Some (non)vanishing theorems for nilpotent Leibniz algebras

Friedrich Wagemann
j/w Jörg Feldvoss (U South Alabama)

Laboratoire de Mathématiques Jean Leray, Université de Nantes, France

Wien, August 19th, 2022

Motivation I

Why bother about Leibniz algebras ?

(1) Leibniz algebras arise in physics in gauge theory as symmetry algebras, see for example work of Strobl, Kotov-Strobl, Strobl-Wagemann ("enhanced Leibniz algebras").

Motivation I

Why bother about Leibniz algebras ?

(1) Leibniz algebras arise in physics in gauge theory as symmetry algebras, see for example work of Strobl, Kotov-Strobl, Strobl-Wagemann ("enhanced Leibniz algebras").

(2) L_∞-algebras arise in gauge theory as symmetry algebras, and there are several constructions of an L_∞-algebra from a Leibniz algebra (Lavau, Kotov, Lavau-Palmkvist, Lavau-Stasheff). The construction of Kotov is such that the cohomology of the L_∞-algebra is the Leibniz cohomology.

Motivation II

Theorem (Jacques Dixmier, 1955)

$\mathfrak{g}$ nilpotent, finite dimensional Lie algebra, and M finite dimensional $\mathfrak{g}$-module.
If all $\mathfrak{g}$-modules contained in M are non-trivial, then $H^i(\mathfrak{g}, M) = 0$ for all $i \geq 0$.

Motivation II

Theorem (Jacques Dixmier, 1955)

$\mathfrak{g}$ nilpotent, finite dimensional Lie algebra, and M finite dimensional $\mathfrak{g}$-module.
If all $\mathfrak{g}$-modules contained in M are non-trivial, then $H^i(\mathfrak{g}, M) = 0$ for all $i \geq 0$.

Theorem (Jacques Dixmier, 1955)

$\mathfrak{g}$ nilpotent, finite dimensional Lie algebra of dimension n, and M finite dimensional $\mathfrak{g}$-module.
If M contains a trivial $\mathfrak{g}$-module, then

$$\dim(H^i(\mathfrak{g}, M)) \geq 1 \quad \text{for} \quad i = 0 \quad \text{and} \quad i = n$$

and

$$\dim(H^i(\mathfrak{g}, M)) \geq 2 \quad \text{for} \quad 1 \leq i \leq n - 1.$$

Proof of first Dixmier theorem

Donald Barnes shows the theorem using the Hochschild-Serre spectral sequence.
First step: M irreducible, induction on $\dim(\mathfrak{g})$. There exists an ideal $\mathfrak{h}$ of dimension 1 in $\mathfrak{g}$. $M^{\mathfrak{h}} \subset M$, thus either $M^{\mathfrak{h}} = M$ or $M^{\mathfrak{h}} = 0$. If $M^{\mathfrak{h}} = 0$, then $H^p(\mathfrak{h}, M) = 0$ for all $p \geq 0$.
If $M^{\mathfrak{h}} = M$, then M is a $\mathfrak{g}/\mathfrak{h}$-module and $H^p(\mathfrak{g}/\mathfrak{h}, M) = 0$ for all $p \geq 0$. Thus in any case, we have

$$H^q(\mathfrak{g}/\mathfrak{h}, H^p(\mathfrak{h}, M)) = 0$$

for all $p, q \geq 0$, and by the HS spectral sequence, $H^\bullet(\mathfrak{g}, M) = 0$.

Proof of first Dixmier theorem

Donald Barnes shows the theorem using the Hochschild-Serre spectral sequence.
First step: M irreducible, induction on $\dim(\mathfrak{g})$. There exists an ideal $\mathfrak{h}$ of dimension 1 in $\mathfrak{g}$. $M^{\mathfrak{h}} \subset M$, thus either $M^{\mathfrak{h}} = M$ or $M^{\mathfrak{h}} = 0$. If $M^{\mathfrak{h}} = 0$, then $H^p(\mathfrak{h}, M) = 0$ for all $p \geq 0$.
If $M^{\mathfrak{h}} = M$, then M is a $\mathfrak{g}/\mathfrak{h}$-module and $H^p(\mathfrak{g}/\mathfrak{h}, M) = 0$ for all $p \geq 0$. Thus in any case, we have

$$H^q(\mathfrak{g}/\mathfrak{h}, H^p(\mathfrak{h}, M)) = 0$$

for all $p, q \geq 0$, and by the HS spectral sequence, $H^\bullet(\mathfrak{g}, M) = 0$.

