Dirichlet improvability in L_p-norms

by **Nikolay Moshchevitin**

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joint work with Nikita Shulga
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Nikita Shulga

Hello and welcome!

My name is Nikita Shulga, I am currently working at La Trobe University, Australia with Mumtaz Hussain.

Previously, I finished postgraduate studies and a bachelor's plus master's degree at Moscow State University under the supervision of Nikolay Moshchevitin.

My research interests lie in ***number theory***, ***additive combinatorics****, broadly understood* ***dynamical systems,*** *and how these topics interact with each other. In particular, I work in Diophantine approximation theory, in both metrical and regular approximation problems.*

Mathematical biology/neuroscience enthusiast.

You can find a full list of my publications on the page ***Publications****.*

I am in the postdoctoral/lecturer job market starting in 2025.

Dirichlet improvability

Dirichlet Theorem. *Let $\alpha \in \mathbb{R}$. For any real $t \geq 1$ there exists positive integer q such that*

$$\begin{cases} ||q\alpha|| < \frac{1}{t}, \\ 1 \leq q \leq t. \end{cases}$$

$|| \cdot ||$ - distance to the nearest integer

A real number α is called *Dirichlet improvable* (notation $\alpha \in \mathrm{DI}_\infty$) if there exists a constant $c < 1$, such that the system

$$\begin{cases} ||q\alpha|| < \frac{c}{t}, \\ 1 \leq q \leq t. \end{cases}$$

can be solved in $q \in \mathbb{Z}_+$ for any large real number t.

Irrationality measure function and continued fractions

Dirichlet: $$\psi_\alpha(t) = \min_{1\le x\le t} ||x\alpha|| < \frac{1}{t} \quad \forall\, t \ge 1$$

Dirichlet improvability: $$\limsup_{t\to\infty} t\cdot\psi_\alpha(t) < 1$$

Tools: continued fractions

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots + \cfrac{1}{a_n + \dots}}}} = [a_0; a_1, a_2, a_3, ..., a_n, ...]$$

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, ..., a_n] - \text{ convergents}$$

Lagrange: $$\psi_\alpha(t) = ||q_n\alpha|| \text{ for } q_n \le t < q_{n+1}$$

Lagrange and Dirichlet constants

$$\alpha = [a_0; a_1, a_2, a_3, ..., a_n, ...], \quad \frac{p_n}{q_n} = [a_0; a_1, a_2, ..., a_n],$$

$$\lambda(\alpha) = \liminf_{t\to\infty} t \cdot \psi_\alpha(t) = \liminf_{n\to\infty} q_n \cdot ||q_n\alpha|| = \liminf_{n\to\infty} \frac{1}{\alpha_{n+1} + \alpha_n^*}$$

$$d(\alpha) = \limsup_{t\to\infty} t \cdot \psi_\alpha(t) = \limsup_{n\to\infty} q_{n+1} \cdot ||q_n\alpha|| = \limsup_{n\to\infty} \frac{1}{1 + \frac{\alpha_n^*}{\alpha_{n+1}}}$$

here $\quad \alpha_{n+1} = [a_{n+1}; a_{n+2}, ...], \quad \alpha_n^* = \frac{q_{n-1}}{q_n} = [0; a_n, a_{n-1}, ..., a_1]$

Hurwitz, Szekeres:

$$0 \le \lambda(\alpha) \le \frac{1}{\sqrt{5}}, \qquad \frac{1}{2} + \frac{1}{2\sqrt{5}} \le d(\alpha) \le 1.$$

Badly approximable numbers

Dirichlet: $$\psi_\alpha(t) = \min_{1\le x\le t} ||x\alpha|| < \frac{1}{t} \quad \forall\, t \ge 1$$

α is called badly approximable if

- $\inf_t t \cdot \psi_\alpha(t) > 0$
- $\sup_n a_n < \infty$
- $\lambda(\alpha) = \liminf_{t\to\infty} t \cdot \psi_\alpha(t) > 0$
- $d(\alpha) = \limsup_{t\to\infty} t \cdot \psi_\alpha(t) < 1$

Davenport and **Schmidt**:
An irrational number α satisfies $\alpha \in \mathrm{DI}_\infty$ (= Dirichlet improvable = $d(\alpha) < 1$) if and only if it is badly approximable.

Of course almost all numbers are not in DI_∞, but DI_∞ is winning and $\mathrm{HD}(\mathrm{DI}_\infty) = 1$.

