Adaptive Multilevel Stochastic Galerkin Finite Element Approximation Using Hierarchical Error Estimation

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M\cr NA

Outline

Parametric Elliptic PDEs: Find $u : D \times \Gamma \to \mathbb{R}$ such that

$$-\nabla \cdot (a(\boldsymbol{x}, \mathbf{y})\nabla u(\boldsymbol{x}, \mathbf{y})) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in D \subset \mathbb{R}^{2,3}, \quad \boldsymbol{y} \in \Gamma,$$

where

$$a(\mathbf{x}, \boldsymbol{y}) = a_0(\mathbf{x}) + \sum_{m=1}^{\infty} a_m(\mathbf{x}) y_m$$

and $y_m = \xi_m(\omega)$ are images of **independent** + **bounded** **random variables**.

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- ☐ Galerkin Approximation Basics
- ☐ **Hierarchical Error Estimation: single & multi-level**
- ☐ Adaptivity
- ☐ Numerical Results + Software

Disclaimers + Other Work

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- Much earlier work on **multilevel** SGFEM, e.g.,
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  - **Cohen, DeVore, Schwab**. Analytic regularity and polynomial approx. of parametric and stochastic elliptic PDE's. Anal. Appl., 9(1), 2011.

Background + Collaborators

- Worked presented here based on PhD work of **Adam Crowder**:
  - **PhD thesis**, University of Manchester, 2020.
  - Efficient adaptive multilevel SG approx. using implicit a posteriori error estimation. Crowder, Powell, Bespalov, **SISC. 41(3)**, 2019.
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Stochastic Galerkin Approximation

1. **Parametric PDE:** Find $u : D \times \Gamma \to \mathbb{R}$ such that

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2. **Weak Problem:** Find $u \in V := L^2_\pi(\Gamma, H^1_0(D))$ satisfying

$$\int_\Gamma \left(\int_D a \nabla u \cdot \nabla v \, d\mathbf{x} \right) d\pi(\mathbf{y}) = \int_\Gamma \left(\int_D f \, v \, d\mathbf{x} \right) d\pi(\mathbf{y}) \qquad \forall v \in V,$$

where π is a **probability measure**.

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where $A(\cdot, \cdot)$ is an **inner product** that induces an **energy norm** $\| \cdot \|_A$.

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4. **Error Estimation:** Estimate $\|e\|_A$ where $e := u - u_X \in V$ satisfies:

$$A(e, v) = \underbrace{\ell(v) - A(u_X, v)}_{\text{residual } R(v)} \qquad \forall\, v \in V.$$

Galerkin Approximation Space X

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- J_P is a **finite set** of **finitely supported** sequences:

$$J_P \subset J := \left\{ \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots) \in \mathbb{N}_0^{\mathbb{N}} \mid \# \operatorname{supp} \boldsymbol{\alpha} < \infty \right\}.$$

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If $H_1^{\boldsymbol{\alpha}} = H_1$ for all $\boldsymbol{\alpha} \in J_P$ then

$$X := H_1 \otimes P$$

where $P := \operatorname{span}\{\psi_{\boldsymbol{\alpha}}(\boldsymbol{y}), \boldsymbol{\alpha} \in J_P\}$.

Polynomial-based Approximation

$$u_X(\boldsymbol{x},\mathbf{y}) \;=\; \sum_{\boldsymbol{\alpha}\in J_P}\left(\sum_{i=1}^{n_{\boldsymbol{\alpha}}} u_{i,\boldsymbol{\alpha}}\phi_i^{\boldsymbol{\alpha}}(\boldsymbol{x})\right)\psi_{\boldsymbol{\alpha}}(\mathbf{y}) = \sum_{\boldsymbol{\alpha}\in J_P}\underbrace{u_{\boldsymbol{\alpha}}(\boldsymbol{x})}_{\in H_1^{\boldsymbol{\alpha}}}\psi_{\boldsymbol{\alpha}}(\mathbf{y}).$$

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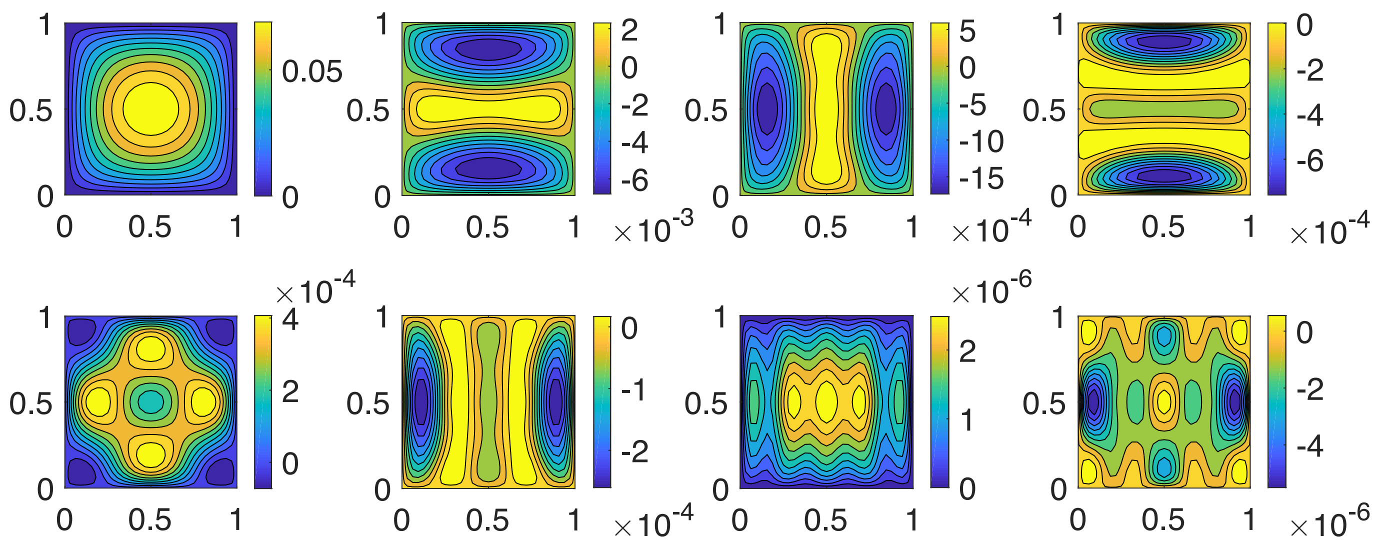
Test Problem: 8 spatial modes $u_{\boldsymbol{\alpha}}(\boldsymbol{x})$



