Super-symmetry of geometric structures

Boris Kruglikov
Andrea Santi ⋄ Dennis The
(UiT the Arctic University of Norway)

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A super-vector space is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\oplus 1}$.
Dim: $(m|n) \equiv m + \epsilon n$ ($\epsilon^2 = 1$), real dim: $m + n$, superdim: $m - n$.
Changing parity yields a superspace ΠV of dimension $(n|m)$.

Define the tensor product $V \otimes W$ by

$$(V \otimes W)_{\bar{0}} = (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}}),\ (V \otimes W)_{\bar{1}} = (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{0}} \otimes W_{\bar{1}})$$

and similarly define $\mathrm{Hom}(V, W)$.

A superalgebra structure on $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is defined via $\mathbb{Z}_2$-homogeneous $\mu \in \mathrm{Hom}(A \otimes A, A)$. It is commutative if

$$ab = (-1)^{|a||b|} ba \quad \text{(sign rule)}.$$

An example is the Grassmann algebra in n variables $\Lambda(n) = \Lambda^{\text{even}} \oplus \Lambda^{\text{odd}}$ of dimension $(2^{n-1}|2^{n-1})$.
Another example is the tensorial algebra $T(V)$. In particular,

$$\dim(S^2 V) = \left(\tbinom{m+1}{2} + \tbinom{n}{2} \middle| mn\right),\ \dim(\Lambda^2 V) = \left(\tbinom{m}{2} + \tbinom{n+1}{2} \middle| mn\right).$$

Lie superalgebras

A Lie superalgebra structure on $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is defined by $\mathbb{Z}_2$-homogeneous bracket $\beta \in \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ that is skew-symm and satisfies the Jacobi in super-sense (sign rule). Examples:

$$\mathfrak{sl}(m|n) = \left\{ A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{End}(\mathbb{R}^{m|n}) : \mathrm{str}(A) = \mathrm{tr}(\alpha) - \mathrm{tr}(\delta) = 0 \right\}.$$

- $\mathfrak{osp}(m|2n)$ preserves even ndg symmetric structure on $\mathbb{R}^{m|2n}$ ($\simeq$)
- ◦ $\mathfrak{spo}(2n|m)$ preserves even ndg skew-symm structure on $\mathbb{R}^{2n|m}$,
- $\mathfrak{pe}(n)$ preserves odd ndg symmetric structure on $\mathbb{R}^{n|n}$ ($\simeq$)
- ◦ $\mathfrak{pe}^{\mathsf{sk}}(n)$ preserves odd ndg skew-symmetric structure on $\mathbb{R}^{n|n}$,
- $\mathfrak{q}(n)$ preserves odd ndg complex structure on $\mathbb{R}^{n|n}$,
- $G(3) = (\mathfrak{g}(2) \oplus \mathfrak{sl}(2) | \mathbb{R}^7 \otimes \mathbb{R}^2)$,
- $F(3|1) = (\mathfrak{spin}(7) \oplus \mathfrak{sl}(2) | \mathbb{R}^8 \otimes \mathbb{R}^2)$,
- $D(2,1;\alpha) = (\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) | \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2)$.

These (with modifications $\mathfrak{psl}(n|n)$, $\mathfrak{psq}(n)$, ...) are classical LSA. Killing form, Cartan subalgebras, gradings, parabolics, ...

Supermanifolds

A supermanifold in the sense of Berezin–Kostant–Leites is a ringed space $M = (M_o, \mathcal{O}_M)$ such that $\mathcal{O}_M|_{\mathcal{U}_o} \cong C^\infty_{M_o}|_{\mathcal{U}_o} \otimes \Lambda^\bullet \mathbb{S}^*$ as sheaves of superalgebras for any sufficiently small open subset $\mathcal{U}_o \subset M_o$. Here $\mathbb{S}$ is a vector space of fixed dimension. We set $\dim(M) = (m|n) = (\dim M_o | \dim \mathbb{S})$, call M_o the reduced manifold and $\mathcal{O}_M = (\mathcal{O}_M)_{\bar 0} \oplus (\mathcal{O}_M)_{\bar 1}$ the structure sheaf.

