Hall conductance as a topological invariant of 2d gapped systems.

N. Sopenko + A.K., 2006.14151

(also S. Bachmann et al., 1810.07351)

1. Lattice systems with $U(1)$ symmetry.

$$\underset{\text{Hamiltonian}}{H} = \sum_{p\in\Lambda} H_p \quad , \quad \underset{\text{Electric charge}}{Q} = \sum_{p\in\Lambda} Q_p .$$

$H_p \in \mathcal{A}_{B_p(r)}$ $\quad Q_p \in \mathcal{A}_p, \ \forall p\in\Lambda$ (on-site symmetry)

$$e^{2\pi i Q_p} = 1 \quad \forall p\in\Lambda$$

$U(1)$-invariance: $[Q, H_p] = 0 \quad \forall p\in\Lambda$

More generally, $\delta_Q A = i[Q, A]$, $A\in\mathcal{A}_{loc}$.

If A is localized on $X\subset\Lambda$, then so is $\delta_Q A$.

Let $\langle \ldots \rangle : \mathcal{A} \to \mathbb{C}$ be the ground state of H.

If H is gapped, then $\langle \ldots \rangle$ is automatically $U(1)$-invariant.

<u>Th</u>. (Goldstone theorem)

$$\langle \delta_Q A \rangle = 0 \quad \forall A \in \mathcal{A}_{al} \subset \mathcal{A}.$$

<u>Remark</u>

δ_Q is an unbounded derivation of $\mathcal{A}$ and is not defined on the whole $\mathcal{A}$.

$$\delta_Q (A \cdot B) = (\delta_Q A) \cdot B + A \cdot (\delta_Q B)$$

<u>Def</u>. $A \in \mathcal{A}_{al}$ iff it can be approximated by a local observable $A_r \in \mathcal{A}_{B_r(p_0)}$ s.t. that $\| A - A_r \| = O(r^{-\infty})$.

$\mathcal{A}_{al}$ is a dense $*$-sub-algebra of $\mathcal{A}$, and $\delta_Q : \mathcal{A}_{al} \to \mathcal{A}_{al}$.

Key idea: although $\pi_{GNS}(Q_p)|0\rangle \neq 0$, one can find "improved" observables $\tilde{Q}_p$ s.t.

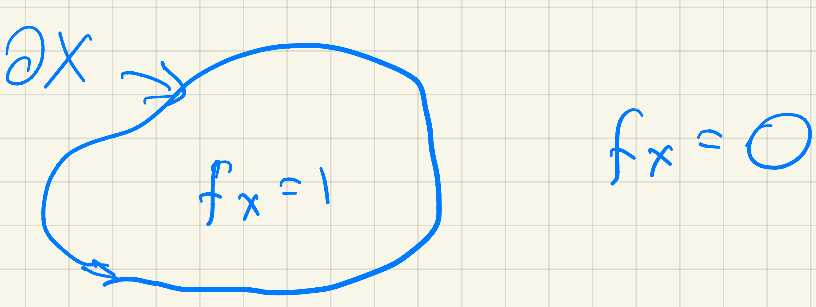
- $\tilde{Q}_p \in \mathcal{A}_{al}$
- $\pi_{GNS}(\tilde{Q}_p)|0\rangle = 0$
- $\tilde{Q}_p = Q_p + (\partial K)_p = Q_p + \sum_q K_{qp}$

(I view Q_p, $\tilde{Q}_p$ as $\mathcal{A}_{al}$-valued 0-chains, and K is an $\mathcal{A}_{al}$-valued 1-chain.)

Then let $X \subset \Lambda$ be a finite subset.

$$Q_X = \langle Q, \underbrace{f_X}_{\text{"characteristic 0-cochain of } X\text{"}} \rangle, \quad \tilde{Q}_X = \langle \tilde{Q}, f_X \rangle$$

$$\tilde{Q}_X = Q_X + \langle K, \delta f_X \rangle = Q_X + \underbrace{K_{\partial X}}_{\text{supported near } \partial X.}$$



$$\langle [Q,A] \rangle = \lim_{\longrightarrow X} \langle [Q_X, A] \rangle =$$

$$= \lim_{\longrightarrow X} \langle [K_{\partial X}, A] \rangle = 0.$$

$\square$

So how does construct the 1-chain K?

