UNIVERSITY OF OXFORD

# Calabi-Yau metrics with full moduli dependence

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Based on

- [2603.12384, MLST] with Andrei Constantin and Andre Lukas
- [2606.28487] with Andrei Constantin, Seung-Joo Lee and Andre Lukas

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CY to get $\mathcal{N} = 1$ SUSY in 4D

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Holomorphic and poly-stable vector bundle

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❋ $\tau$ depends on matter field Kahler metric

$$K_{IJ} \sim \int_X \nu_I \wedge \star\, (H_J \bar{\nu}_J) \longrightarrow$$ It requires the Ricci-flat metric

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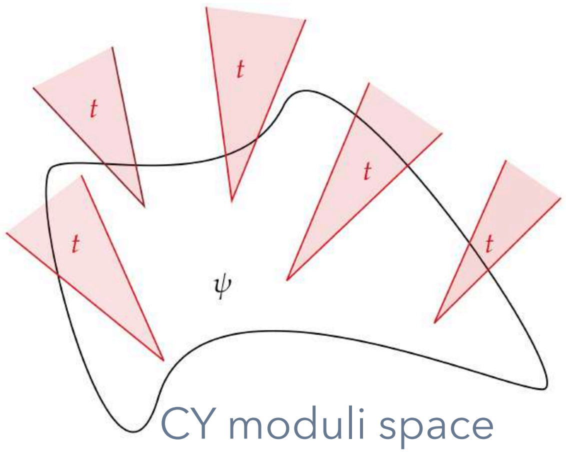

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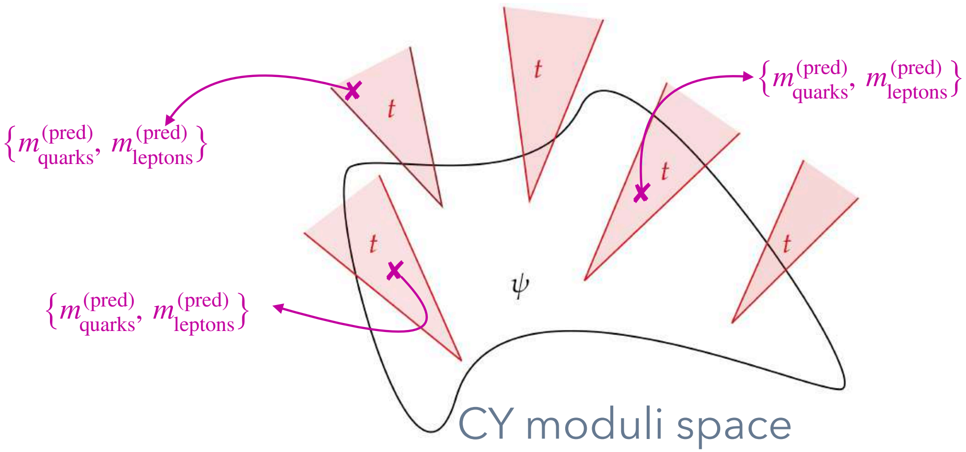

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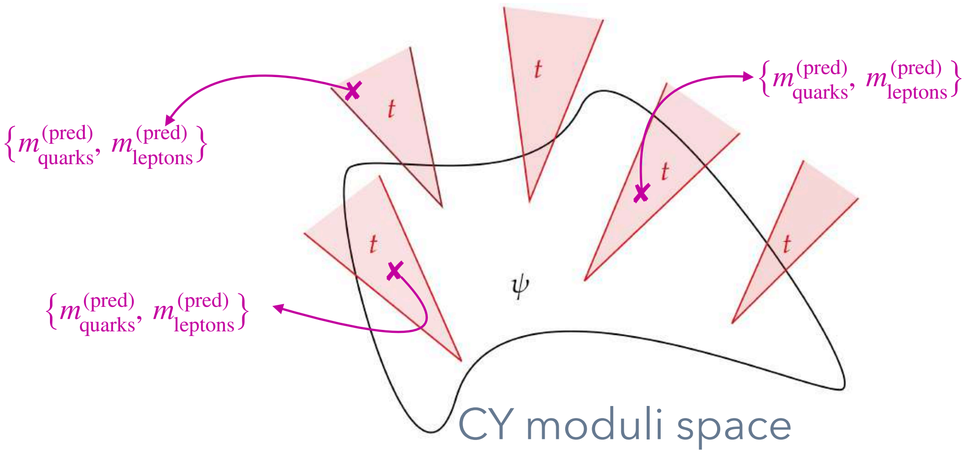