Dirichlet improvability in arbitrary norm

- N. Andersen; W. Duke, *On a theorem of Davenport and Schmidt*, Acta Arithmetica 198 (2021), 37-75.
- D. Kleinbock; A. Rao, *A zero-one law for uniform Diophantine approximation in Euclidean norm*, Int. Math. Res. Not. IMRN, 8, (2022), 5617–5657.
  D. Kleinbock; A. Rao, *A dichotomy phenomenon from Bad minus normed Dirichlet*, Mathematika, 69:4 (2023), 1145-1164.
- D. Kleinbock, *Simultaneously dense and non-dense orbits in homogeneous dynamics and Diophantine approximation*, talk at "Diophantine Approximation, Fractal Geometry and Related Topic" Univ. Gustave Eiffel (Paris), 3rd — 7th June 2024.

Diophantine Approximation, Fractal Geometry and Related Topic, 3rd — 7th June 2024

Dirichlet improvability in arbitrary norm

- Strongly symmetric norm $F(x, y)$:
  $F(x, y) = F(|x|, |y|),\ F(1, 0) = F(0, 1) = 1.$
- unit disc : $\mathcal{B}_F = \{(x, y) \in \mathbb{R}^2 : F(x, y) \le 1\}.$
- lattice : $\Lambda_\alpha(t) = G_t A_\alpha \mathbb{Z}^2, G_t = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, A_\alpha = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}$
- successive minima : $\lambda_i(t) = \lambda_i(\Lambda_\alpha(t), \mathcal{B}_F),\ \ i = 1, 2$
- critical determinant:

$$\Delta_F = \inf\{\det \Lambda : \text{ there are no non-zero points of } \Lambda \text{ inside } \mathcal{B}_F\}.$$

- infimum is attained on some lattice - *critical lattice*.
  critical locus: $\mathfrak{L}_F$ - set of all critical lattices
- Dirichlet constant of α for the norm F:

$$d_F(\alpha) = \limsup_{t \to \infty} \lambda_1(t) = \limsup_{t \to \infty} \lambda_1(\Lambda_\alpha(t), \mathcal{B}_F).$$

- α is called *F-Dirichlet improvable* if $d_F(\alpha) < \frac{1}{\sqrt{\Delta_F}}$

Dirichlet improvability in arbitrary norm

Theorem (**Andersen** and **Duke**). *For every strongly symmetric norm F, almost all α in the sense of Lebesgue measure are not F-Dirichlet improvable, that is, for almost all α we have the equality* $d_F(\alpha) = \frac{1}{\sqrt{\Delta_F}}$.

Theorem (**Kleinbock** and **Rao**). *If F is an irreducible norm on $\mathbb{R}^2$ whose unit ball is not a parallelogram, then the set of all badly approximable F-Dirichlet non-improvable numbers $\mathrm{DI}_F^c \cap \mathrm{BA}$ has full Hausdorff dimension. In particular, the set of all badly approximable L_2-Dirichlet non-improvable numbers has full Hausdorff dimension.*

Theorem (**Kleinbock** and **Rao**). *For each norm F the set DI_F is of measure zero but winning. In particular, $\mathrm{HD}(\mathrm{DI}_F) = 1$.*

Kleinbock and **Rao**: many quiestions

Dirichlet improvability in L_p-norm: selected results

Theorem 1. *For any $p \in [1, \infty)$, the set* $\mathrm{HD}(\mathrm{DI}_p \setminus \mathrm{BA}) = 1$.

Theorem 2. $\mathrm{HD}(\mathrm{DI}_2 \setminus \mathrm{DI}_1) = \mathrm{HD}(\mathrm{DI}_1 \setminus \mathrm{DI}_2) = 1$.

Theorem 3. *For $p \in (2, p_0)$ the set of DI_p^c contains no badly approximable numbers.*

Theorem 4. *For $p \in (1, 2) \cup (p_0, \infty)$ the set $\mathrm{DI}_p^c \cap \mathrm{BA} \neq \emptyset$ if and only if the number σ_p is badly approximable.*

Theorem 5. *The set*
$\mathfrak{P} = \{p \in [1, \infty) :$
$\exists$ *p-Dirichlet non-improvable badly approximable numbers* $\alpha\}$
has zero Lebesgue measure, is dense in $(1, 2) \cup (p_0, \infty)$, is absolutely winning in any interval $[a, b] \subset (1, 2) \cup (p_0, \infty)$.

