# Shifted substitution in noncommutative power series

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# Introduction: Power series

Given a sequence of (real) numbers $m_0, m_1, \ldots$ we encode it as an (exponential) power series:

$$M(t) := \sum_{n=0}^{\infty} m_n \frac{t^n}{n!}.$$

Consider the ring $R = \mathbb{K}[[t]]$ with the usual product

$$PQ(t) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} p_j q_{n-j} \right) \frac{t^n}{n!}.$$

The set $G^1 := \{M \in R : m_0 = 1\}$ is a group for this product.

# Introduction: Power series

Each $M \in G^1$ has a logarithm:

$$K(t) := \log M(t) = \sum_{n=1}^{\infty} k_n \frac{t^n}{n!} \in G^0 := \{K \in R : k_0 = 0\}.$$

where

$$k_n = \sum_{j=1}^{n} (-1)^{j-1} (j-1)! B_{n,j}(m_1, \ldots, m_{n-j}).$$

# Introduction: Hopf algebras (Umbral calculus?)

We can also encode the sequence $m_0, m_1, \ldots$ as a linear map $\mu \colon \mathbb{K}[x] \to \mathbb{K}$ via

$$\mu(x^n) := m_n,$$

and we get a linear map $\Lambda\colon \operatorname{Hom}_{\mathrm{Vect}}(\mathbb{K}[x], \mathbb{K}) \to R,\ \phi \mapsto \sum_{n=0}^{\infty} \phi(x^n) \frac{t^n}{n!}$.

Endow $H = \mathbb{K}[x]$ with a Hopf algebra structure where the product is the usual polynomial product and the coproduct is

$$\Delta x^n := \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}.$$

# Introduction: Hopf algebras

**Theorem 1**

Endow $H^* = \mathrm{Hom}_{\mathrm{Vect}}(\mathbb{K}[x], \mathbb{K})$ with the convolution product

$$\phi * \psi := (\phi \otimes \psi) \circ \Delta.$$

The map $\Lambda : H^* \to R$ is an algebra isomorphism.

In particular, we can recover the previous formulas from known Hopf-algebraic statements.

# Introduction: Hopf algebras

The set $\mathcal{G}(H) := \{\phi \in H^* : \phi(1) = 1\}$ is group under $*$, unit $\varepsilon(p) = p(0)$ and inverses given by

$$\phi^{-1} = \sum_{j\geq 0} (\varepsilon - \phi)^{*j},$$

that is

$$\phi^{-1}(x^n) = \sum_{j=0}^{n} (-1)^j \sum_{n_1+\ldots+n_j=n} \frac{n!}{n_1!\cdots n_j!} \phi(x^{n_1})\cdots\phi(x^{n_j}).$$

There is a logarithm $\log_* : \mathcal{G}(H) \to \mathcal{L}(H) =: \{\phi \in H^* : \phi(1) = 0\}$ given by

$$\log_*(\phi) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} (\phi - \varepsilon)^{*j}.$$

# Introduction: Character trick

$\mathrm{Hom}_{\mathrm{Alg}}(H, \mathbb{K}) =: \mathrm{Char}(H)$ is a group under $*$ with unit $\varepsilon$, inverse is given by $\phi^{-1} = \phi \circ S$ where $S : H \to H$ is the antipode, satisfying

$$S * \mathrm{id} = \mathrm{id} * S = 1\varepsilon.$$

Solving yields $S(x^n) = (-1)^n x^n$.

The Lie algebra of $\mathrm{Char}(H)$ is formed by the *infinitesimal characters*, such that

$$\gamma(pq) = \varepsilon(p)\gamma(q) + \gamma(p)\varepsilon(q).$$

It can be shown that $\log_* : \mathrm{Char}(H) \to \mathrm{InfChar}(H)$.

# Introduction: Character trick

Trick: extend $H$ to $\hat{H} := \mathrm{Sym}(H)$ whose product we denote by $x^n \mid x^m \neq x^{n+m}$, with antipode

$$\hat{S}(x^n) = \sum_{k=1}^{n} (-1)^k \sum_{n_1+\cdots+n_k=n} \frac{n!}{n_1!\cdots n_k!} x^{n_1} \mid \cdots \mid x^{n_k}.$$

**Theorem 2 (EFPTZ 2018)**

$H$ is a (left) comodule over $\hat{H}$, and the restriction $\mathrm{Char}(\hat{H}) \to \mathcal{G}(H)$ is a group isomorphism.