Hierarchical Error Estimation

For $u_X \in X \subset V$, we know $e := u - u_X \in V$ satisfies:

$$A(e, v) = R(v) \qquad \forall\, v \in V.$$

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1. Consider $e_W \in W \supset X$ such that:

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$$W = X \;\oplus \underbrace{Y}_{\textbf{`detail'}}, \qquad X \cap Y = \{0\}$$

   and define **error estimate** $\eta := \|e_Y\|_A$ where

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Two-sided Error Bounds

If we can choose the **detail space** Y so that

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1. **Saturation Assumption:** $\exists\, \beta \in [0,1)$ such that

$$\|u - u_W\|_A \leq \beta\, \|u - u_X\|_A$$

2. **CBS Inequality:** $\exists\, \gamma \in [0,1)$ such that

$$|A_0(u,v)| \leq \gamma\, \|u\|_{A_0} \|v\|_{A_0}, \qquad \forall u \in X, \forall v \in Y$$

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Single-level Case

$$X = H_1 \otimes P$$

- Choose H_2 **(FEM space)** such that $H_1 \cap H_2 = \{0\}$.
- Choose J_Q **(set of multi-indices)** with $J_P \cap J_Q = \emptyset$. Then

$$P \cap Q = \{0\}, \qquad Q := \mathrm{span}\left\{\psi_{\boldsymbol{\beta}}(\boldsymbol{y}), \boldsymbol{\beta} \in J_Q\right\}.$$

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$$Y = (H_2 \otimes P) \oplus (H_1 \otimes Q) =: Y_1 \oplus Y_2.$$

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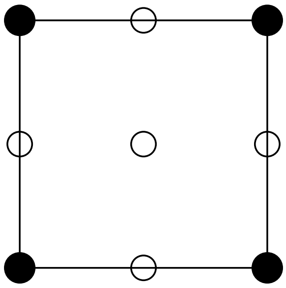
$$\eta := \|e_Y\|_{A_0} = \left(\|e_{Y_1}\|_{A_0}^2 + \|e_{Y_2}\|_{A_0}^2\right)^{1/2},$$

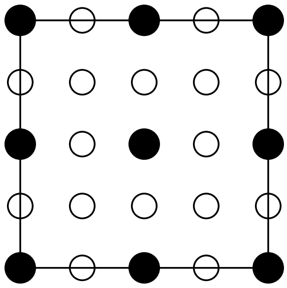
where problem **decouples:**

$$\begin{aligned} e_{Y_1} \in Y_1 : \quad A_0(e_{Y_1}, v) &= R(v), \quad \forall\, v \in Y_1, \\ e_{Y_2} \in Y_2 : \quad A_0(e_{Y_2}, v) &= R(v), \quad \forall\, v \in Y_2. \end{aligned}$$

How to Choose H_2?

$H_1 = \mathbb{Q}_1(h)$, $H_1 = \mathbb{Q}_2(h)$





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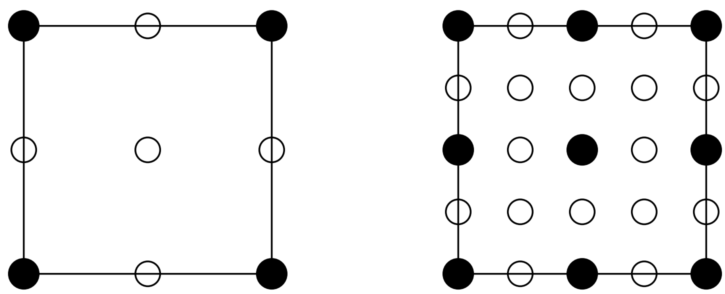

Example: Let $H_1 = \mathbb{Q}_1(h)$ **(piecewise bilinear)** on a mesh $\mathcal{T}_h$. Define H_2 using 'bubble' functions associated with **five nodes** ($\circ$).

H_1	H_2	γ^2	$\sqrt{1-\gamma^2}$
$\mathbb{Q}_1(h)$	$\mathbb{Q}_2(h)$	**0.4545**	0.7385
$\mathbb{Q}_1(h)$	$\mathbb{Q}_1(h/2)$	**0.3750**	0.7905
$\mathbb{Q}_1(h)$	$\mathbb{Q}_2(h/2)$	**0.0446**	0.9774
$\mathbb{Q}_1(h)$	$\mathbb{Q}_4(h)$	**0.0121**	0.9939

What about the Saturation Constant?

$$X = H_1 \otimes P, \qquad Y = (H_2 \otimes P) \oplus (H_1 \otimes Q)$$

To get an **accurate** error estimate, need to choose H_2 and Q so that

$$\sqrt{1-\beta^2}\,\sqrt{1-\gamma^2} \approx 1$$

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- ☐ Define Q by choosing

$$J_Q = \{\boldsymbol{\beta} \in J \setminus J_P,\ \boldsymbol{\beta} = \boldsymbol{\alpha} \pm \mathbf{e}^m, \boldsymbol{\beta} \in J_P, m = 1, \ldots, M_P + \Delta_M\}$$

where M_P is # highest parameter activated in P.