Second step: M arbitrary: induction on the dimension of M and the long exact sequence in cohomology.

Proof of second Dixmier theorem I

Proposition

$\mathfrak{h}$ codimension 1 ideal in $\mathfrak{g}$, and $x \in \mathfrak{g}$ with $x \notin \mathfrak{h}$. There is a long exact sequence

$$\ldots \stackrel{u_{i-1}}{\to} H^{i-1}(\mathfrak{h}, M) \stackrel{s_i}{\to} H^i(\mathfrak{g}, M) \stackrel{r_i}{\to} H^i(\mathfrak{h}, M) \stackrel{u_i}{\to} H^i(\mathfrak{h}, M) \to \ldots$$

where u_i is given by the action of x.

The l.e.s. comes from

$$0 \to D^\bullet(\mathfrak{g}, M) \to C^\bullet(\mathfrak{g}, M) \stackrel{\mathrm{res}}{\to} C^\bullet(\mathfrak{h}, M) \to 0.$$

(1) Cartan's formula implies that $\rho_i : D^i(\mathfrak{g}, M) \to C^{i-1}(\mathfrak{h}, M)$, $f \mapsto \mathrm{res}(i_x f)$ is a morphism of complexes.
(2) The connecting homomorphism is given by $c \mapsto d\tilde{c}$ and identifies, thanks to Cartan's formula, to $c \mapsto \mathrm{res}(i_x d\tilde{c}) = \mathrm{res}(L_x \tilde{c})$ up to a coboundary.

Proof of second Dixmier theorem II

One may suppose that all $x \in \mathfrak{g}$ act nilpotently on M (Fitting decomposition).
We reason by induction on $\dim(\mathfrak{g})$. The endomorphisms u_i and u_{i-1} are nilpotent.

$$\ldots \stackrel{u_{i-1}}{\to} H^{i-1}(\mathfrak{h}, M) \stackrel{s_i}{\to} H^i(\mathfrak{g}, M) \stackrel{r_i}{\to} H^i(\mathfrak{h}, M) \stackrel{u_i}{\to} H^i(\mathfrak{h}, M) \to \ldots$$

For $0 < i < n$, we have $H^i(\mathfrak{h}, M) \neq 0$ and $H^{i-1}(\mathfrak{h}, M) \neq 0$. But $\ker(u_i) \neq 0$ implies $r_i \neq 0$, and $\mathrm{im}(u_{i-1}) \neq H^{i-1}(\mathfrak{h}, M)$ implies $s_i \neq 0$.
Thus $r_i \neq 0$ with non-trivial kernel, and therefore

$$\dim(H^i(\mathfrak{g}, M)) \geq 2.$$

Leibniz bimodules

Let $\mathfrak{g}$ be a left Leibniz algebra.
A vector space M is a *Leibniz $\mathfrak{g}$-bimodule* in case there are operators $\lambda_x, \rho_x \in \mathrm{End}(M)$ for any $x \in \mathfrak{g}$ such that

(LLM) $\lambda_{[x,y]} = \lambda_x \circ \lambda_y - \lambda_y \circ \lambda_x$,

(LML) $\rho_y \circ \lambda_x = \lambda_x \circ \rho_y - \rho_{[x,y]}$,

(MLL) $\rho_y \circ \rho_x = \rho_{[x,y]} - \lambda_x \circ \rho_y$.

We write also $\lambda_x(m) =: x \cdot m$ and $\rho_x(m) =: m \cdot x$ for all $x, y \in \mathfrak{g}$ and all $m \in M$.
(LML) and (MLL) imply that $(x \cdot m + m \cdot x) \cdot y = 0$, thus $M_0 := \langle x \cdot m + m \cdot x \,|\, x \in \mathfrak{g}, m \in M \rangle$ is an antisymmetric subbimodule, and we have a short exact sequence

$$0 \to M_0 \to M \to M/M_0 =: M^{\mathrm{sym}} \to 0.$$

Fitting decomposition for one operator T

V vector space, $T \in \mathrm{End}(V)$. T acts *locally nilpotently* on V if $\forall v \in V$, $\exists n(v) \in \mathbb{N}$ such that $T^{n(v)}(v) = 0$.

Fitting decomposition for T and finite dimensional V:

$$V = V_0(T) \oplus V_1(T)$$

such that

$$V_0(T) := \bigcup_{n \geq 1} \ker(T^n), \quad \text{and} \quad V_1(T) := \bigcap_{n \geq 1} \mathrm{im}(T^n)$$

are T-stable subspaces of V. T acts nilpotently on $V_0(T)$ and invertibly on $V_1(T)$.