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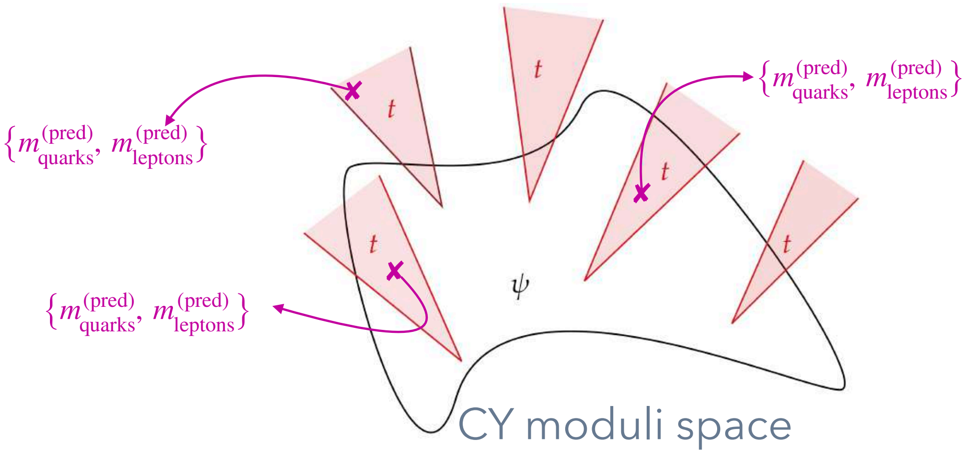
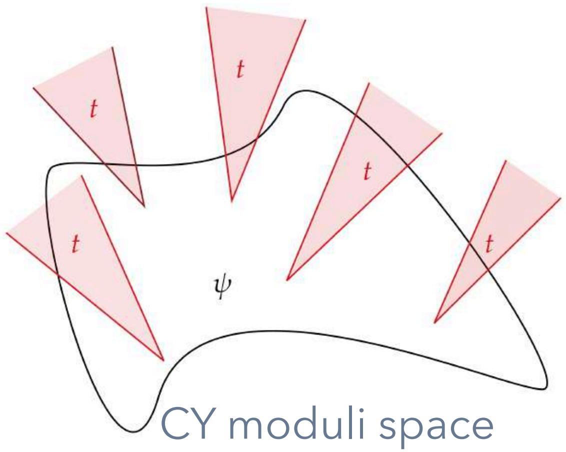



- We need the **Ricci-flat metric as a function of the moduli** => $g_{i\bar{j}}(\psi, t)$
- $t = \{t_1, \ldots, t_{h^{1,1}}\}$ <=> Kahler class

*∃ Unique Ricci-flat metric by Yau's theorem*

# Motivations

- **Complex structure moduli dependence** can easily be incorporated in analytic ansatzes for $g_{i\bar{j}}$.
- In cases where $h^{(1,1)} = 1$ **Kähler moduli dependence is easy**




- The single **Kähler modulus $t$ parametrises the overall volume**, therefore

$$g_{i\bar{j}}(x, \psi, t) \sim t\, \hat{g}(x, \psi)$$

- **What about $h^{(1,1)} > 1$?**

# Outline

- Numerical approximate Ricci-flat metric with Neural Networks
- Analytic approaches when $h^{(1,1)} = 1$
- Hybrid numerical-analytic approach for cases with $h^{(1,1)} > 1$
- Conclusions and outlook

# Which tools do we have?

- In recent years, **neural networks** have been extensively used to get **approximate numerical Ricci-flat metrics**

[Ashmore, He, Ovrut, 2020]
[Jejjala, Mayorga Peña, Mishra, 2020]
[Anderson, Gerdes, Gray, Krippendorf, Raghuram, Ruehle, 2021]
[Larfors, Lukas, Ruehle, Schneider, 2022]
[Berglund, Butbaia, Hübsch, Jejjala, Mayorga Peña, Mishra, Tan, 2022]
[Gerdes, Krippendorf, 2023]
[Challenger, Mirjanić, 2025]

# How to find numerical Ricci-flat metrics with a NN?

- Ricci-flatness condition is encoded as a fourth-order, highly non-linear system of PDEs