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- Δ_M needs to be **larger** for problems where $\|a_m\|_\infty$ decays **slowly**.

Example: Single-level Approximation

Test Problem: (very) slow decay case

Let $D := [-1,1]^2$, $f(\mathbf{x}) = \frac{1}{8}(2 - x_1^2 - x_2^2)$ and

$$a(\mathbf{x}, \boldsymbol{y}) = 1 + \sigma\sqrt{3}\sum_{m=1}^{\infty}\sqrt{\lambda_m}\,\varphi_m(\mathbf{x})\,y_m$$

where $\{(\lambda_m, \varphi_m)\}_{m=1}^{\infty}$ are the eigenpairs associated with the covariance

$$C(\mathbf{x}, \mathbf{x}') = \exp\left(-\tfrac{1}{2}\|\mathbf{x} - \mathbf{x}'\|_1\right).$$

□ Fix Galerkin approximation space X:

- $H_1 = \mathbb{Q}_1(h)$ on a uniform partition $\square_h$ of D
- $P =$ polynomials of total degree ≤ 4 in $y_1, \ldots, y_5$ $(M = 5)$.

Example: Single-level Approximation

Test Problem: (very) slow decay case

Let $D := [-1, 1]^2$, $f(\mathbf{x}) = \frac{1}{8}(2 - x_1^2 - x_2^2)$ and

$$a(\mathbf{x}, \boldsymbol{y}) = 1 + \sigma\sqrt{3}\sum_{m=1}^{\infty}\sqrt{\lambda_m}\,\varphi_m(\mathbf{x})\,y_m$$

where $\{(\lambda_m, \varphi_m)\}_{m=1}^{\infty}$ are the eigenpairs associated with the covariance

$$C(\mathbf{x}, \mathbf{x}') = \exp\left(-\tfrac{1}{2}\|\mathbf{x} - \mathbf{x}'\|_1\right).$$

- Fix Galerkin approximation space X:
  - $H_1 = \mathbb{Q}_1(h)$ on a uniform partition $\square_h$ of D
  - P = polynomials of total degree ≤ 4 in $y_1, \ldots, y_5$ $(M = 5)$.
- Vary error estimation detail space Y:
  - $H_2 = \mathbb{Q}_2(h), \mathbb{Q}_1(h/2), \mathbb{Q}_4(h)$ (5 element bubble functions).
  - Define Q by choosing J_Q with $\Delta_M = 1$ or 3.

Effectivity Indices

$$E_\eta := \frac{\eta}{\|u_{\mathrm{ref}} - u_X\|_A}$$

- Q defined with $\Delta_M = 1$

h	$\|u_{\mathrm{ref}} - u_X\|_A$	$H_2 = \mathbb{Q}_2(h)$	$H_2 = \mathbb{Q}_1(h/2)$	$H_2 = \mathbb{Q}_4(h)$
$1/8$	1.0244e-03	0.93	0.84	0.22
$1/16$	6.2277e-03	0.80	0.73	0.31
$1/32$	4.7187e-03	0.62	0.59	0.39

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- Q defined with $\Delta_M = 3$

h	$\|u_{\mathrm{ref}} - u_X\|_A$	$H_2 = \mathbb{Q}_2(h)$	$H_2 = \mathbb{Q}_1(h/2)$	$H_2 = \mathbb{Q}_4(h)$
1/8	1.0244e-03	**0.96**	**0.88**	0.35
1/16	6.2277e-03	**0.92**	**0.86**	0.54
1/32	4.7187e-03	**0.86**	**0.83**	0.71

Multi-level Case (1)

$$X = \bigoplus_{\boldsymbol{\alpha} \in J_P} H_1^{\boldsymbol{\alpha}} \otimes P^{\boldsymbol{\alpha}}$$

- For each $\boldsymbol{\alpha} \in J_P$ choose $H_2^{\boldsymbol{\alpha}}$ **(FEM space)** such that

$$H_1^{\boldsymbol{\alpha}} \cap H_2^{\boldsymbol{\alpha}} = \{0\}.$$

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$$J_P \cap J_Q = \emptyset$$

and for each $\boldsymbol{\beta} \in J_Q$ define $Q^{\boldsymbol{\beta}} := \mathrm{span}\{\psi_{\boldsymbol{\beta}}(\boldsymbol{y})\}$.

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Define **detail space**:

$$Y = \left(\bigoplus_{\boldsymbol{\alpha} \in J_P} H_2^{\boldsymbol{\alpha}} \otimes P^{\boldsymbol{\alpha}} \right) \oplus \left(\bigoplus_{\boldsymbol{\beta} \in J_Q} H \otimes Q^{\boldsymbol{\beta}} \right) := Y_1 \oplus Y_2$$

where H is **one** of the **FEM spaces** $H_1^{\boldsymbol{\alpha}}$.

Multi-level Case (2)

Re-write detail space as

$$Y = \left(\bigoplus_{\boldsymbol{\alpha} \in J_P} Y_{1,\boldsymbol{\alpha}} \right) \oplus \left(\bigoplus_{\boldsymbol{\beta} \in J_Q} Y_{2,\boldsymbol{\beta}} \right)$$

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$$Y_{1,\boldsymbol{\alpha}} = H_2^{\boldsymbol{\alpha}} \otimes P^{\boldsymbol{\alpha}}, \qquad Y_{2,\boldsymbol{\beta}} = H \otimes Q^{\boldsymbol{\beta}}.$$

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$$Y_{1,\alpha} = H_2^{\alpha} \otimes P^{\alpha}, \qquad Y_{2,\beta} = H \otimes Q^{\beta}.$$

Now, $\gamma := \max_{\alpha \in J_P} \{\gamma_\alpha\}$ where γ_α is associated with H_2^{α} and H_1^{α} and