2. Currents on a lattice.

$$\frac{dQ_p}{dt} = i[H, Q_p] = \sum_q J_{pq} = -(\partial J)_p$$

J_{pq} = current from $q \in \Lambda$ to $p \in \Lambda$.

An obvious solution:

$$J_{pq} = i[H_q, Q_p] - i[H_p, Q_q].$$

Unique up to $J \mapsto J + \partial N$,

$N \in C_2(\Lambda; \mathcal{A}_{al})$

(To prove this, need to show that $H_1(\Lambda; \mathcal{A}_{al}) = 0$. Local coefficients!)

3. Quasi-adiabatic map. (Hastings, 1001.5280)

Let $0 < \Delta < E_{gap}$.

Pick a continuous real function $W_\Delta(t)$ s.t.

- $W_\Delta(t) = O(|t|^{-\infty})$ for $|t| \to \infty$
- $\hat{W}_\Delta(\omega) = \int e^{i\omega t} W_\Delta(t)\, dt$ satisfies

$$\hat{W}_\Delta(\omega) = -\frac{i}{\omega} \text{ for } |\omega| > \Delta.$$

$\ell_\Delta : \mathcal{A} \to \mathcal{A}$ is defined by

$$\ell_\Delta(A) = \int_{-\infty}^{+\infty} W_\Delta(t)\, \tau_t(A)\, dt,$$

$$\tau_t(A) = e^{iHt} A e^{-iHt}.$$

<u>Th</u> ℓ_Δ maps $\mathcal{A}_{al}$ to $\mathcal{A}_{al}$.

Then <u>can define</u> $\underline{\tilde{Q} = Q + \partial K}$

by letting $K_{pq} = \ell_\Delta(J_{pq})$.

This proves Goldstone's theorem for any lattice system with short-range interactions

4. Defining Hall conductance.

$$[Q_p, Q_q] = 0 \qquad \forall p, q.$$

$$[\tilde{Q}_p, \tilde{Q}_q] \neq 0.$$

However:

$$2\pi i\, [\tilde{Q}_p, \tilde{Q}_q] = -(\partial M)_{pq} \quad ,$$

where

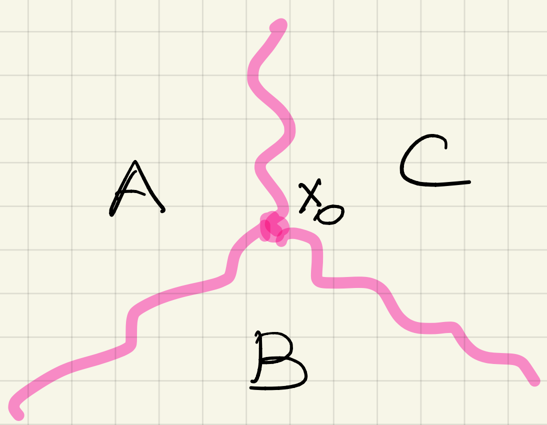
$$M_{pqr} = i\pi \left([Q_p + \tilde{Q}_p, K_{qr}] + \text{cyclic} \right)$$

contraction of a chain and a cochain

<u>Def</u>. $$\sigma = \sum_{\substack{p \in A \\ q \in B \\ r \in C}} \langle M_{pqr} \rangle = (\langle M \rangle, \alpha)$$

vacuum expectation value

normalized generator of $H^2(\Lambda, \mathbb{R})$



$x_0 \notin \Lambda$.

<u>Claim</u> : $\sigma = 2\pi\, \sigma_{Hall}$

Properties of σ.

- σ is independent of the choice of the point x_0 and the paths.

Indeed, $\partial\langle M\rangle \sim \partial\langle[\tilde{Q}_p, \tilde{Q}_q]\rangle = 0$

$$\Rightarrow (\langle M\rangle, \alpha + \delta\beta) - (\langle M\rangle, \alpha) =$$

$$= (\partial\langle M\rangle, \beta) = 0.$$

$$\forall \beta \in C^1(\Lambda, \mathbb{R}).$$

- σ depends only on the state $\langle\ldots\rangle$ and the charge Q_p, not on the concrete Hamiltonian.

Relies on the following:

If we define the operator-valued 1-chain K by $\langle Q + \partial K ; A\rangle = 0 \quad \forall A \in \mathcal{A}$, then the solution is unique up to

$$K \mapsto K + \partial N + K_0,$$

where $\langle K_0 ; A\rangle = 0 \quad \forall A \in \mathcal{A}$

(i.e. K_0 annihilates $|0\rangle$)

N.B. This a special case of a more general statement. It shows that ground states of gapped local Hamiltonians are quite special among all clustering states.