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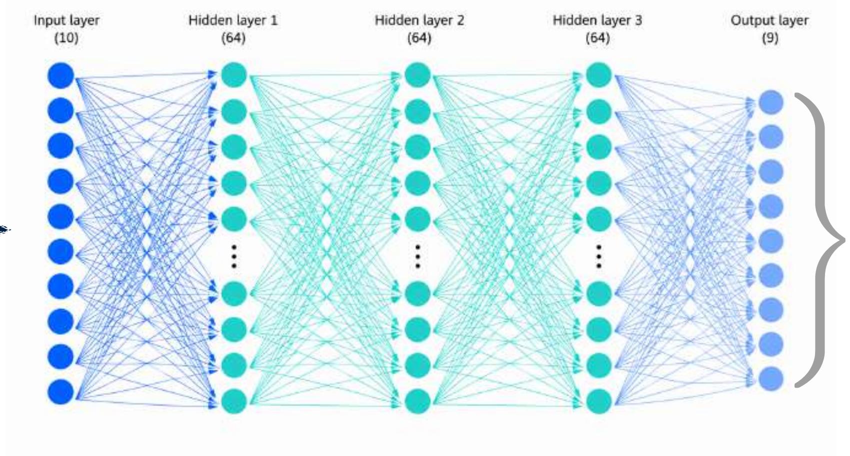
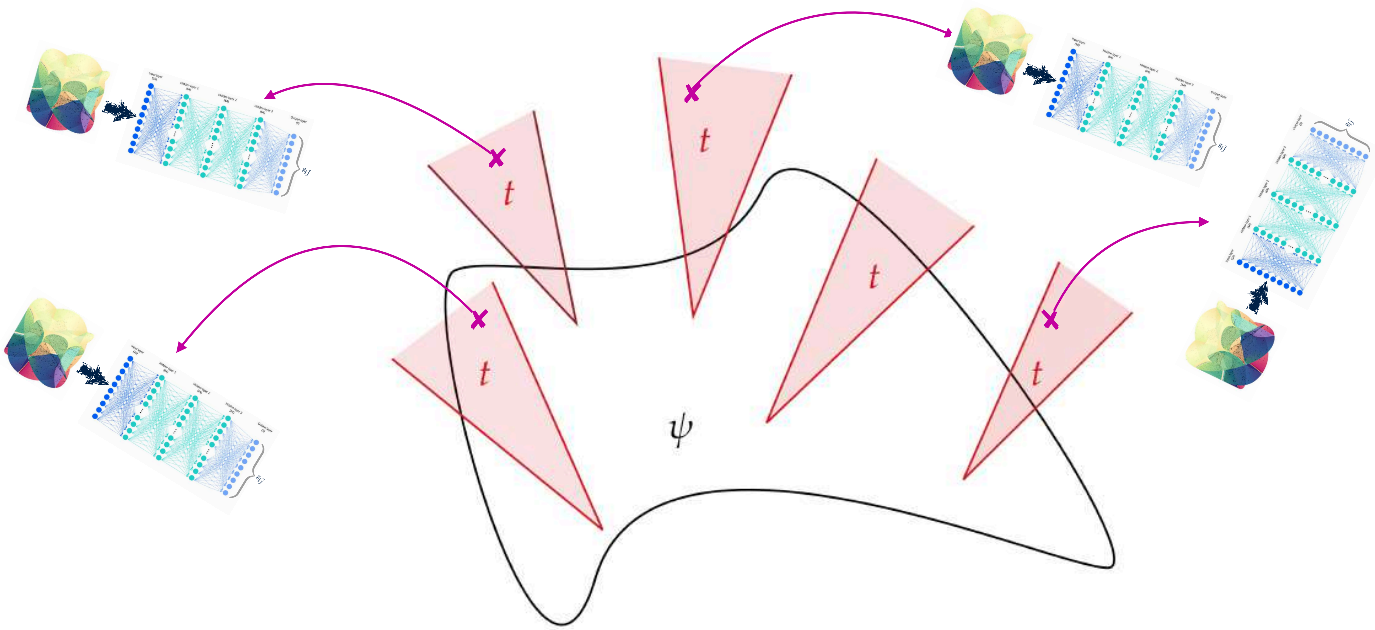



$\Phi$

$$\mathrm{Loss} \;=\; \frac{1}{N_{\mathrm{pts}}} \sum_{\mathrm{points}} \left| 1 - \frac{\det g}{\kappa\, \Omega \wedge \overline{\Omega}} \right| + \ldots$$