Theorem 6. *For $p \in [1, \infty]$, the number $e = 2.71828...$ satisfies $e \in \mathrm{DI}_p$ if and only if $p \in (1, 2) \cup (p_0, \infty)$.*

σ_p the unique root of the equation $\sigma^p + (1 + \sigma)^p = 2$.

$p_0 = 2.57...$ - Davis' constant.

Complete structural theorem for L_p

(**a**) Let $2 < p < p_0$. Then $\alpha \in \mathbf{DI}_p^c$ if and only if in the continued fraction for α there are patterns of the type

$$x, 1, 1, y \quad \text{or} \quad x, 2, y$$

with $\min(x, y) \to \infty$.

(**b**) Let $p \in (1, 2) \cup (p_0, \infty)$.

(**b1**) If $\sigma_p \in Q$, consider its regular finite continued fraction expansion

$$\sigma_p = [0; s_1, s_2, \dots, s_k], s_k \geq 2.$$

Then $\alpha \in \mathbf{DI}_p^c$ if and only if in its continued fraction expansion of α there occur patterns of at least one of the following four forms:

$$x, s_k, s_{k-1}, \dots, s_2, s_1, 1\, s_1, s_2, \dots, s_{k-1}, s_k, y;$$

$$x, 1, s_k - 1, s_{k-1}, \dots, s_2, s_1, 1\, s_1, s_2, \dots, s_{k-1}, s_k, y;$$

$$x, s_k, s_{k-1}, \dots, s_2, s_1, 1\, s_1, s_2, \dots, s_{k-1}, s_k - 1, 1, y;$$

$$x, 1, s_k - 1, s_{k-1}, \dots, s_2, s_1, 1\, s_1, s_2, \dots, s_{k-1}, s_k - 1, 1, y$$

with $\min(x, y) \to \infty$.

(**b2**) If $\sigma_p \notin Q$, consider its regular continued fraction expansion

$$\sigma_p = [0; s_1, s_2, \dots, s_\nu, \dots].$$

Then $\alpha \in \mathbf{DI}_p^c$ if and only if in its continued fraction expansion of α there occur palindromic patterns of the form

$$s_\nu, s_{\nu-1}, \dots, s_2, s_1, 1\, s_1, s_2, \dots, s_{\nu-1}, s_\nu$$

with arbitrary large values of ν.

(**c**) Number $\alpha \in \mathbf{DI}_1^c$ if and only if there exists a sequence of positive integers $\{b_n\}_{n \in Z_+}$, such that either the continued fraction expansion of α contains almost symmetric patterns

$$b_\nu, b_{\nu-1}, \dots, b_2, b_1, 1, 1, b_1 + 1, b_2, \dots, b_{\nu-1}, b_\nu$$

or

$$b_\nu, b_{\nu-1}, \dots, b_2, b_1 + 1, 1, 1, b_1, b_2, \dots, b_{\nu-1}, b_\nu$$

with arbitrary large values of ν

or a sequence of patterns of at least one of the following eight forms:

$$x, b_\nu, b_{\nu-1}, \dots, b_2, b_1, 1, 1, b_1 + 1, b_2, \dots, b_{\nu-1}, b_\nu, y;$$

$$x, b_\nu, b_{\nu-1}, \dots, b_2, b_1, 1, 1, b_1 + 1, b_2, \dots, b_{\nu-1}, b_\nu - 1, 1, y;$$

$$x, 1, b_\nu - 1, b_{\nu-1}, \dots, b_2, b_1, 1, 1, b_1 + 1, b_2, \dots, b_{\nu-1}, b_\nu, y;$$

$$x, 1, b_\nu - 1, b_{\nu-1}, \dots, b_2, b_1, 1, 1, b_1 + 1, b_2, \dots, b_{\nu-1}, b_\nu - 1, 1, y;$$

$$x, b_\nu, b_{\nu-1}, \dots, b_2, b_1 + 1, 1, 1, b_1, b_2, \dots, b_{\nu-1}, b_\nu, y;$$

$$x, 1, b_\nu - 1, b_{\nu-1}, \dots, b_2, b_1 + 1, 1, 1, b_1, b_2, \dots, b_{\nu-1}, b_\nu, y;$$

$$x, b_\nu, b_{\nu-1}, \dots, b_2, b_1 + 1, 1, 1, b_1, b_2, \dots, b_{\nu-1}, b_\nu - 1, 1, y;$$

$$x, 1, b_\nu - 1, b_{\nu-1}, \dots, b_2, b_1 + 1, 1, 1, b_1, b_2, \dots, b_{\nu-1}, b_\nu - 1, 1, y$$

with fixed $\nu, b_1, \dots, b_\nu$ and $\min(x, y) \to \infty$,
or patterns $x, 2, y$ with $\min(x, y) \to \infty$, or patterns $x, 1, 1, y$ with $\min(x, y) \to \infty$.