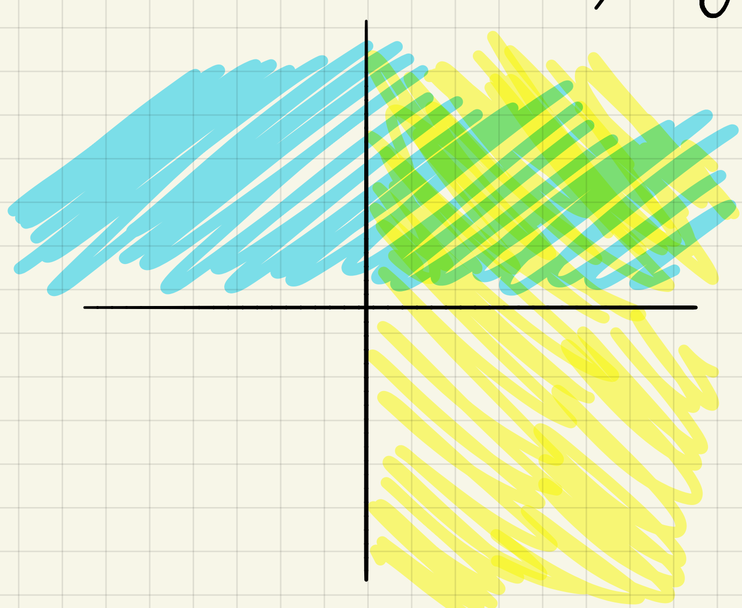
- $\sigma = 2\pi \sigma_{Hall}$.

This follows from a version of Kubo formula derived in 1905.06488 :
(Spodyneiko + AK)

$$\sigma_{Hall} = i \langle 0 | (\hat{J}, \delta f)\, G_0^2\, (\hat{J}, \delta g) | 0 \rangle - (f \leftrightarrow g)$$

Here $G_0 = (1-P) \frac{1}{H} (1-P)$, $P = |0\rangle\langle 0|$

$f(p) = \theta(x(p))$, $g(p) = \theta(y(p))$



With a little work, one can express it in terms of the 1-chain K:

$$\sigma_{Hall} = i \langle [(K, \delta f), (K, \delta g)] \rangle$$

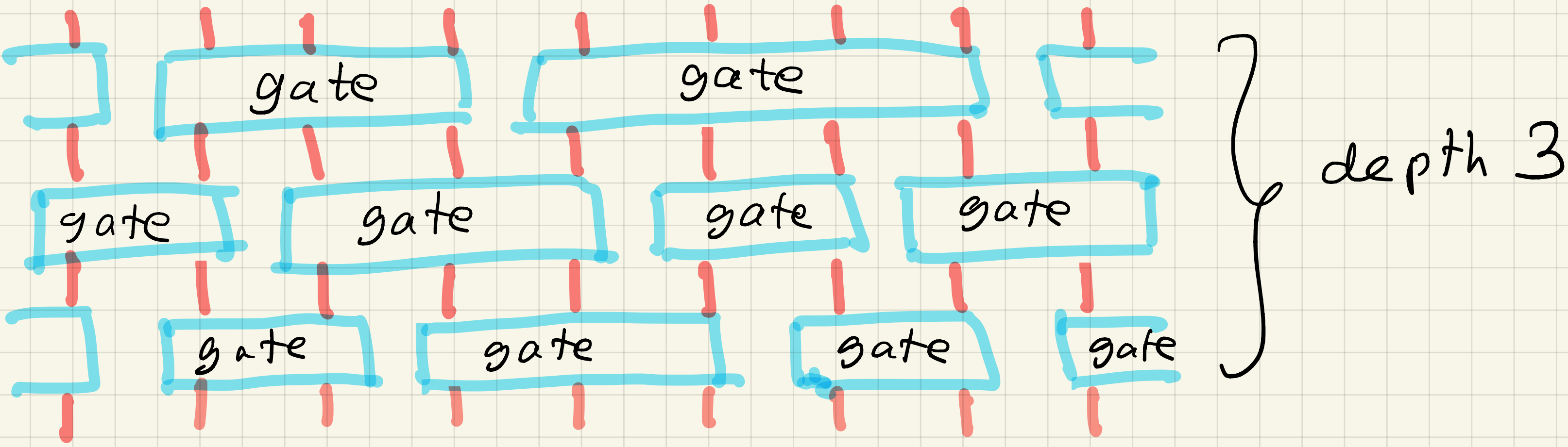
With more work, can show $\sigma = 2\pi \sigma_{Hall}$.