# How to find numerical Ricci-flat metrics with a NN?

- This approach can be applied at every point in moduli space



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- **Machine-learning** approaches are extremely flexible but **obscure the interpretability** of the results.
- **Donaldson** devised an iterative algorithm that gives analytic approximate Ricci-flat metrics with increasingly higher accuracy.
- It works by iteratively refining a Kähler potential ansatz $K = \frac{1}{\pi k} \log\Big(H_{IJ}(\psi)\, s^I\, \bar{s}^{\bar{J}}\Big)$ where $g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^{\bar{j}}}$

  - $\{s_I\} \in H^0(X, O_X(k))$ – a **basis of global holomorphic sections of a line bundle** $L = \mathcal{O}_X(k) \to X$, for $k$ integer.
  - **Coefficients** $H_{IJ}$, **to be determined** iteratively for each $\psi$.
  - For each $\psi$, **the** $k \to \infty$ **limit gives the Ricci-flat metric,** but slow numerical convergence.

[Donaldson, 2005]
[Douglas, Karp, Lukic, Reinbacher, 2006]
[Headrick, Nasar, 2010], …

# An example with $h^{(1,1)} = 1$

Consider the **Dwork one-parameter family of quintic** in $\mathbb{P}^4$

$$X \;=\; \left\{[x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^4 \;\middle|\; P_\psi(x) = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi\, x_0 x_1 x_2 x_3 x_4 = 0\right\}.$$

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- Using **Donaldson's Ansatz one can get approximate analytic Ricci-flat Kahler potential**

$$K \;=\; \frac{t}{2\pi} \log\left( I_0 \;+\; f_1(|\psi|)\, I_1 \;+\; f_2(|\psi|)\, I_2 \right)$$

[Lee, Lukas, 2025]

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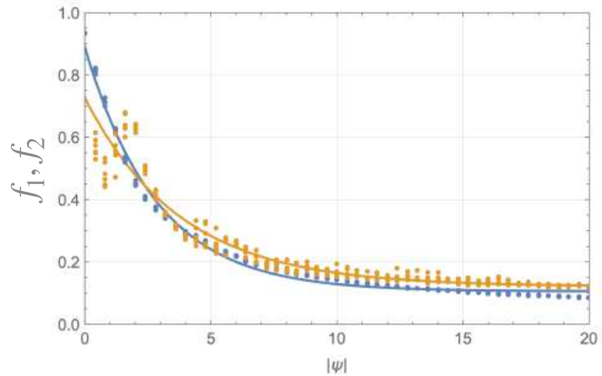
$$I_0 = \sum_{A<B<C} |x_A|^2 |x_B|^2 |x_C|^2$$

$$I_1 = \sum_{A \neq B} |x_A|^4 |x_B|^2$$

$$I_2 = \sum_{A} |x_A|^6$$

$$f_1(|\psi|) \simeq 0.121 + 0.605\, e^{-0.264\,|\psi|},$$

$$f_2(|\psi|) \simeq 0.106 + 0.783\, e^{-0.352\,|\psi|}$$

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- If we want to move continuously in the Kähler cone, **we need another Ansatz.**

# An Ansatz for arbitrary $h^{(1,1)}$

$\mathscr{A} = \prod_{r=1}^{m} \mathbb{P}^{n_r}$ – Ambient space

$\{t_1, \ldots, t_m\}$ – Kähler moduli

$\{x_a^{(r)}\}_{a=0,\ldots,n_r}$ – coordinates on the r^th projective space

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cf. [Berglund, Butbaia, Hübsch, Jejjala, Mayorga Peña, Mishra, Tan, 2022]

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**Fubini-Study metric**:

$$K_{\mathrm{FS}} = \sum_{r=1}^{m} \frac{t_r}{\pi} \log\left( \sum_{a=0}^{n_r} |x_a^{(r)}|^2 \right),$$

- Metric of the ambient space restricted to the CY
- Serves as a reference metric

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Parametrises the required **deformation** needed to approximate the Ricci-flat CY metric

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- $\alpha_{IJ}(\psi, t_1/t_m, \ldots, t_{m-1}/t_m)$ **incorporate the full moduli dependence**

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  *($\phi$ is, by construction, a globally-defined function on the CY)*

