; (in fact, $\hat{G} = \tilde{G}_e$, and $\tilde{G}_e \xrightarrow{\pi} G_e \xrightarrow{i} G$ with $l(\pi \circ i) = \mathrm{id}$)

**Lie II** Given $G, H$, $\mathfrak{g} = l(G)$, $\mathfrak{h} = l(H)$, and a hom. $\varphi : \mathfrak{g} \to \mathfrak{h}$
If $G$ is 1-connected, $\exists!\ \Phi : G \to H$ s.t. $l(\Phi) = \varphi$
(i.e. $l : \mathrm{Hom}(G,H) \to \mathrm{Hom}(\mathfrak{g},\mathfrak{h})$ is a bijection when $G$ is 1-conn.)

**Lie III** Given a $\mathfrak{g}$, $\exists$ a $G$ with $l(G) \simeq \mathfrak{g}$.

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To summarize:

**THM** (LIE THEORY) $\exists$ an adjunction

$$\begin{tikzcd} \mathrm{LALG} \arrow[r, shift left=1.5ex, "L"] \arrow[r, phantom, "\perp"] & \mathrm{LGP} \arrow[l, shift left=1.5ex, "l"] \end{tikzcd}$$

whose unit $\mu_{\mathfrak{g}} : \mathfrak{g} \to l(L(\mathfrak{g}))$ is an iso, and
whose counit $\epsilon_G : L(l(G)) \to G$ is $\begin{tikzcd} \tilde{G}_e \arrow[r, "\pi"] \arrow[rr, bend right, "\epsilon_G"'] & G_e \arrow[r, "i"] & G \end{tikzcd}$
(the universal cover of $G_e$)

**RMK** $L$ exhibits LALG as a full coreflective subcategory of LGP, whose objects are the 1-connected Lie groups

# WHERE IT FAILS

- LIE III fails(*) for $\infty$-dim'l Lie algebras ($\pi_2(G)$) (van Est)
- LIE III fails(*) for Lie Algebroids ($\pi_2$(leaf)) (Mackenzie)
- LIE II fails(*) for higher cocycles/cohomology ($\pi_{i>1}$) (van Est)
- LIE II fails(*) for R.U.T.H. (Igusa, Block-Smith, Arias Abad, Schätz, Velez) ...
- What about Lie Theory for higher symmetries (CA, str(g)..) ?

(*) in its naive form.

$\rightsquigarrow$

## Need a new framework!

Has to do with/come from homotopy theory

(late '90's - early '00's)

$$
\begin{array}{ccc}
& \text{Sullivan} & \text{Lurie} \\
& \rightsquigarrow & \rightsquigarrow \\
\text{Crainic-Fernandes, Zhu,} & \text{Ševera,} & \text{Henriques, Getzler} \\
\swarrow & & \searrow \\
\text{"Integration = Homotopy"} & & \text{Simplicial manifolds}
\end{array}
$$

## HLT: Introducing the main characters

Lie algebras $\rightsquigarrow$ $dg_{\substack{(+)\\(\geq 0)}}$-(super) manifolds (NQ-manifolds)
(higher Lie algebroids)

Lie groups $\rightsquigarrow$ (Kan) simplicial (super) manifolds
(higher Lie groupoids)

1-connected $\rightsquigarrow$ horn-contractible (wait!)

Categories $\rightsquigarrow$ $(\infty, 1)$-categories

Hom-sets $\rightsquigarrow$ RHom-spaces (homotopy types, s. sets)

bijection $\rightsquigarrow$ homotopy equivalence

functors, adjunctions $\rightsquigarrow$ $\infty$-cat versions.

RMK: Need to embrace $\infty$-dim'l (super) manifolds!

HLT: $\exists$ an adjunction betw. $(\infty,1)$ cat's

$$\begin{tikzcd} \mathrm{LALGD}_\infty \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathrm{LGPD}_\infty \arrow[l, shift left, "\ell"] \end{tikzcd}$$

whose unit $\mu_g : g \to \ell(L(g))$ is an equivalence, and
whose counit $\epsilon_G : L(\ell(G)) \to G$ is the horn-contractible cover