Let $\mathcal{J} = \langle \mathcal{O}_{\bar 1} \rangle = \mathcal{J}_{\bar 0} \oplus \mathcal{J}_{\bar 1}$ be the subsheaf generated by nilpotents: $\mathcal{J}_{\bar 1} = \mathcal{O}_{\bar 1}$ and $\mathcal{J}_{\bar 0} = \mathcal{O}_{\bar 1}^2$. For any sheaf $\mathcal{E}$ of $\mathcal{O}_M$-modules on M_o the evaluation is $\mathrm{ev} : \mathcal{E} \to \mathcal{E}/(\mathcal{J} \cdot \mathcal{E})$. Thus $\mathrm{ev} : \mathcal{O}_M \to C^\infty_{M_o}$, $f \mapsto \mathrm{ev}(f)$, yields the canonical morphism $\imath : M_o \hookrightarrow M$, with evaluation $\mathrm{ev}(f)$ at $x \in M_o$ being $\mathrm{ev}_x(f)$. We stress, however, that there is no canonical morphism from M to M_o.

A Lie supergroup is a supermanifold $G = (G_o, \mathcal{O}_G)$ that is also a group object in the category of supermanifolds. (The reduced manifold G_o is a Lie group.) It can be represented by a Harish-Chandra pair $(G_o, \mathfrak{g})$, $\mathrm{Lie}(G_o) = \mathfrak{g}_{\bar 0}$.

Superbundles

A fiber bundle (FB) is a submersion $E \to M$ with typical fiber F.

A geometric vector bundle (VB) is a FB with vector fiber F, given via a cocycle $\varphi_{ij} : \mathcal{U}_{ij} \to \mathrm{GL}(F)$. A locally free sheaf $\mathcal{E}$ on M_o of $\mathcal{O}_M$-modules of finite rank is an algebraic VB over M.

Theorem

The category of the geometric VBs with morphisms of VBs is equivalent to the category of algebraic VBs with morphisms of locally free coherent sheaves, provided the bases are connected.

A geometric principal bundle (PB) with structure group G is a FB $\pi : P \to M$ with typical fiber G with the transition cocycle: $\varphi_{ij} : \mathcal{U}_{ij} \to G \subset \mathrm{Aut}(G)$. An algebraic PB over M is a sheaf $\mathcal{P}$ of right $\mathcal{G}_M$-sets that is locally simply transitive; $\mathcal{G}_M(\mathcal{U}) = G[\mathcal{U}]$.

Theorem

The categories of geometric PBs and algebraic PBs with γ-morphisms are equivalent, for homomorphisms of supergroups γ.

Geometric super structures

Filtered geometric super structures, in particular G-structures are defined through succesive frame bundles reductions. In particular, super tensors, connections and differential equations are such.

Example (nondegenerate even symmetric form)

The supermanifold $M = \mathbb{R}^{m|2n}(x,\xi)$ with the metic $g = (1+k\|x\|^2)^{-2} \cdot \sum_{i=1}^m dx_i^2 + \sum_{i=1}^n d\xi_i d\xi_{i+n}$ has symmetry:

$$\mathfrak{g} = \begin{cases} \mathfrak{osp}(m+1|2n) & k>0 \\ \mathfrak{osp}(m|2n) \ltimes \mathbb{R}^{m|2n} & k=0 \\ \mathfrak{osp}(m,1|2n) & k<0. \end{cases}$$

Example (nondegenerate odd symmetric form)

The supermanifold $M = \mathbb{R}^{2n|m}(x,\xi)$ with the symplectic form $\omega = \sum_{i=1}^n dx_i \wedge dx_{i+n} + \sum_{i=1}^m d\xi_i \wedge d\xi_i$ has infinite-dim symmetry $\mathfrak{symp}(\omega) \simeq \mathcal{O}_M/\mathbb{R}$ – prolongation of $\mathfrak{spo}(2n|m)$ (see below).