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- The **Kähler class is looked and correctly parametrised by the FS one:** $J = \sum_{r=1}^{m} t_r J_r^{(\mathrm{FS})}$ *($\phi$ is, by construction, a globally-defined function on the CY)*
- **For $k_1,\ldots,k_m \to \infty$ this is an expansion in a complete basis of functions on the projective space.**

cf. [Berglund, Butbaia, Hübsch, Jejjala, Mayorga Peña, Mishra, Tan, 2022]

## Example with $h^{(1,1)} = 2$

- Consider the **bi-cubic hypersurface** in $\mathbb{P}^2 \times \mathbb{P}^2$ (coordinates $x = (x_0, x_1, x_2)$ and $y = (y_0, y_1, y_2)$, respectively)

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(\mathbb{Z}_3^{(x)})_a: &\quad x_a \mapsto \omega x_a, &\quad x_b \mapsto x_b \text{ for } b \neq a, &\quad y_c \mapsto y_c \text{ for } c = 0,1,2,\\
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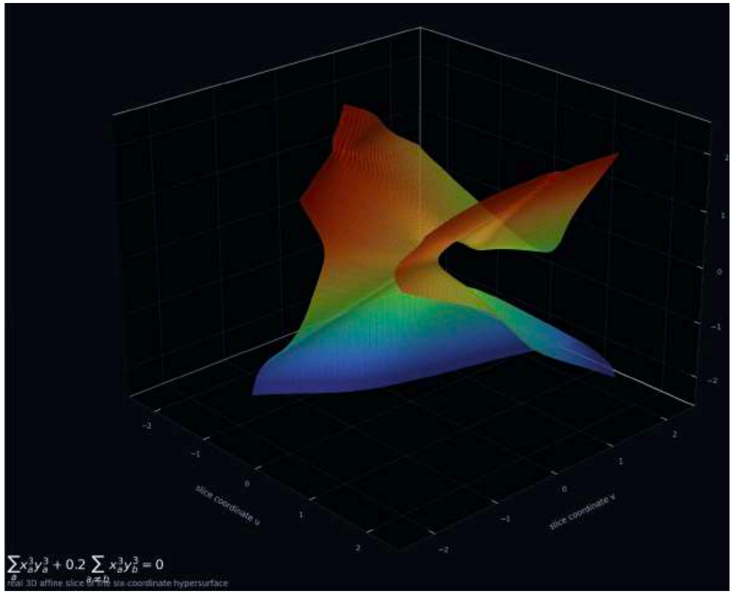
- The group is $G \cong \left((\mathbb{Z}_3^{(x)})^2 \times (\mathbb{Z}_3^{(y)})^2\right) \rtimes (S_3 \times \mathbb{Z}_2)$ and has order 972.

## Example with $h^{(1,1)} = 2$

The most general $G$-invariant polynomial in $\mathbb{P}^2 \times \mathbb{P}^2$ is

$$\sum_a x_a^3 y_a^3 + \psi \sum_{a \neq b} x_a^3 y_b^3 \;=\; 0 \qquad \rightarrow \qquad \text{one-parameter family of } \textbf{bi-cubics}$$



## The Ansatz specialises to

$$K(x, y, \psi, t_1, t_2) \;=\; t_1\, K_{\mathrm{FS}}(x) \;+\; t_2\, K_{\mathrm{FS}}(y) \;+\; \phi(x, y, \psi, t_1, t_2)$$

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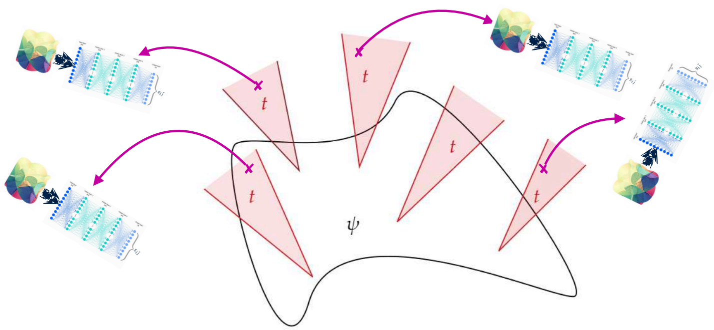
- Meaning that $\alpha_{IJ}$ **is a 36x36 Hermitian matrix => 1296 real parameters**