- σ is a numerical invariant of a phase
- $\sigma \in \mathbb{Z}$ for SRE phases
- $\sigma \in 2\mathbb{Z}$ for bosonic SRE phases
- σ controls the "statistics" of flux insertions

5. What is a quantum phase of matter?

Based on ideas from "Quantum information meets quantum matter", (B. Zeng, X. Chen, D. Zhou, X-G Wen)

Local unitary quantum circuit:



• Two pure states are declared equivalent if they are connected by a finite-depth LUQ circuit.

• Stacking a state with a "trivial" (factorized) state is also an equivalence.

N. B. In a finite system all states are (approximately) equivalent.

a more "physical" definition:
replace finite-depth LUQ circuits with a path in the space of local Hamiltonians, $H(\lambda)$, s.t. $H(\lambda)$ is gapped for all λ and $\langle \ldots \rangle_0$ and $\langle \ldots \rangle_1$ are ground states of $H(0)$ & $H(1)$.

<u>Th</u>. (Moon, Ogata, 1906.05479)
If $\langle A \rangle_\lambda$ is a smooth function of λ $\forall A \in \mathcal{A}_{al}$, then this is equivalent to:

$\exists$ a path of automorphisms $\beta_\lambda : \mathcal{A} \to \mathcal{A}$ s.t. $\langle \beta_\lambda(\ldots) \rangle_0 = \langle \ldots \rangle_\lambda$ and

$$\frac{d}{d\lambda} \langle A \rangle_\lambda = \langle i [G(\lambda), A] \rangle_\lambda .$$

for some 0-chain $G(\lambda) = \sum_p G_p(\lambda)$.

$\forall p \quad G_p(\lambda) \in \underline{\mathcal{A}_{al}}$.

In other words:

$$\beta_\lambda = P\exp\left(i \int ad_{G(\lambda)}\right).$$

Let's call such automorphisms $\beta = \beta_1$ "locally generated".

A locally generated β is an evolution automorphism generated by some local Hamiltonian.

<u>a trivial observation</u>:

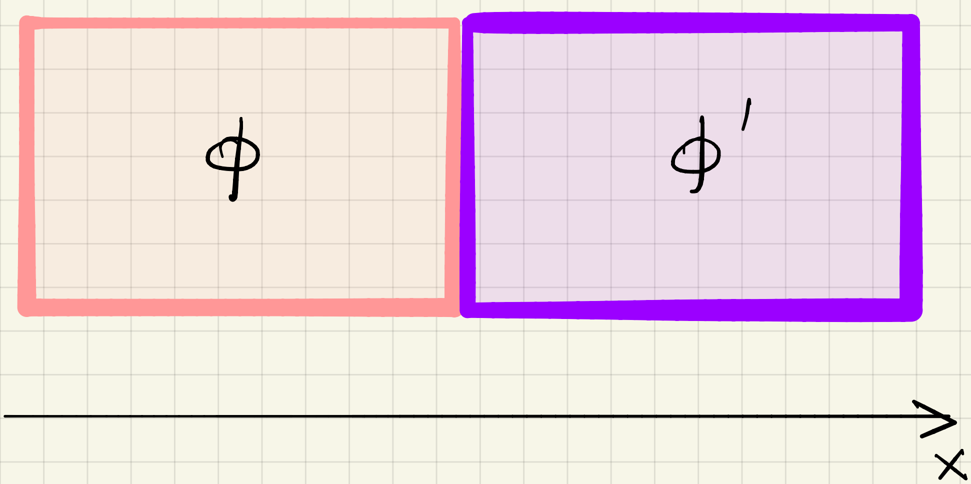
If $\langle \ldots \rangle$ is a ground state of a gapped Hamiltonian, then $\langle \beta(\ldots) \rangle$, where β is locally generated, is also a ground state of a gapped Hamiltonian.

<u>Def</u>. $\langle \ldots \rangle_\phi \sim \langle \ldots \rangle_{\phi'}$ if $\exists$ locally generated β s.t. $\langle \ldots \rangle_{\phi'} = \langle \beta(\ldots) \rangle_\phi$

Locally generated automorphisms map $\mathcal{A}_{al} \to \mathcal{A}_{al}$ and are "fuzzy" analogs of finite-depth LUQ circuits.