So, the task is to define those $\infty$-cat's and that adjunction, and to show it satisfies the above properties – sounds simple enough :)

in particular, need to define what an equivalence is, for dg-manifolds

# DG (NQ) - MANIFOLDS

- "NQ-Mfld" (Ševera): a supermfld with a smooth right action
  (dg$_+$- or dg$_{\geq 0}$-manifold) $M \times \mathbb{R}^{0|1}$ ... $\mathbb{R}^{0|1} \longrightarrow M$
  ... infinitesimally given by vector fields on $M$: $\epsilon$, $d$ with
  $$[\epsilon,\epsilon]=0, \ [\epsilon,d]=d, \ [d,d]=0$$

- $M_0 = M\cdot 0$, a supermanifold $(0\cdot 0=0)$, the fixed pt. locus of the action
  $$M_0 \underset{i}{\overset{r}{\leftrightarrows}} M$$

- linearize the action on $T_{M_0}M$: the tangent complex of $M$

$$T_{M_0}M = \big(\underbrace{\cdots \xrightarrow{\delta^{-3}} E^{-2} \xrightarrow{\delta^{-2}} E^{-1}}_{N_{M_0}M} \xrightarrow{\delta^{-1}} E^0 = TM_0$$

$\delta^{-1} = \rho : E^{-1} \to E_0 = TM_0$ the anchor map; $\mathrm{Im}\,\rho \subset TM_0$ is an integrable (b/c $d^2=0$) singular distribution $\rightsquigarrow$ the orbit foliation.

- A splitting is an iso

$$\begin{tikzcd} M \arrow[r, "\cong"] \arrow[dr, shift left, "p"'] & N_{M_0}M = \bigoplus_{i\geq 0} E^{-i}[i] \arrow[d, shift left, "0"] \\ & M_0 \arrow[ul, shift left, "i"] \arrow[u, shift left, "i"] \end{tikzcd}$$

$\{$

- Lie $\infty$-algebroid (anchor to $TM_0$ + $[\cdot,\cdot]$, $[\cdot,\cdot,\cdot]$, ...)

- In particular, if $N_{M_0}M = E^{-1}[1]$, the splitting is canonical, $\rightsquigarrow$ Lie algebroids: $(A \to M_0, \rho, [\cdot,\cdot]) \mapsto (A[1], d)$

- Category $\mathbf{Man}_{dg}$, morphisms respect tangent cx's and orbits foliations, but not splittings.

- Contains Lie algebroids as a full subcat., but also many other interesting objects and morphisms: $L_\infty$-algebras and algebroids an $L_\infty$-morphisms between them.

# SIMPLICIAL (SUPER) MANIFOLDS

A simplicial (super) manifold is a functor $X : \Delta^{op} \to (S)Man$ where $\Delta$ is the category of nonempty finite ordinals.

Alternatively, $X_\bullet = (X_0 \leftleftarrows X_1 \leftleftarrows X_2 \cdots)$

$$X_n = X([n]) = X^{\Delta[n]}, \quad f : [n] \to [m] \Rightarrow X_m \xleftarrow{f^*} X_m \quad (X^{\Delta[n]} \leftarrow X^{\Delta[m]})$$

- $\forall$ (finite) s.set $K$, can form the sheaf $X^K$ (a finite limit of the $X_i$'s), and for every map $j : K \to L$, have the dual map $X^K \xleftarrow{j^*} X^L$ of sheaves

- In particular, have horn inclusions

$$j_{n,k} : \Lambda[n,k] \longrightarrow \Delta[n], \quad n \geq 1, \quad k = 0, 1, \ldots, n$$

and corresponding projections $p_{n,k} = j_{n,k}^* : X_n = X^{\Delta[n]} \longrightarrow X^{\Lambda[n,k]}$

Eg. $\Lambda[2,1] = [1] \coprod_{[0]} [1]$ $\begin{pmatrix} 2 \leftarrow 1 \\ \uparrow \\ 0 \end{pmatrix} \hookrightarrow \begin{pmatrix} 2 \leftarrow 1 \\ \nwarrow \uparrow \\ 0 \end{pmatrix} = \Delta[2]$

$$X^{\Lambda[2,1]} = X_1 \times_{X_0} X_1 = \{ x_2 \xleftarrow{x_{21}} x_1 \xleftarrow{x_{10}} x_0 \}, \quad X_2 \xrightarrow{(\partial_0, -, \partial_2) = p_{2,1}} X_1 \times_{X_0} X_1$$

$$x_{210} \longmapsto (x_{21}, x_{10})$$

- The Kan condition: $p_{n,k}$ is surjective for all $n, k$. ("every horn can be filled")

$$\begin{tikzcd} X_2 \arrow[r, "\partial_1"] \arrow[d, "p_{2,1}"'] & X_1 \\ X_1 \times_{X_0} X_1 \arrow[u, dotted, bend left, "s_{2,1}"] \arrow[ur, dashed, "\partial_1 \circ s_{2,1} =: m_{2,1}"'] & \end{tikzcd}$$

(sel- -th coreche)

a composition; also get left & right division from other horns, and then their higher analogues $\rightsquigarrow$ "a weak $\infty$-groupoid"

Smoothness: surjectivity is not enough, $p_{n,k}$ must be submersions (so local smooth sections exist).