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Now, $\gamma := \max_{\alpha \in J_P}\{\gamma_\alpha\}$ where γ_α is associated with H_2^α and H_1^α and

$$\eta := \|e_Y\|_{A_0} = \left(\sum_{\alpha \in J_P} \|e_{Y_{1,\alpha}}\|_{A_0}^2 + \sum_{\beta \in J_Q} \|e_{Y_{2,\beta}}\|_{A_0}^2 \right)^{1/2},$$

where error estimation problem **decouples further**:

$$\begin{aligned} e_{Y_{1,\alpha}} \in Y_{1,\alpha} : \quad A_0(e_{Y_{1,\alpha}}, v) &= R(v), \quad \forall v \in Y_{1,\alpha}, \\ e_{Y_{2,\beta}} \in Y_{2,\beta} : \quad A_0(e_{Y_{2,\beta}}, v) &= R(v), \quad \forall v \in Y_{2,\beta}. \end{aligned}$$

Error Reduction Indicators

Define the spaces

$$W_1 := X \oplus \left(\bigoplus_{\boldsymbol{\alpha} \in \overline{J_P}} Y_{1,\boldsymbol{\alpha}} \right), \qquad W_2 := X \oplus \left(\bigoplus_{\boldsymbol{\beta} \in \overline{J_Q}} Y_{2,\boldsymbol{\beta}} \right)$$

where $\overline{J_P} \subset J_P$ and $\overline{J_Q} \subset J_Q$ and consider the Galerkin approximations:

- $u_{W_1} \in W_1$ **(spatial refinement)**, $u_{W_2} \in W_2$ **(parametric enrichment)**.

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- $u_{W_1} \in W_1$ **(spatial refinement)**, $u_{W_2} \in W_2$ **(parametric enrichment)**.

Define the error estimates

$$\eta_1 := \sum_{\alpha \in \overline{J_P}} \| e_{Y_{1,\alpha}} \|_{A_0}^2, \qquad \eta_2 := \sum_{\beta \in \overline{J_Q}} \| e_{Y_{2,\beta}} \|_{A_0}^2.$$

Then one can prove that

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Then one can prove that

$$\lambda \, \eta_1 \le \quad \| u_{W_1} - u_X \|_E^2 \quad \le \frac{\Lambda}{1 - \gamma^2} \, \eta_1$$

$$\lambda \, \eta_2 \le \quad \| u_{W_2} - u_X \|_E^2 \quad \le \Lambda \, \eta_2.$$

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- **INITIALIZE:** J_P and $\{H_1^{\boldsymbol{\alpha}}, \forall \boldsymbol{\alpha} \in J_P\}$
- **SOLVE:** Find $u_X \in X$

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$$\{\|e_{Y_{1,\boldsymbol{\alpha}}}\|_{A_0}, \ \boldsymbol{\alpha} \in J_P\}, \qquad \{\|e_{Y_{2,\boldsymbol{\beta}}}\|_{A_0}, \ \boldsymbol{\beta} \in J_Q\}$$

- **ENERGY ERROR ESTIMATE:**

$$\eta = \left(\sum_{\boldsymbol{\alpha} \in J_P} \|e_{Y_{1,\boldsymbol{\alpha}}}\|_{A_0}^2 + \sum_{\boldsymbol{\beta} \in J_Q} \|e_{Y_{2,\boldsymbol{\beta}}}\|_{A_0}^2 \right)^{1/2}$$

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- ☐ **IF** $\eta \leq TOL$ **STOP**;
- ☐ **ELSE**

Compute **ESTIMATED ERROR REDUCTION RATIOS:**

$$\left\{ \frac{\|e_{Y_{1,\boldsymbol{\alpha}}}\|_{A_0}^2}{\dim(Y_{1,\boldsymbol{\alpha}})}, \, \boldsymbol{\alpha} \in J_P \right\}, \qquad \left\{ \frac{\|e_{Y_{2,\boldsymbol{\beta}}}\|_{A_0}^2}{\dim(Y_{2,\boldsymbol{\beta}})}, \, \boldsymbol{\beta} \in J_Q \right\}$$

Basic Adaptive Algorithm (2)

- **IDENTIFY** 'important' subsets

$$\overline{J_P} \subset J_P, \qquad \overline{J_Q} \subset J_Q$$

- **ERROR REDUCTION INDICATORS**

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- **DECIDE** whether to do **SPATIAL** or **PARAMETRIC** enrichment

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IF SPATIAL

- Freeze J_P
- **improve** $H_1^{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha} \in \overline{J_P}$

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IF PARAMETRIC

- Freeze $H_1^{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha} \in J_P$
- initialize H for new $\boldsymbol{\alpha} \in \overline{J_Q}$
- $J_P \leftarrow J_P \cup \overline{J_Q}$;

END

Example 1: 2D in space + slow algebraic decay

Test Problem: slow decay case

$D = [0,1]^2$, $f(\boldsymbol{x}) = 1$, $y_m \sim U(-1,1)$ and

$$a(\boldsymbol{x},\boldsymbol{y}) = 1 + \sum_{m=1}^{\infty} a_m(\boldsymbol{x}) y_m, \qquad \|a_m(\boldsymbol{x})\|_{\infty} \sim m^{-2}.$$