(**d**) Number $\alpha \in \mathbf{DI}_2^c$ if and only if either in continued fraction for α there are patterns of the type

$$x, 1, 1, y \quad \text{or} \quad x, 2, y$$

with $\min(x, y) \to \infty$ or
there exist two irrational numbers

$$\beta^* = [b_0^*; b_1^*, b_2^*, \dots, b_{\nu-1}^*, b_\nu^*, \dots],$$

$$\beta = [b_0; b_1, b_2, \dots, b_{\nu-1}, b_\nu, \dots], \quad b_0^*, b_0 \geq 0$$

satisfying the equation such that in the continued fraction expansion of α there exist patterns

$$b_\nu^*, \dots, b_1^*, b_0^* + 1, 1, b_0 + 1, b_1, \dots, b_\nu$$

or

$$b_\nu, \dots, b_1, b_0 + 1, 1, b_0^* + 1, b_1^*, \dots, b_\nu^*$$

with arbitrary large values of ν or

there exist two rational numbers β^* and β satisfying and in the continued fraction expansion of α there exists a sequence of one of the eight patterns with $\min(x, y) \to \infty$ constructed from $\beta^* = [b_0^*; b_1^*, \dots, b_k^*], \beta = b_0; b_1, \dots, b_\nu]$ similarly to those from statement (**c**)[1].

[1] If β^*, β are both rational, then their continued fractions are not necessarily of the same length. One should interpret the pattern structure in the following way.

For example, one of the eight patterns is $x, b_k^*, \dots, b_1^*, b_0^* + 1, 1, b_0 + 1, b_1, \dots, b_\nu, y$ with $\min(x, y) \to \infty$ and $b_k^*, b_\nu \geq 2$ and the rest are constructed from this one in the same way as in the case (**c**) by changing last partial quotient b_ν to $b_\nu - 1, 1$ and so on. Note that for $\beta = 1$ there's no partial quotient ≥ 2, but it has still two representations $\beta = [0; 1] = [1]$.

Structural theorem: L_1

Number $\alpha \in \mathrm{DI}_1^c$ if and only if there exists a sequence of positive integers $\{b_n\}_{n\in\mathbb{Z}_+}$, such that either the continued fraction expansion of α contains almost symmetric patterns

$$b_\nu, b_{\nu-1}, \ldots, b_2, b_1, 1, 1, b_1+1, b_2, \ldots, b_{\nu-1}, b_\nu \quad \text{or}$$

$$b_\nu, b_{\nu-1}, \ldots, b_2, b_1+1, 1, 1, b_1, b_2, \ldots, b_{\nu-1}, b_\nu$$

with arbitrary large ν, or a sequence of patterns of at least one of the following eight forms:

$$x, b_\nu, b_{\nu-1}, \ldots, b_2, b_1, 1, 1, b_1+1, b_2, \ldots, b_{\nu-1}, b_\nu, y;$$

$$x, b_\nu, b_{\nu-1}, \ldots, b_2, b_1, 1, 1, b_1+1, b_2, \ldots, b_{\nu-1}, b_\nu-1, 1, y;$$

$$x, 1, b_\nu-1, b_{\nu-1}, \ldots, b_2, b_1, 1, 1, b_1+1, b_2, \ldots, b_{\nu-1}, b_\nu, y;$$

$$x, 1, b_\nu-1, b_{\nu-1}, \ldots, b_2, b_1, 1, 1, b_1+1, b_2, \ldots, b_{\nu-1}, b_\nu-1, 1, y;$$

$$x, b_\nu, b_{\nu-1}, \ldots, b_2, b_1+1, 1, 1, b_1, b_2, \ldots, b_{\nu-1}, b_\nu, y;$$

$$x, 1, b_\nu-1, b_{\nu-1}, \ldots, b_2, b_1+1, 1, 1, b_1, b_2, \ldots, b_{\nu-1}, b_\nu, y;$$

$$x, b_\nu, b_{\nu-1}, \ldots, b_2, b_1+1, 1, 1, b_1, b_2, \ldots, b_{\nu-1}, b_\nu-1, 1, y;$$

$$x, 1, b_\nu-1, b_{\nu-1}, \ldots, b_2, b_1+1, 1, 1, b_1, b_2, \ldots, b_{\nu-1}, b_\nu-1, 1, y$$

or patterns $x, 2, y$, or patterns $x, 1, 1, y$ with $x, y \to \infty$.