Def. A submersion $q: U \to V$ of (super)manifolds is

- surjective if its fibers are nonempty
- étale 0-dim'l
- contractible contractible

Rmk Each of these classes of maps gives rise to a Grothendieck pretopology on the category (S)Man

Def. $X_\bullet$ is said to be Kan – or a Lie $\infty$-groupoid – if $p_{n,k}$ are surjective submersions; it is étale (resp. horn-contractible) if $p_{n,k}$ are étale (resp. contractible).

Eg. For a Lie groupoid $G_1 \rightrightarrows G_0$, its nerve $N_\bullet(G)$ is a Kan simplicial manifold. Furthermore, it is étale (resp. horn-contractible) iff $G$ is étale (resp. source-contractible).

Equivalence. Kan simplicial mflds have homotopy groups, $\pi_i$, (in the category of sheaves, or better yet, diffeological spaces — so they have Lie algebras), and the assignment is functorial

$f: X_\bullet \to Y_\bullet$ is said to be a w.e. if it induces isomorphisms on all the $\pi_i$'s (equivalent definitions are possible)

Let $W$ be the class of these w.e.'s. Define the $\infty$-category of higher Lie groupoids (or smooth stacks) to be the localization

$$LGpd_\infty := Man_\Delta^{Kan}[W]^{-1}$$

# What about the infinitesimal side?

We have functors (not adjoint) between the categories

$$\begin{tikzcd} \mathrm{Man}_{dg} \arrow[r, shift left, "\mathrm{Int}"] & \mathrm{Man}_\Delta^{Kan} \arrow[l, shift left, "\mathrm{diff}"] \end{tikzcd}$$

Thom-Sullivan

Ševera

$\mathrm{Int}(M)_\bullet = \mathrm{Map}_{\mathrm{Man}_{dg}}(T[1]\Delta^\bullet, M)$ ; $\left(\mathrm{diff}(X_\bullet) = \mathrm{Map}_{\mathrm{Man}_\Delta}(E_\bullet(\mathbb{R}^{0|1}), X_\bullet)\right)$

$N_\bullet(\mathrm{Pair}(\mathbb{R}^{0|1}))$

Ševera-Širan: representable in the category of Frechet manifolds, and is Kan.

Moreover, is horn-contractible $\Rightarrow$ the universal integration!

**Def.** A dg-map $f: M \to N$ is a w.e. if $\mathrm{Int}(f)$ is

(Later: internal characterization of these)

Let $\widetilde{W}$ be the class of these, and define the $\infty$-category of higher Lie algebroids to be the localization

$$\mathrm{LALGD}_\infty := \mathrm{MAN}_{dg}[\widetilde{W}]^{-1}$$

**Problem** While Int does induce a functor $L: \mathrm{LALGD}_\infty \to \mathrm{LGPD}_\infty$, diff does not: it does not even take $W$ to $\widetilde{W}$!

Eg. Let $M_0$ be a manifold with $\Gamma = \pi_1(M_0, m)$; then the inclusion $\Gamma \to \Pi_1(M_0)$ is in $W$, but after applying diff we have $* \to T[1]M_0$ (inclusion of the point $m$), which is in $\widetilde{W}$ iff $M_0$ is contractible (and then in particular $\Gamma = \{e\}$)

(this has to do with (the lack of) horn-contractibility!)