Super distributions

A distribution on a supermanifold M is a graded $\mathcal{O}_M$-subsheaf $\mathcal{D} = \mathcal{D}_{\bar{0}} \oplus \mathcal{D}_{\bar{1}} \subset \mathcal{T}M$ that is locally a direct factor. Any such sheaf is locally free, associating the VB $D = \mathrm{ev}(\mathcal{D}) \subset TM$. This induces a reduced subbundle $D|_{M_o} \subset TM|_{M_o}$ that does not determine $\mathcal{D}$.

The weak derived flag of (bracket-generating) $\mathcal{D}$ is defined so:

$$\mathcal{D}^1 = \mathcal{D} \subset \mathcal{D}^2 \subset \cdots \subset \mathcal{D}^i \subset \cdots, \qquad \mathcal{D}^i = [\mathcal{D}, \mathcal{D}^{i-1}],$$

where each $\mathcal{D}^i \subset \mathcal{T}M$ is a graded $\mathcal{O}_M$-subsheaf, also assumed locally direct factor (regularity).

Example (non-regular superextension of Hilbert–Cartan equation)

Let $M = \mathbb{R}^{5|2}(x, u, p, q, z \,|\, \theta, \nu)$ be endowed with superdistribution

$$\mathcal{D} = \langle D_x = \partial_x + p\partial_u + q\partial_p + q^2\partial_z,\ \partial_q \,|\, D_\theta = \partial_\theta + q\partial_\nu + \theta\partial_p + 2\nu\partial_z \rangle$$

of rank $(2|1)$. We directly compute

$$\mathcal{D}^2 = \langle D_x,\ \partial_q,\ \partial_p + 2q\partial_z \,|\, D_\theta,\ \partial_\nu,\ \theta\partial_u \rangle.$$

This is not a superdistribution, due to the presence of a nilpotent.

Tanaka-Weisfeiler prolongation

For regular $\mathcal{D}$ we get filtration $\mathcal{D}^i$ of $\mathcal{T}M$, compatible with brackets of supervector fields: $[\mathcal{D}^i(\mathcal{U}), \mathcal{D}^j(\mathcal{U})] \subset \mathcal{D}^{i+j}(\mathcal{U})$. Setting $\mathrm{gr}(\mathcal{T}M)_{-i} = \mathcal{D}^i/\mathcal{D}^{i-1}$ for $i > 0$, we get a locally free sheaf of $\mathcal{O}_M$-modules and Lie superalgebras over M_o:

$$\mathrm{gr}(\mathcal{T}M) = \bigoplus_{i<0} \mathrm{gr}(\mathcal{T}M)_i.$$

It is strongly regular if there exists Lie superalgebra $\mathfrak{m} = \bigoplus_{-\mu \le i < 0} \mathfrak{g}_i$ such that $\mathrm{gr}(\mathcal{T}_x M) \cong (\mathcal{O}_M)_x \otimes \mathfrak{m} \ \forall\ x \in M_o$.

Assuming strong regularity, non-degeneracy (no center in $\mathfrak{g}_{-1}$) and fundamental property ($\mathfrak{g}_{-1}$ generates $\mathfrak{m}$) define Tanaka–Weisfeiler prolongation of $\mathfrak{m}$ as the maximal $\mathbb{Z}$-graded LSA $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ s.t:

- $\mathfrak{g}_- = \mathfrak{m}$,
- $\mathrm{Ker}\big(\mathrm{ad}(\mathfrak{g}_{-1})\,|_{\mathfrak{g}_i}\big) = 0\ \forall i \ge 0$.

It exists and unique, and is denoted $\mathfrak{g} = \mathrm{pr}(\mathfrak{m})$. There is prolongation version $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ and also other reductions.