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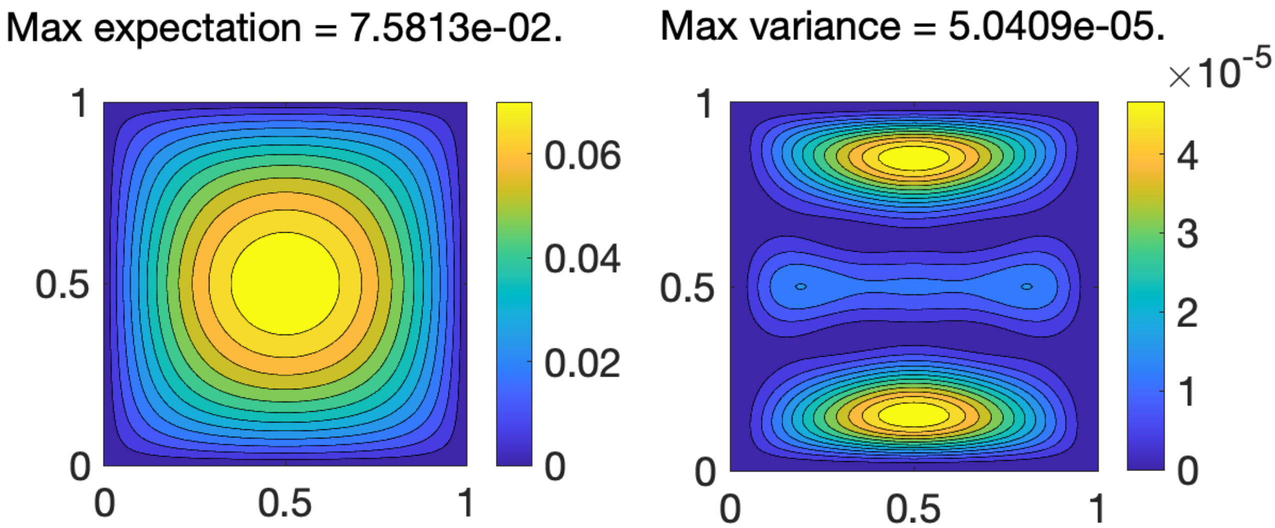
$$a(\mathbf{x},\mathbf{y}) = 1 + \sum_{m=1}^{\infty} a_m(\mathbf{x}) y_m, \qquad \|a_m(\mathbf{x})\|_\infty \sim m^{-2}.$$

- **INITIALIZE:** $J_P = \{\mathbf{0}, (1,0,\ldots,0)\}$ and $H_1^\alpha = \mathbb{Q}_1(h)$ on uniform mesh, with $h = 2^{-4}$ **('level' 4)**.
- **DETAIL SPACE:** Choose $H_2^\alpha = \mathbb{Q}_2(h)$ and $\Delta_M = 5$.
- Choose $TOL = 1.5\text{e-}3$.
- **Target convergence rate:** $N_{\rm dof}^{-1/2}$.

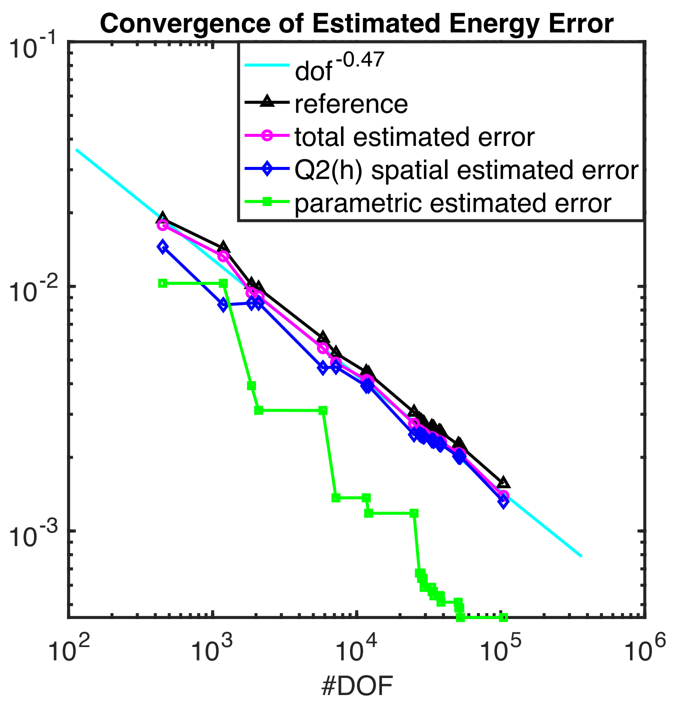
Eigel, Gittelson, Schwab, Zander. Adaptive stochastic Galerkin FEM. **CMAME** (2014).