Fitting decomposition for a Leibniz bimodule

For a subset $S \subset \mathfrak{g}$, define

$$M_0(S) := \bigcap_{s \in S} M_0(\lambda_s), \quad \text{and} \quad M_1(S) := \sum_{s \in S} M_1(S).$$

Theorem (Feldvoss-W)

Let M be a Leibniz $\mathfrak{g}$-bimodule via operators (λ_x, ρ_x) for $x \in \mathfrak{g}$. Let $S \subset \mathfrak{g}$ be a subset such that ad_s acts locally nilpotently on $\mathfrak{g}$ for all $s \in S$. Then

(a) *$M_0(S)$ is a $\mathfrak{g}$-subbimodule of M.*

(b) *$\forall s \in S$: (λ_s, ρ_s) acts nilpotently on $M_0(S)$.*

Moreover, if $\dim(M) < \infty$, then

(c) *$M_1(S)$ is a $\mathfrak{g}$-subbimodule of M.*

(d) *$M = M_0(S) \oplus M_1(S)$.*

Cohomology vanishing for nilpotent Leibniz algebras I

Lemma

V, W left $\mathfrak{g}$-modules. Suppose there exists $x \in \mathfrak{g}$ such that

(a) *x acts locally nilpotently on V,*

(b) *x acts invertibly on W.*

Then x acts invertibly on $\mathrm{Hom}(V, W)$.

(proof due to Rolf Farnsteiner).

For $\varphi : W \to W$, define $\alpha_\varphi : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ by $f \mapsto \varphi \circ f \circ \lambda_x^V$. Observe that for any φ the sum $\sum_{n \in \mathbb{N}} (\alpha_\varphi^n(f)) \in \mathrm{Hom}(V, W)$ exists. Thus $\mathrm{id} - \alpha_\varphi$ is invertible with inverse $\sum_{n \in \mathbb{N}} (\alpha_\varphi^n(f))$.

Therefore $\Gamma := L_{\lambda_x^W} \circ (\mathrm{id} - \alpha_{(\lambda_x^W)^{-1}})$ is invertible, where L_β denotes the left multiplication with β. But

$$\Gamma(f) = \lambda_x^W \circ f - f \circ \lambda_x^V .$$

Cohomology vanishing for nilpotent Leibniz algebras II

Theorem

Suppose there exists $a \in \mathfrak{g}$ such that

(a) ad_a *acts locally nilpotently (on* $\mathfrak{g}$*),*

(b) λ_a *acts invertibly (on* M*).*

Then $HL^n(\mathfrak{g}, M) = 0$ *for all* $n \geq 0$.

(proof adapted from Rolf Farnsteiner).

We are in the conditions of the previous lemma with

$$CL^n(\mathfrak{g}, M) := \mathrm{Hom}(\otimes^n \mathfrak{g}, M),$$

thus a acts invertibly on $CL^\bullet(\mathfrak{g}, M)$ and on $HL^\bullet(\mathfrak{g}, M)$.
But the Cartan relation $\theta_a = d \circ i_a + i_a \circ d$ implies that a acts trivially on its cohomology. One concludes that $HL^n(\mathfrak{g}, M) = 0$ for all $n \geq 0$. □

Cohomology vanishing for nilpotent Leibniz algebras III

Theorem

$S \subset \mathfrak{g}$ subset such that

(a) ad_s *acts locally nilpotently for all* $s \in S$,

(b) $\dim(M/M_0(S)) < \infty$.

Then $HL^n(\mathfrak{g}, M) \cong HL^n(\mathfrak{g}, M_0(S))$ *for all* $n \geq 0$.

(proof adapted from Rolf Farnsteiner).

Induction on $d = \dim(M/M_0(S))$. $d = 0$ OK. For $d > 0$, there exists $s_0 \in S$ such that $N := M_0(\lambda_{s_0}) \neq M$. N is a subbimodule of M with $N_0(S) = M_0(S)$ (easy to see).

By induction hypothesis applied to N, we have for all $n \geq 0$ that $HL^n(\mathfrak{g}, N) = HL^n(\mathfrak{g}, N_0(S)) = HL^n(\mathfrak{g}, M_0(S))$. Then use the l.e.s. associated to $0 \to N \to M \to M/N \to 0$ in order to conclude, because λ_{s_0} acts invertibly on M/N. □

Cohomology vanishing for nilpotent Leibniz algebras IV

Corollary (Feldvoss-W)

Let $\mathfrak{g}$ be a Leibniz algebra with a nilpotent ideal $\mathfrak{n} \subset \mathfrak{g}$, and M be a finite dimensional Leibniz $\mathfrak{g}$-bimodule such that $M^{\mathfrak{n}} = 0$. Then

$$HL^{\bullet}(\mathfrak{g}, M) = 0.$$

Proof.