Structural theorem: L_2

Number $\alpha \in \mathrm{DI}_2^c$ if and only if either in continued fraction for α there are patterns of the type $x, 1, 1, y$ or $x, 2, y$ with $\min(x, y) \to \infty$ or

there exist two irrational numbers

$$\beta^* = [b_0^*; b_1^*, b_2^*, \ldots, b_{\nu-1}^*, b_\nu^*, \ldots],$$

$$\beta = [b_0; b_1, b_2, \ldots, b_{\nu-1}, b_\nu, \ldots], \quad b_0^*, b_0 \geq 0$$

satisfying the equation

$$\beta \cdot \beta^* = 3,$$

such that in the continued fraction expansion of α there exist patterns $b_\nu^*, \ldots, b_1^*, b_0^* + 1, 1, b_0 + 1, b_1, \ldots, b_\nu$
with arbitrary large values of ν,

or ... (8+ cases similar to those from L_1).

Critical lattices for L_p-disc

After Minkowski classification of critical lattices for $\mathcal{B}_p$ was dealt by

C.S. Davis, *Note on a conjecture by Minkowski*, J. London Math. Soc. 23, (1948), 172-175,
G.L. Watson, *Minkowski's conjectures on the critical lattices of the region* $|x|^p + |y|^p \leq 1$. *I*, J. London Math. Soc. 28, (1953). 305-309.
G.L. Watson, *Minkowski's conjectures on the critical lattices of the region* $|x|^p + |y|^p \leq 1$. *II*, J. London Math. Soc. 28, (1953). 402-410.

. . . .

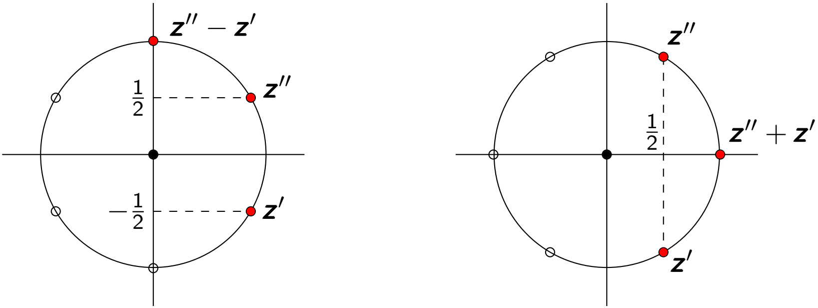
and finalised by Glazunov, Golovanov, and Malyshev:

Н. М. Глазунов; А. С. Голованов; А. В. Малышев, *Доказательство гипотезы Минковского о критическом определителе области* $|x|^p + |y|^p < 1$, Исследования по теории чисел. 9, Зап. научн. сем. ЛОМИ, 151, Изд-во «Наука», Ленинград. отд., Л., 1986, 40-53 (in Russian).

Case $2 < p < p_0$. In this case the the only two (congruent) critical lattices for the ball $\mathcal{B}_p$ are $\Lambda_1 = \Omega_1 \cdot \mathbb{Z}^2$ and $\Lambda_1' = \Omega_1' \cdot \mathbb{Z}^2$, where

$$\Omega_1 = \begin{pmatrix} \left(1 - \frac{1}{2^{1/p}}\right)^{\frac{1}{p}} & \left(1 - \frac{1}{2^{1/p}}\right)^{\frac{1}{p}} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

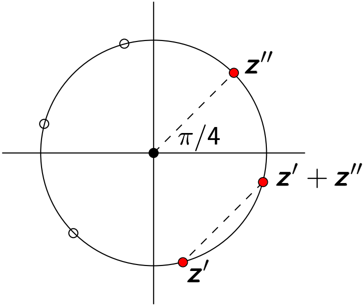
$$\Omega_1' = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\left(1 - \frac{1}{2^{1/p}}\right)^{\frac{1}{p}} & \left(1 - \frac{1}{2^{1/p}}\right)^{\frac{1}{p}} \end{pmatrix}.$$