Application to Wick polynomials:

$$G(t, X) := \sum_{n=0}^{\infty} :X^n: \frac{t^n}{n!} = \frac{e^{tX}}{\mathbb{E}[e^{tX}]} \longleftrightarrow \mathrm{W} = \left(\mathrm{id} \otimes \phi^{-1}\right) \circ \Delta.$$

# Noncommutative power series

Let $R = A\langle\langle x_1, x_2, \ldots \rangle\rangle$ for some commutative $\mathbb{K}$-algebra $A$. A generic element:

$$f(\boldsymbol{x}) = f_0 + \sum_{k=1}^{\infty} \sum_{(i_1,\ldots,i_k)\in\mathbb{N}^k} f_{i_1\ldots i_k} x_{i_1} \cdots x_{i_k}$$
$$= \sum_{w\in\mathbb{N}^\star} f_w x_w,$$

where $x_{i_1\ldots i_k} = x_{i_1} \cdots x_{i_k}$.

**Definition 1**

We set

$$G^1 := \{f \in R : f_0 = 1\}, \quad G^0 := \{f \in R : f_0 = 0\}.$$

# Shifted composition

**Definition 2**

For $f, g \in G^1$, we set

$$(f \bullet g)(\boldsymbol{x}) = g(\boldsymbol{x}) f(\boldsymbol{x} g(\boldsymbol{x}))$$

where

$$(\boldsymbol{x} g(\boldsymbol{x}))_i = x_i g(\boldsymbol{x}) = x_i + \sum_{w \in \mathbb{N}_+^\star} g_w x_{iw}$$

and

$$(\boldsymbol{x} g(\boldsymbol{x}))_{i_1 \dots i_k} = x_{i_1} g(\boldsymbol{x}) x_{i_2} g(\boldsymbol{x}) \cdots x_{i_k} g(\boldsymbol{x}).$$

In the univariate case this reduces to composition of “tangent-to-identity” formal diffeomorphisms ($f_0 = 0, f_1 = 1$).

# Hopf algebras again

Let $V$ be the linear span of $\mathbb{N}_+^\star$ and $H = T(V)$ whose product we denote again by $|$. $H$ is linearly spanned by "sentences" $w_1 \mid w_2 \mid \cdots \mid w_k$, where $w_j \in \mathbb{N}^\star$.

**Definition 3 (Ebrahimi-Fard, Patras 2014)**

Define $\Delta\colon V \to V \otimes H$

$$\Delta(i_1 \cdots i_n) = \sum_{S \subseteq [n]} i_S \otimes i_{J_1^S} \mid \cdots \mid i_{J_k^S}$$

where $[n] \setminus S = J_1^S \cup \cdots \cup J_k^S$ and each is an interval. Extend to $\Delta\colon H \to H \otimes H$ multiplicatively.

For example

$$\begin{aligned}\Delta(i_1 i_2 i_3) = \mathbf{1} \otimes i_1 i_2 i_3 + i_1 \otimes i_2 i_3 + i_2 \otimes i_1 \mid i_3 + i_3 \otimes i_1 i_2 \\ + i_1 i_2 \otimes i_3 + i_1 i_3 \otimes i_2 + i_2 i_3 \otimes i_1 + i_1 i_2 i_3 \otimes \mathbf{1}\end{aligned}$$

# Hopf algebras and group laws

**Theorem 3 (EFPTZ 2023)**

$\Lambda :\ \mathrm{Char}(H) \to G^1$ defined by

$$\Lambda(\phi)(\boldsymbol{x}) = \phi(\mathbf{1}) + \sum_{w \in \mathbb{N}_+^\star} \phi(w) x_w$$

is a group isomorphism, that is, if $f = \Lambda(\phi), g = \Lambda(\gamma)$ then

$$\Lambda(\phi * \gamma)(\boldsymbol{x}) = (f \bullet g)(\boldsymbol{x}) = g(\boldsymbol{x}) f(\boldsymbol{x} g(\boldsymbol{x})).$$

# Half (co)products

**Definition 4 (Ebrahimi-Fard, Patras 2014)**

Define

$$\Delta_{\prec}(i_1 \cdots i_n) = i_1 \cdots i_n \otimes \mathbf{1} + \sum_{1 \in S \subseteq [n]} i_S \otimes i_{J_1^S} \mid \cdots \mid i_{J_k^S}$$

and

$$\Delta_{\succ}(i_1 \cdots i_n) = \mathbf{1} \otimes i_1 \cdots i_n + \sum_{1 \notin S \subseteq [n], S \neq \emptyset} i_S \otimes i_{J_1^S} \mid \cdots \mid i_{J_k^S}.$$

Then, $\Delta = \Delta_{\prec} + \Delta_{\succ}$ and they endow $H$ with the structure of an unshuffle bialgebra.