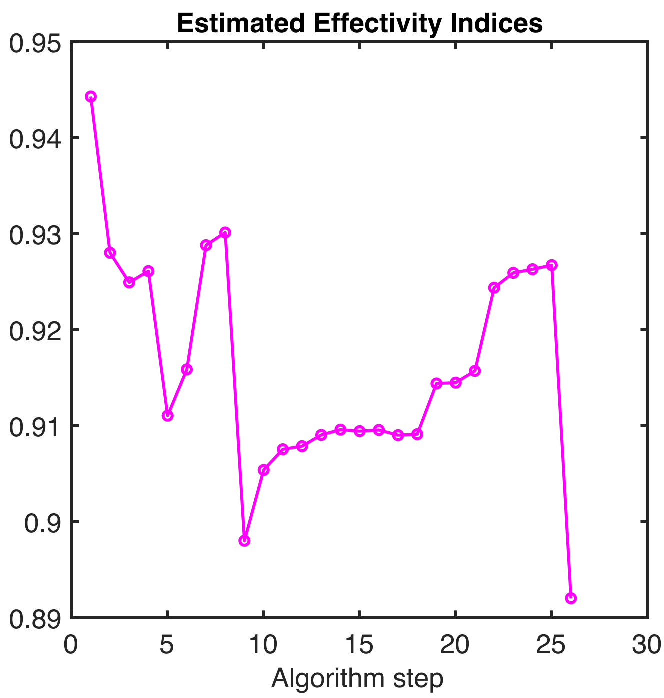
Example 1: Final Mean & Variance



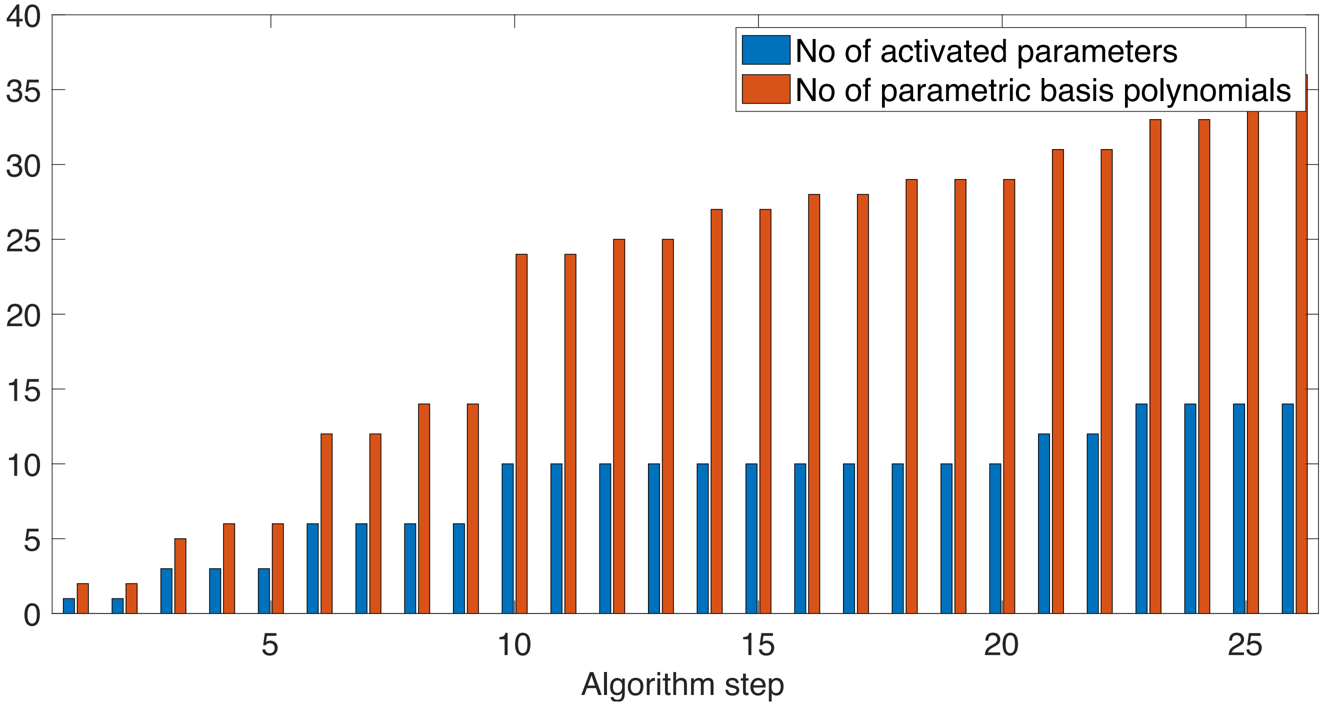

```
Time = 20.90 seconds.
Reference error = 1.5602e-03
Estimated error = 1.3917e-03
Total iterations = 26.
No. parametric polynomials = 36
No. activated variables = 14.
Total DOF = 104,452.
```

Example 1: Convergence & Accuracy





Example 1: Final Approximation Space



At the final step $X := \bigoplus_{\boldsymbol{\alpha} \in J_P} H_1^{\boldsymbol{\alpha}} \otimes P^{\boldsymbol{\alpha}}$

- J_P contains 36 multi-indices, ($M = 14$ **activated parameters**)
- $H_1^{\boldsymbol{\alpha}} = \mathbb{Q}_1(h)$ with $h = 2^{-8}, 2^{-7}, 2^{-6}, 2^{-5}, 2^{-4}$ **(1,1,3,6,25 terms)**

Example 2: 2D in space + exponential decay

Test Problem: exponential decay case

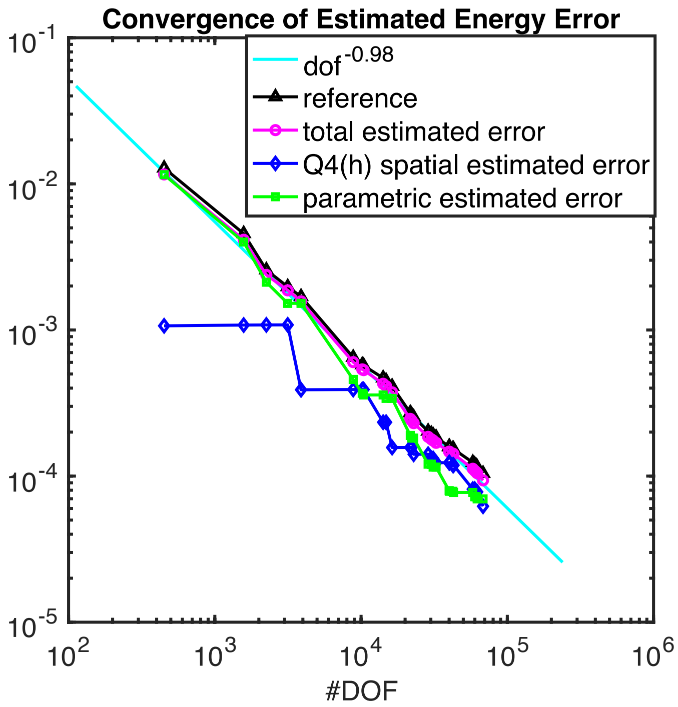
$D = [0,1]^2$, $f(\boldsymbol{x}) = 1$, $y_m \sim U(-1,1)$ and