# What do we do with this Ansatz?

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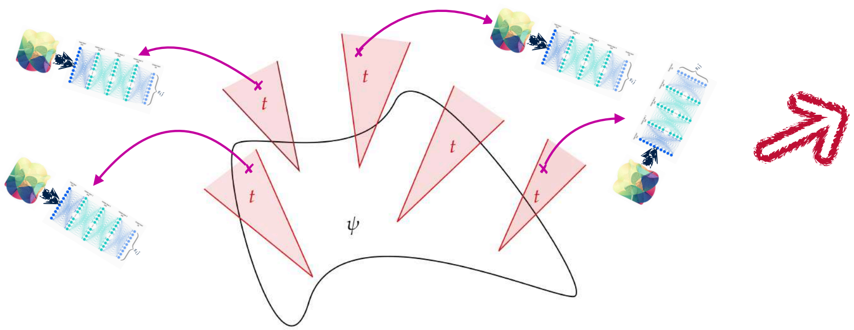


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***To determine $\alpha_{IJ}$ as functions of the moduli***

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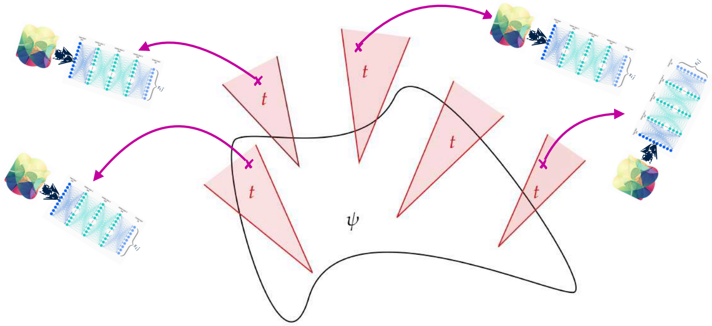

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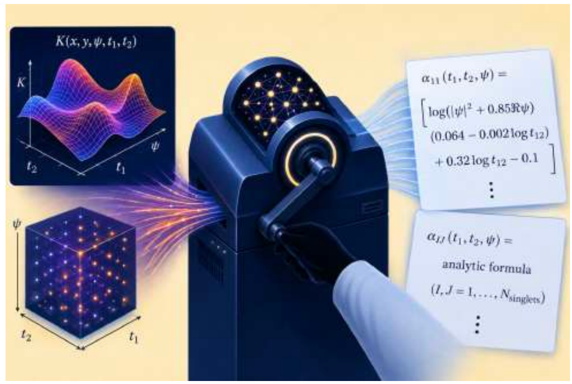


*[GPT generated with inspiration from Quanta Magazine]*

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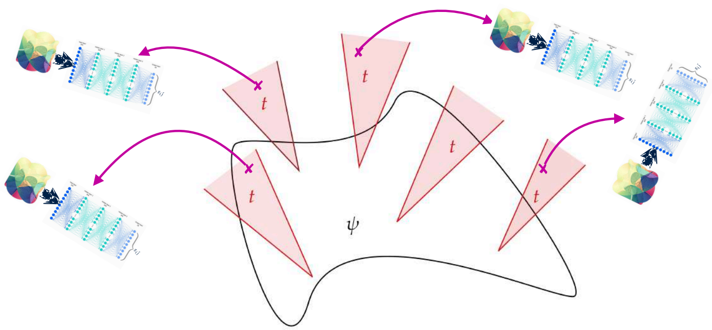


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**No general expectation for $\alpha_{IJ}(\psi, t_1/t_2)$ to be a smooth function of the Kähler moduli**

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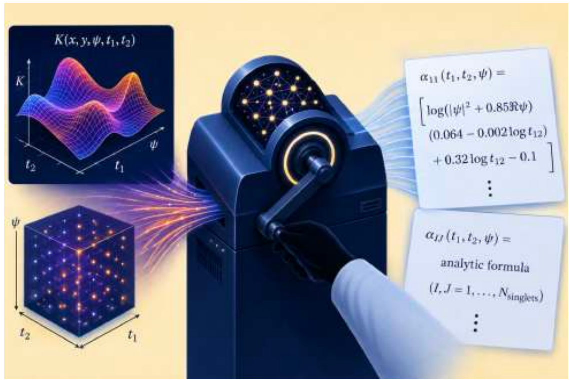