In turn, this splits the convolution product into two "half-shuffles":

$$\phi \prec \gamma := (\phi \otimes \gamma) \circ \Delta_{\prec}, \quad \phi \succ \gamma := (\phi \otimes \gamma) \circ \Delta_{\succ}.$$

**Theorem 4 (EFPTZ 2023)**

Let $\phi \in \mathrm{Hom}_{\mathrm{Vect}}(H_+, A)$, $\gamma \in \mathrm{Char}(H)$ so that $f \in G^0$, $g \in G^1$. Then

$$\Lambda(\phi \prec \gamma) = f(\boldsymbol{x}g(\boldsymbol{x})), \quad \Lambda(\phi \succ \gamma) = (g(\boldsymbol{x}) - 1)f(\boldsymbol{x}g(\boldsymbol{x})).$$

# Lie structure

The group $G^1$ is left-linear, thus its tangent space at 1 $G^0$ has a pre-Lie structure.

**Theorem 5**

The operation

$$x_{i_1}\cdots x_{i_n} \triangleleft x_{j_1}\cdots x_{j_m} := \sum_{k=0}^{n} x_{i_1}\cdots x_{i_k} x_{j_1}\cdots x_{j_m} x_{i_{k+1}}\cdots x_{i_n}$$

is (right) pre-Lie, i.e.,

$$\mathrm{a}_{\triangleleft}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) := (\boldsymbol{x}\triangleleft\boldsymbol{y})\triangleleft\boldsymbol{z} - \boldsymbol{x}\triangleleft(\boldsymbol{y}\triangleleft\boldsymbol{z})$$

is symmetric in $\boldsymbol{y}$ and $\boldsymbol{z}$.

In the single variable case $x^n \triangleleft x^m = (n+1)x^{n+m}$ (Faà di Bruno).

**Theorem 6 (EFPTZ 2023)**

Let $\phi \in \mathrm{Hom}_{\mathrm{Vect}(H,A)}$, $\gamma \in \mathcal{L}(A)$ so that $f \in R$, $g \in G^0$. Then

$$\Lambda(\phi * \gamma) = f \triangleleft g.$$

# Link with noncommutative probability

Let $(B, \varphi)$ a ncps and $(b_1, \ldots)$ be nc rvs on $B$.
We define $\phi : V \to \mathbb{C}$ by $\phi(i_1 \cdots i_n) = \varphi(b_{i_1} \cdot_B \cdots \cdot_B b_{i_n})$. It extends uniquely to $\Phi \in \mathrm{Char}(H)$.

**Theorem 7 (Ebrahimi-Fard, Patras 2016 & 2018)**

The equations

$$\Phi = \varepsilon + \Phi \succ \beta, \quad \Phi = \varepsilon + \kappa \prec \Phi$$

define $\beta, \kappa \in \mathrm{InfChar}(H)$, and

$$\Phi(i_1 \cdots i_n) = \sum_{\pi \in \mathrm{NC}(n)} \prod_{B \in \pi} \kappa(i_B) = \sum_{\pi \in \mathrm{Int}(n)} \prod_{B \in \pi} \beta(i_B).$$

# Link with noncommutative probability

**Corollary 1 (EFPTZ 2023)**

Denoting $M, K, B$ the generating series of $\Phi, \kappa, \beta$ respectively, we have

$$M(\boldsymbol{x}) = 1 + K(\boldsymbol{x}M(\boldsymbol{x})) = 1 + B(\boldsymbol{x})M(\boldsymbol{x}).$$

**Theorem 8 (Ebrahimi-Fard, Patras 2018)**

$\rho = \log_* \Phi$ defines an infinitesimal character corresponding to monotone cumulants.

Introduce a formal parameter $t$ and set $\Phi_t = \exp_*(t\rho)$.

# Trees

**Theorem 9 (EFPTZ 2023, Hasebe-Saigo 2011)**

Let $R$ be the generating function of $\rho$. Then

$$\dot{M}_t(\boldsymbol{x}) = R(\boldsymbol{x}) + ((M_t - 1) \lhd R)(\boldsymbol{x}),$$

that is

$$\begin{aligned} M_t &= 1 + tR + (R \lhd R)\frac{t^2}{2} + ((R \lhd R) \lhd R)\frac{t^3}{6} + \cdots \\ &= \exp_{\lhd}(R) \\ &= 1 + \sum_{\tau} \frac{1}{\tau! \sigma(\tau)} P_h(\tau) t^{|\tau|}, \end{aligned}$$

where e.g. $P_h(\mathbin{\vert}) = R \lhd R$, $P_h(\vee) = (R \lhd R) \lhd R - R \lhd (R \lhd R)$, etc.