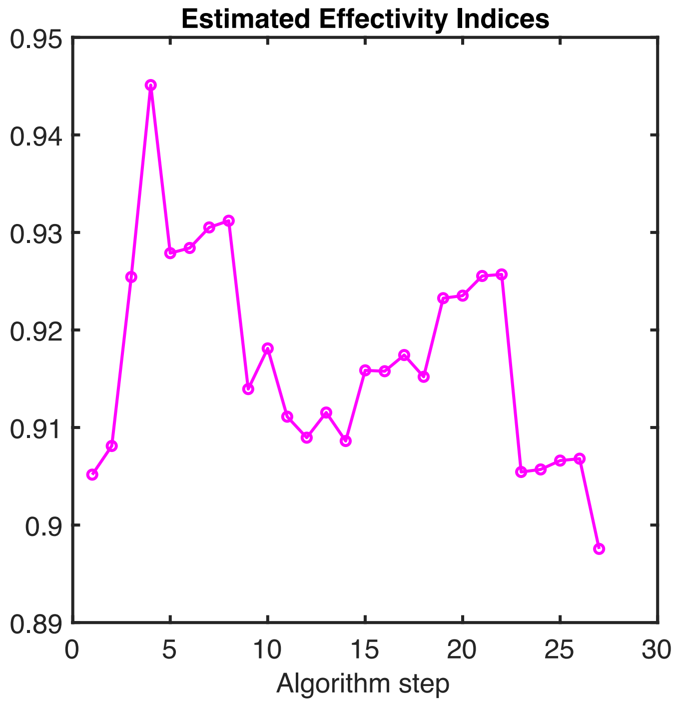
$$a(\boldsymbol{x},\boldsymbol{y}) = 1 + \sum_{m=1}^{\infty} a_m(\boldsymbol{x}) y_m, \quad , \quad \|a_m(\boldsymbol{x})\|_\infty \sim e^{-m}.$$

- **INITIALIZE:** $J_P = \{\boldsymbol{0}, (1,0,\ldots,0)\}$ and $H_1^{\alpha} = \mathbb{Q}_2(h)$ on uniform mesh with $h = 2^{-4}$ **(level 4)**.
- **DETAIL SPACE:** Choose $H_2^{\alpha} = \mathbb{Q}_4(h)$ and $\Delta_M = 2$.
- Choose $TOL = 1\text{e}{-4}$.
- **Target convergence rate:** N_{dof}^{-1}.

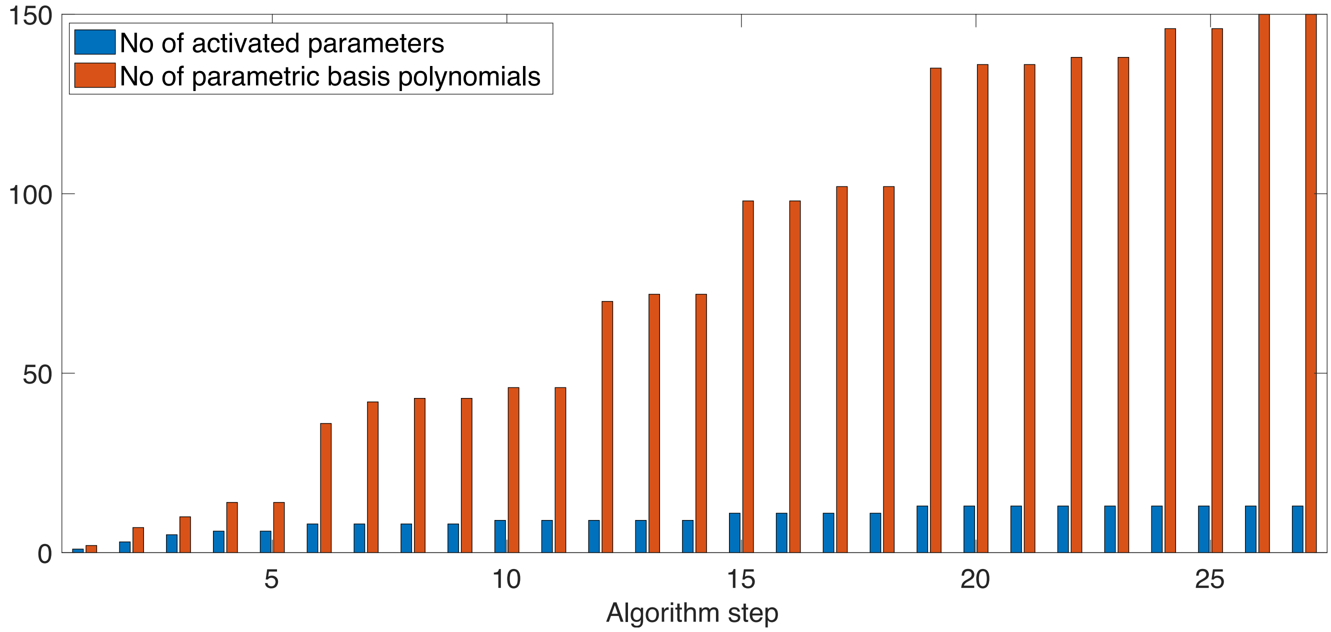
Lord, Powell, Shardlow. Introduction to Computational Stochastic PDEs, **CUP**, 2014.