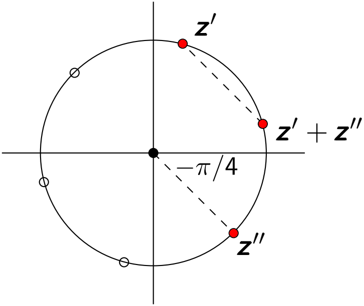
Lattice Λ_1 Lattice Λ_1'

Case $1 < p < 2$ and $p > p_0$.

$$\Lambda_2^{\pm} = \Omega_2^{\pm} \cdot \mathbb{Z}^2 \quad \text{where} \quad \Omega_2^{\pm} = \begin{pmatrix} \frac{\sigma_p}{2^{1/p}} & \frac{1}{2^{1/p}} \\ \mp\frac{1+\sigma_p}{2^{1/p}} & \pm\frac{1}{2^{1/p}} \end{pmatrix}.$$



Lattice Λ_2^+

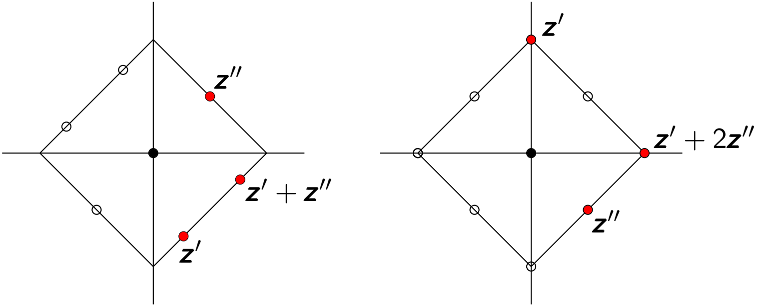


Lattice Λ_2^-

Case $p = 1$.

$a \in \left[0, \frac{1}{2}\right)$,

$$\Lambda_3^{\pm}(a) = \Omega_3^{\pm}(a) \cdot \mathbb{Z}^2, \quad \text{where} \quad \Omega_3^{\pm}(a) = \begin{pmatrix} a & \frac{1}{2} \\ \pm(a-1) & \pm\frac{1}{2} \end{pmatrix}.$$



Lattice $\Lambda_3^{+}(a)$

Lattice $\Lambda_3^{-}(0)$

Case $p = 2$.

$\varphi \in \left[0, \frac{\pi}{6}\right]$ and $u = \sin\varphi \in \left[0, \frac{1}{2}\right]$, consider the lattices

$$\Lambda_4^{\pm}(\varphi) = \Omega_4^{\pm}(\varphi)\cdot\mathbb{Z}^2,\ \Omega_4^{\pm}(\varphi) = \begin{pmatrix} \sin\varphi & \cos\left(\frac{\pi}{6}+\varphi\right) \\ \pm\cos\varphi & \mp\sin\left(\frac{\pi}{6}+\varphi\right) \end{pmatrix} =$$

$$= \begin{pmatrix} u & \frac{\sqrt{3-3u^2}-u}{2} \\ \pm\sqrt{1-u^2} & \mp\frac{\sqrt{1-u^2}+u\sqrt{3}}{2} \end{pmatrix}.$$

This parametrisation and some manipulations $u \mapsto \beta, \beta^*$ lead to equation

$$\beta(\Omega_4^{\pm}(\varphi)) \cdot \beta^*(\Omega_4^{\pm}(\varphi)) = 3.$$

L_1: Minkowski diagonal fraction and spectrum

Those denominators of convergent for which $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{2q_n^2}$:

$$Q_1, Q_2, ..., Q_n, ...$$

$$\mu_\alpha(t) = \frac{Q_{n+1} - t}{Q_{n+1} - Q_n} \cdot ||Q_n\alpha|| + \frac{t - Q_n}{Q_{n+1} - Q_n} \cdot ||Q_{n+1}\alpha||, \quad Q_n \le t \le Q_{n+1}$$

Minkowski: $\mu_\alpha(t)$ is convex.

$$\mathfrak{m}(\alpha) = \limsup_{t \to +\infty} t \cdot \mu_\alpha(t).$$

$$\mathbb{M} = \{m \in \mathbb{R} : \ \exists \alpha \in \mathbb{R} \text{ such that } m = \mathfrak{m}(\alpha)\}.$$

Theorem.

$$\min \mathbb{M} = \frac{1}{4}, \quad \max \mathbb{M} = \frac{1}{2}.$$

Open problem: is it true that

$$\mathbb{M} = \text{ or } \neq \left[\frac{1}{4}, \frac{1}{2}\right].$$

THANK YOU!