Symmetry dimension bound

Theorem (BK, A.Santi, D.The ⋄ 2021)

Let $\mathfrak{s}$ be the symmetry superalgebra of a bracket-generating, strongly regular filtered structure $(M, \mathcal{D}, q)$, with the Tanaka–Weisfeiler prolongation $\mathfrak{g} = \mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$. If the reduced manifold M_o is connected, then $\dim \mathfrak{s} \leq \dim \mathfrak{g}$ in the strong sense: the inequality applies to both even and odd dimensions.

The LSA $\mathfrak{s}$ can be considered as a superalgebra of vector fields localized in a fixed neighborhood $U_o \subset M_o$ or as germs of those.

Assuming $\dim \mathfrak{g}$ is finite, the above bound is sharp: there exists a standard model G/P with the required symmetry dimension.

Theorem (—)

With the above assumptions $\mathrm{Aut}(M, \mathcal{D}, q)$ is a Lie supergroup. If M_o is connected, then $\dim \mathrm{Aut}(M, \mathcal{D}, q) \leq \dim \mathfrak{g}$ in strong sense.

Idea of the proof

Similar to the Cartan method we construct a tower of bundles

$$M \leftarrow \mathcal{P}_0 \leftarrow \mathcal{P}_1 \leftarrow \dots$$

consisting of partial frames. (No functor of points.)

A care should be taken as we cannot assume frames to be (classical) points of these fiber bundles. Therefore we revise the Tanaka construction and its version by I.Zelenko, adapting it to the super-context and using a uniform normalization via the generalized Spencer complex (equiv: Lie superalgebra cohomology with the Chevalley–Eilenberg differential):

$$\delta : \mathfrak{m}^* \otimes \mathfrak{g} \to \Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g}.$$

Again, the bundles $\mathcal{P}_k \to \mathcal{P}_{k-1}$ are principal with the structure group $\mathfrak{g}_k$ for $k > 0$ (and for $k = 0$: G_0). The final bundle $\mathcal{P} \to M$ has a canonical connection (abs parallelism), whence the claim.

Holonomic examples

- Super-Riemann structures (M,g) are G_0-structures with $G_0=\mathrm{OSp}(m|2n)$. For $\mathfrak{g}_0=\mathrm{Lie}(G_0)=\mathfrak{osp}(m|2n)$ we have $\mathfrak{g}_1=\mathfrak{g}_0^{(1)}=0$. Hence the Lie superalgebra of Killing supervector fields satisfies

$$\dim\mathfrak{s}\leq\dim\mathfrak{g}_{-1}+\dim\mathfrak{g}_0=\Bigl(\tbinom{m+1}{2}+\tbinom{2n+1}{2}\,|\,2n+2mn\Bigr).$$

- Almost super-symplectic structures (M,ω) are G_0-structures with $G_0=\mathrm{SpO}(2n|m)$. For $\mathfrak{g}_0=\mathrm{Lie}(G_0)=\mathfrak{spo}(2n|m)$ we have: $\mathfrak{g}_i=\mathfrak{g}_0^{(i)}=S^{i+2}V^*$, $V=TM$ (in the super-sense), so $\mathfrak{g}_0\subset\mathfrak{gl}(V)$ is of infinite type unless M is purely odd:

$$\mathfrak{g}=\mathrm{pr}(\mathfrak{g}_0)\simeq\oplus_{i=1}^{\infty}S^iV^*.$$

In the case M is purely odd $(n=0)$, the Lie superalgebra of symplectic supervector fields satisfies:

$$\dim\mathfrak{s}\leq\sum_{i=-1}^{m-2}\dim\mathfrak{g}_i=(2^{m-1}-1|2^{m-1}).$$

Holonomic examples

• Periplectic structures (M, q) with q odd ndg bilinear form on TM are irreducible G_0-structures with $G_0 = \mathrm{Pe}(n)$. For $\mathfrak{g} = \mathrm{Lie}(G_0) = \mathfrak{pe}(n)$ we have $\mathfrak{g}^{(1)} = 0$. Hence the Lie superalgebra of symmetries satisfies:

$$\dim \mathfrak{s} \leq (n^2 + n|n^2 + n).$$

(There are some other periplectic-related structures for which the prolongations are different/longer.)