Example 2: Convergence & Accuracy





Example 2: Final Approximation Space

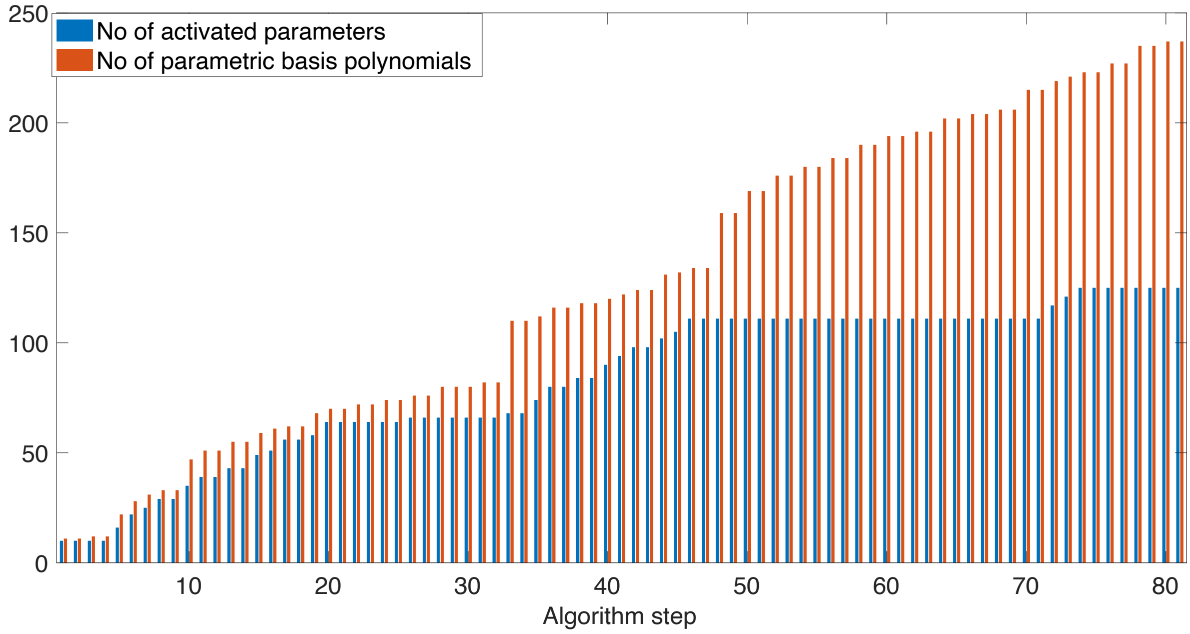


At the final step $X := \bigoplus_{\alpha \in J_P} H_1^{\alpha} \otimes P^{\alpha}$

- J_P contains 150 multi-indices, ($M = 13$ **activated parameters**)
- $H_1^{\alpha} = \mathbb{Q}_2(h)$ with $h = 2^{-7}, 2^{-6}, 2^{-5}, 2^{-4}$ **(1,3,10,136 terms)**

Example 3: (very) slow decay case

Poor convergence ($N_{\mathrm{dof}}^{-0.32}$) and the error estimator is less accurate, but can compute an approximation to TOL=1.5e-3 in a couple of mins on a laptop.



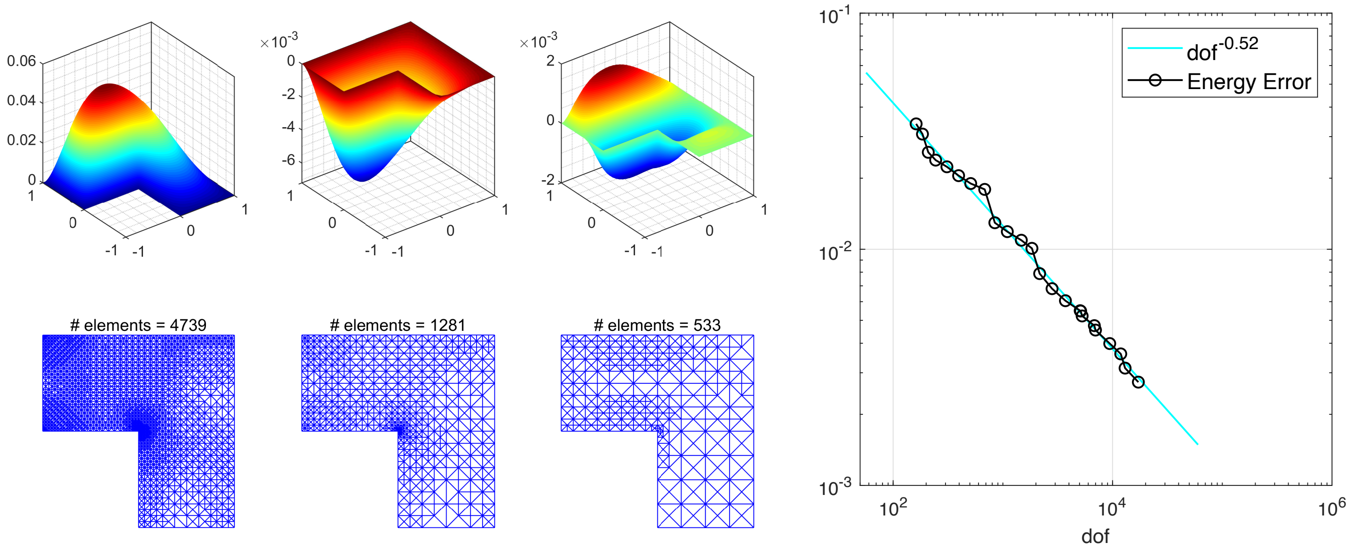
At the final step:

- J_P contains 237 multi-indices, ($M = 111$ **activated parameters**)
- $H_1^\alpha = \mathbb{Q}_1(h)$ with $h = 2^{-8}, 2^{-6}, 2^{-5}, 2^{-4}$ **(1,25,62,149 terms)**

Locally Adapted Meshes

IF **SPATIAL**

- Freeze J_P
- **improve** H_1^{α} for $\alpha \in \overline{J_P}$



Adam Crowder. Adaptive & Multilevel Stochastic Galerkin Finite Element Approximation. PhD Thesis, University of Manchester, (2020).