Suppose $M_0(\mathfrak{n}) \neq 0$. The set $\mathfrak{n}$ acts nilpotently on $M_0(\mathfrak{n})$. By Engel's theorem, we have $M_0(\mathfrak{n})^{\mathfrak{n}} \neq 0$. But then $0 \neq M_0(\mathfrak{n})^{\mathfrak{n}} \subset M^{\mathfrak{n}} = 0$ is a contradiction. □

With some further work, we arrive at: $\mathfrak{g}$ and M finite dimensional. If every composition factor in M is non-trivial, then $HL^n(\mathfrak{g}, M)$ is zero except possibly for $n = 0$, and we have $HL^0(\mathfrak{g}, M) = M_0$.

Non-vanishing theorem I

Let us first consider the trivial 1-dimensional Lie/Leibniz algebra $\mathfrak{g} = k$.

Theorem (Feldvoss-W)

For $\mathfrak{g} = k$ and M any Leibniz $\mathfrak{g}$-bimodule, we have:

(a)

$$HL^n(\mathfrak{g}, M) = \begin{cases} M^{\mathfrak{g}} & \text{for} \quad n = 0 \\ M^0/(M \cdot \mathfrak{g}) & \text{for} \quad n \quad \text{odd} \\ M^{\mathfrak{g}}/M_0 & \text{for} \quad n \quad \text{even}, n \neq 0 \end{cases}$$

where $M^0 := \{m \in M \,|\, x \cdot m + m \cdot x = 0 \quad \forall x \in \mathfrak{g}\}$.

(b) *If M is finite dimensional, then*

$$M^0/(M \cdot \mathfrak{g}) \cong M^{\mathfrak{g}}/M_0.$$

Non-vanishing theorem II

In fact, the previous theorem can be explained (observation by Geoffrey Powell) by the fact that the associative algebra $UL(\mathfrak{g})$ is isomorphic to $k[x,y]/(xy)$ and the bimodule $U\mathfrak{g}_{\rm Lie} \cong k[x]$ has a periodic resolution. Thus any Leibniz cohomology of $\mathfrak{g} = k$ is periodic (in degree $n > 0$), because (as Loday-Pirashvili show)

$$Ext^{\bullet}_{UL(\mathfrak{g})}(U\mathfrak{g}_{\rm Lie}, M) \cong HL^{\bullet}(\mathfrak{g}, M).$$

We showed that the 2-dimensional nilpotent Leibniz non-Lie algebra $\mathfrak{N}$ also has this property.

Which Leibniz algebras have this property ?

Non-vanishing theorem III

We can adapt the reasoning with the l.e.s. of Dixmier to the Leibniz case. The difficulty is here that the kernel complex $D^\bullet(\mathfrak{g}, M) = \mathrm{ker}(\mathrm{res})$ is now much bigger, as the factor kx (which must be present in the kernel of the restriction map) can now show up at any place in the tensor product and even show up several times:

$$\bigoplus_{i=1}^{n} \mathrm{Hom}(\mathfrak{g}^{\otimes(i-1)} \otimes kx \otimes \mathfrak{g}^{\otimes(n-i)}, M) \oplus \ldots \oplus \mathrm{Hom}((kx)^{\otimes n}, M)$$

Theorem (Feldvoss-W)

Let M be a finite dimensional Leibniz $\mathfrak{g}$-bimodule such that $M^{\mathfrak{g}}/M_0 \neq 0$. Then $\dim(HL^n(\mathfrak{g}, M)) \geq 1$ for all $n \geq 1$.

Some remarks:
(1) The trivial module always satisfies the hypothesis $M^{\mathfrak{g}}/M_0 \neq 0$.

(2) The adjoint module does not always satisfy the hypothesis. The hypothesis boils down to the condition that the left center strictly contains the ideal of squares. This is false, for example, for the two dimensional nilpotent Leibniz not-Lie algebra $\mathfrak{N}$. Its adjoint cohomology is zero (in degree > 0).
This answers a question by Bakhrom Omirov who asked whether a finite dimensional nilpotent Leibniz algebra always possesses an outer derivation.