• Projective structures on supermanifolds of $\dim M = (m|n)$ are equivalence classes of torsion-free connections: $\nabla \sim \nabla'$ iff $\nabla - \nabla' = \mathrm{Id} \circ \omega \in \Gamma(S^2\mathcal{T}^*M \otimes \mathcal{T}M)$ for an even $\omega \in \Omega^1(M)$. We have $\mathfrak{g}_0 = \mathfrak{gl}(m|n)$ and its prolongation if $(m|n) \neq (1|0)$ is $\mathfrak{g}_1 = S^2V^* \otimes V = V^* \oplus (S^2V^* \otimes V)_0 = \mathfrak{g}_1' \oplus \mathfrak{g}_1''$. Projective connection reduces this to the first component, further prolongations are trivial. Whence the bound for symmetries:

$$\dim \mathfrak{s} \leq \dim V + \dim \mathfrak{gl}(V) + \dim \mathfrak{g}_1' = \big(2m + n^2 + m^2 \,|\, 2n + 2mn\big).$$

$G(3)$ supergeometries: regular extension of HC equation

There are 19 parabolic supergeometries associated to the simple exceptional LSA $G(3)$. We consider only Hilbert-Cartan type supergeometry $G(3)/P_2^{IV}$, which is equivalent to a gerneric $(2|4)$ superdistribution on a $(5|6)$-dimensional supermanifold. (Similarly a generic rank 2 distribution in 5D gives a $G(2)/P_1$ geometry.)

Tanaka-Weisfeiler prolongation of the symbol of this distribution has the following dimensions of the components:

$$(2|0, 1|2, 2|4, 7|2, 2|4, 1|2, 2|0).$$

Therefore $\dim \mathfrak{s} \leq (17|14)$ and the maximal symmetry is $G(3)$. The corresponding distribution super-extends the Hilbert-Cartan distribution; on $M = \mathbb{R}^{5|6}(x, y, y_x, y_{xx}, z|\nu, \tau, y_\nu, y_\tau, y_{x\nu}, y_{x\tau})$ it is

$$\mathcal{D}_{\bar{0}} = \langle D_x = \partial_x + y_x\partial_y + y_{xx}\partial_{y_x} + \left(\tfrac{y_{xx}^2}{2} + y_{x\nu}y_{x\tau}\right)\partial_z + y_{x\tau}\partial_{y_\tau} + y_{x\nu}\partial_{y_\nu}, \partial_{y_{xx}}\rangle$$

$$\mathcal{D}_{\bar{1}} = \langle D_\tau = \partial_\tau + y_\tau\partial_y + y_{x\tau}\partial_{y_x} + y_{xx}y_{x\tau}\partial_z + y_{xx}\partial_{y_\nu},\ \partial_{y_{x\tau}},$$
$$D_\nu = \partial_\nu + y_\nu\partial_y + y_{x\nu}\partial_{y_x} + y_{xx}y_{x\nu}\partial_z - y_{xx}\partial_{y_\tau},\ \partial_{y_{x\nu}}\rangle.$$

N-extended Poincaré superstructures

Let $(\mathbb{V}, g)$ be a metric vector space and $\mathbb{S}$ be a spin module. Let $\mathfrak{g}_{-2} = \mathbb{V}$, $\mathfrak{g}_{-1} = \underbrace{\mathbb{S} \oplus \cdots \oplus \mathbb{S}}_{N}$ and $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ be a LSA with consistent gradation: $\mathfrak{m}_{\bar{0}} = \mathfrak{g}_{-2}$, $\mathfrak{m}_{\bar{1}} = \mathfrak{g}_{-1}$. Then $\mathfrak{m} \oplus \mathfrak{so}(\mathbb{V})$ is the N-extended Poincaré superalgebra. Brackets $\Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ were classified by D.Alekseevsky-V.Cortes (Λ^2 in super sense).