# Homotopy theory

Calculating $(\ldots)[W]^{-1}$ is extremely difficult if one only has the $W$; an auxiliary structure is required to make the problem tractable. One such structure is that of a category of fibrant objects (CFO): apart from $W$, one also has a class $F$ of fibrations (and therefore also hypercovers $W \cap F$), satisfying a number of axioms, most important of which being the existence of a path object $P(X)$ for every $X$, factorizing the diagonal:

$$\begin{tikzcd} X \arrow[r, "\in W"] \arrow[rr, bend right, "\text{diag}"'] & P(X) \arrow[r, "\in F"] & X \times X \end{tikzcd},$$

THM (Rogers-Zhu, building on Behrend-Getzler) There is an iCFO structure on $\mathrm{Man}_\Delta^{\mathrm{Kan}}$ with $F$ being the Kan fibrations (in the smooth - i.e. surj. submersions - pretopology)

CONJ. There is an iCFO structure on $\mathrm{Man}_\Delta^{\mathrm{hcKan}}$, with $F$ being the Kan fibrations in the contractible submersions pretopology.

CONJ. $\mathrm{diff}|_{\mathrm{Man}_\Delta^{\mathrm{hcKan}}}$ does take $W$ to $\tilde{W}$

CONJ. ("injective resolutions") There is a functor $R : \mathrm{Man}_\Delta^{\mathrm{Kan}} \to \mathrm{Man}_\Delta^{\mathrm{hcKan}}$ and a natural transformation $u : \mathrm{id} \Rightarrow \iota \circ R$ (where $\iota : \mathrm{Man}_\Delta^{\mathrm{hcKan}} \to \mathrm{Man}_\Delta^{\mathrm{Kan}}$ is the inclusion) which is in $W$

COR. The induced $\mathrm{Man}_\Delta^{\mathrm{hcKan}}[W]^{-1} \to \mathrm{Man}_\Delta^{\mathrm{Kan}}[W^{-1}]$ is an equivalence.

COR. There is an iCFO structure on $\mathrm{Man}_{dg}$, transferred along Int

$\mathrm{Int} : \mathrm{Man}_{dg} \to \mathrm{Man}_\Delta^{\mathrm{hcKan}}$

<u>CONJ.</u> (HLT) There is an adjunction

$$\mathrm{LALGD}_\infty \underset{\ell}{\overset{L}{\rightleftarrows}} \mathrm{LGPD}_\infty \quad (\perp)$$

with $L$ (resp. $\ell$) induced by Int (resp. diff $\circ$ R)

<u>COR.</u> This is, in fact, an <u>adjoint</u> <u>equivalence</u>!

(Note the sharp contrast with 1-Lie theory!)

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<u>EVIDENCE</u>

<u>CONJ</u> (<u>THM</u> in the transitive case) $f: M \to M'$ is in $\widehat{W}$ iff

(1) $\underline{\pi_0(f)}: \underline{\pi_0(M)} \longrightarrow \underline{\pi_0(M')}$ is a bijection of sets

(2) For each orbit (leaf) $O \subset M_0$,

$$f_0|_O : O \to f_0(O)$$

is a weak homotopy equivalence

(3) For each $m \in M_0$,

$$T_m f : T_m M \longrightarrow T_{f_0(m)} M'$$

is a quasi-isomorphism

<u>CONJ</u> $f: M \to M'$ is in $\widehat{F} = \mathrm{Int}^{-1}(F)$ iff it satisfies the conditions of Brahic-Zhu and Laurent-Gengoux-Ryvkin in cases those apply.

**Prop.** If $G, H$ are source-contractible Lie groupoids, and $\varphi : G \to H$ is in $W$, then $\operatorname{diff}(\varphi)$ is in $\tilde{W}$.

**Pf.** In this case, for each $m \in G_0$, the principal bundle $O_m \leftarrow s^{-1}(m) \circlearrowleft G_m$ is universal; namely $|BG_m| \leftarrow |EG_m| \circlearrowleft G_m$. The conclusion follows from the l.e.s. for this bundle and the above characterization of $\tilde{W}$.

**Prop.** Let $G$ be a Lie group(oid). Then the groupoid $\operatorname{Gauge}(P_G)$, where $P_G$ is the universal principal $G$-bundle $|BG| \leftarrow |EG| \circlearrowleft G$, is source-contractible, and the inclusion $G \to \operatorname{Gauge}(P_G)$ is in $W$.

**Pf.** Clear.

**Eg.** Let $G = \mathbb{Z}$. Then $P_{\mathbb{Z}}$ is the universal cover $S^1 \leftarrow \mathbb{R} \circlearrowleft \mathbb{Z}$, and $\operatorname{Gauge}(P_{\mathbb{Z}})$ is $\Pi_1(S^1)$, ~~then~~ and is source-contractible. Then $\operatorname{diff} \circ R(\mathbb{Z}) = T[1]S^1$, "the Lie algebroid of $\mathbb{Z}$".