An important property:

if $\langle \ldots \rangle_\phi$ & $\langle \ldots \rangle_{\phi'}$ are in the same phase, one can "glue" them across a line to produce another gapped state $\langle \ldots \rangle_{\phi\phi'}$ in the same phase.



Suppose $\langle \ldots \rangle_{\phi'} = \langle \beta(\ldots) \rangle_\phi$ where β is generated by $G(\lambda) = \sum_p G_p(\lambda)$.

Then let $\langle \ldots \rangle_{\phi'\phi} = \langle \beta_+(\ldots) \rangle$

where β_+ is generated by

$$G_+(\lambda) = \sum_{p\in\Lambda} \underset{\substack{\uparrow \\ \text{step-function}}}{\theta(x(p))}\, G_p(\lambda)$$

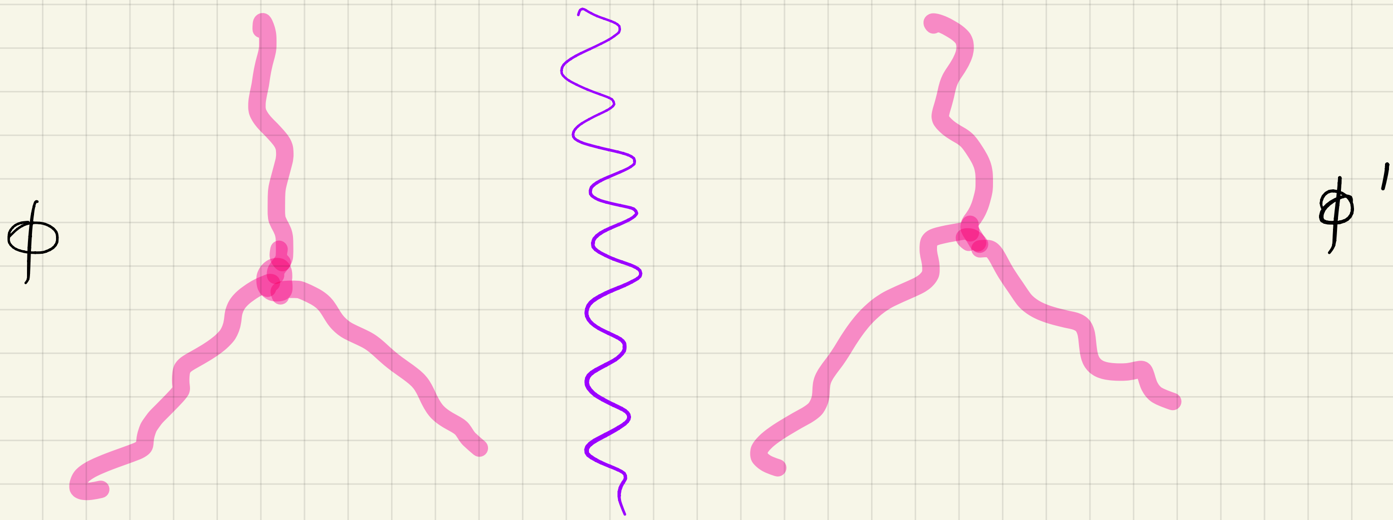
Def. A quantum phase of matter is an equivalence class under equivalence generated by locally generated automorphisms and stacking with factorized systems.

Def. Trivial phase is the equivalence class of a factorized state.

Def. A state $\phi : \mathcal{A} \to \mathbb{C}$ is said to be in an SRE phase if $\exists\, \psi : \mathcal{A}' \to \mathbb{C}$ s.t. $\phi \otimes \psi$ is in a trivial phase.

To show that σ is a numerical invariant of a phase, it is sufficient to show it is invariant under locally-generated automorphisms.

This follows easily from Local computability and the gluing property



$$\sigma = \sigma'$$

In particular, if $\sigma \neq 0$, the system cannot have a gapped edge (i.e. a gapped interface with a factorized state).

Concluding remarks

- Another numerical invariant of 2d gapped systems is thermal Hall conductance (= chiral central charge)

  Not clear if the methods described can be used to define it.

- Our methods can be used to re-write WZW classes in terms of states alone. (N. Sopenko + AK, work in progress)