The prolongation $\mathfrak{g} = \mathrm{pr}(\mathfrak{m})$ was computed by A.Altomani-A.Santi. It equals $\mathfrak{m} \oplus \mathfrak{g}_0$, $\mathfrak{g}_0 = \mathfrak{so}(\mathbb{V}) \oplus \mathbb{R} \oplus \mathfrak{g}_0^\dagger$, except for the cases $A(m|3)/P_{2,m+2}$, $B(m|2)/P_2$, $D(m|2)/P_2$, $F(3|1)/P_2$, where the prolongation is the corresponding semisimple Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$. This gives the symmetry bound

$$\dim \mathfrak{s} \leq \left(\binom{d+1}{2} + 1 + \dim \mathfrak{g}_0^\dagger \,|\, N \cdot 2^{[d/2]} \right),$$

where $d = \dim \mathbb{V}$ (achieved for the homogeneous model).

Odd second & third order ODEs

Recall at first the classical story ($y = y(x)$ even):

- 2nd ord ODEs $y'' = f(x, y, y')$ mod point transformations have at most 8-dim symmetry algebra and max symmetry $\mathfrak{sl}(3)$ corresponds to $y'' = 0$;
- 3rd ord ODEs $y''' = f(x, y, y', y'')$ mod contact transformations have at most 10-dim symmetry algebra and max symmetry $\mathfrak{sp}(4, \mathbb{R})$ corresponds to $y''' = 0$.

Now let us look to super analogs ($\xi = \xi(x)$ odd, x even):

- 2nd ord ODEs $\xi'' = f(x, \xi, \xi')$ mod point transformations have $(4|4)$-dim symmetry algebra and always trivialize to $\xi'' = 0$ with symmetry $\mathfrak{sl}(2|1)$;
- 3rd ord ODEs $\xi''' = f(x, \xi, \xi', \xi'')$ mod contact transformations have at most $(4|4)$-dim symmetry algebra and max symmetry corresponds to $\xi''' = 0$.

	Even part	Odd part
$+2$	$\cdot$	$-\xi\partial_x + \xi'\xi''\partial_{\xi''} + 2\xi'\xi'''\partial_{\xi'''}$
$+1$	$\frac{x^2}{2}\partial_x + x\xi\partial_\xi + \xi\partial_{\xi'}$ $+(\xi' - x\xi'')\partial_{\xi''} - 2x\xi'''\partial_{\xi'''}$	$\cdot$
0	$x\partial_x + \xi\partial_\xi - \xi''\partial_{\xi''} - 2\xi'''\partial_{\xi'''}$ $\xi\partial_\xi + \xi'\partial_{\xi'} + \xi''\partial_{\xi''} + \xi'''\partial_{\xi'''}$	$\cdot$
-1	$-\partial_x$	$\frac{x^2}{2}\partial_\xi + x\partial_{\xi'} + \partial_{\xi''}$
-2	$\cdot$	$x\partial_\xi + \partial_{\xi'}$
-3	$\cdot$	∂_ξ

These are all *point* symmetries. The derived superalgebras of $\mathfrak{g}$:

$$\mathfrak{g}^{(1)} = \mathbb{R}^{0|1} \ltimes \mathfrak{g}^{(2)}, \quad \mathfrak{g}^{(2)} \cong \mathfrak{sl}(2,\mathbb{R})_{\bar 0} \ltimes (S^2\mathbb{R}^2)_{\bar 1}.$$

We also have for non-flat cases:

$$\dim \mathrm{sym}(\xi''' = \xi'') = (2|3), \quad \dim \mathrm{sym}(\xi''' = \xi\xi'\xi'') = (2|2).$$

Thanks for your attention!