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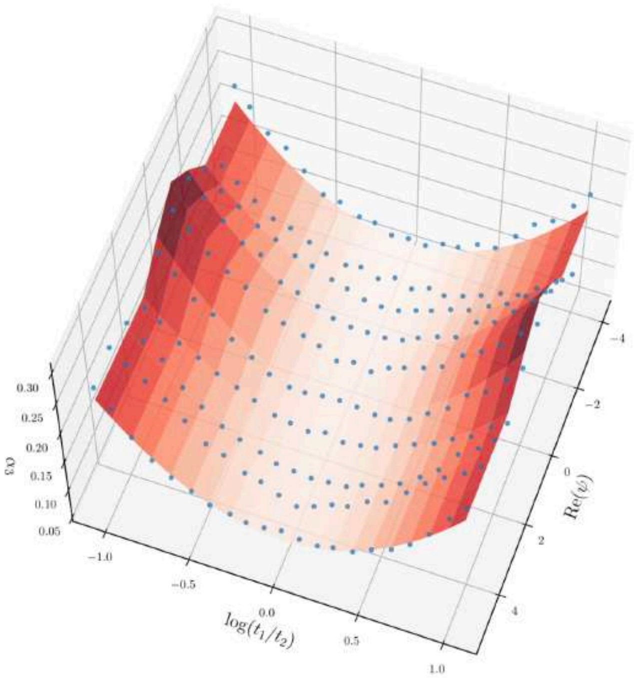
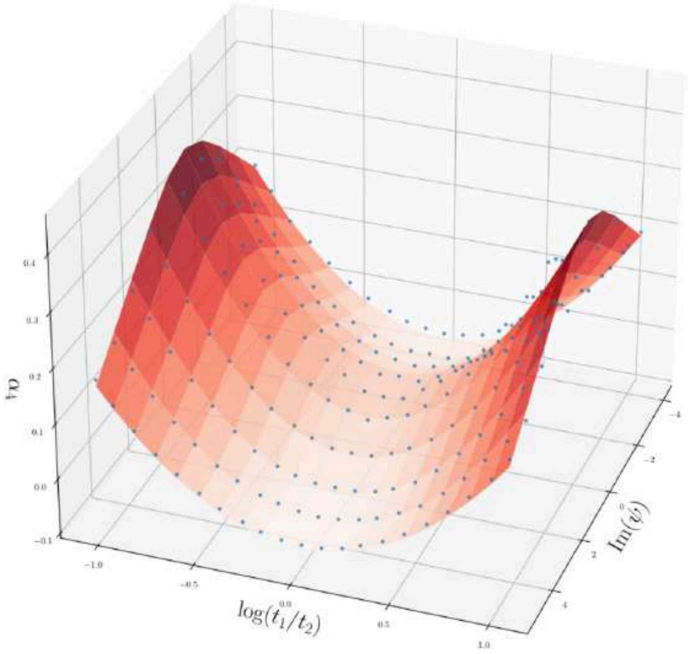
# Results

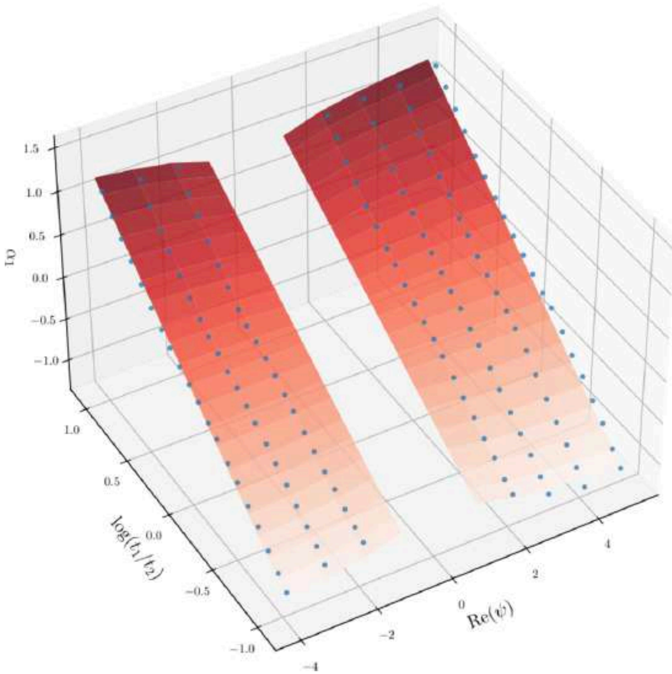
$$K(x,\psi,t_1,t_2) = K_{\rm FS}|_X + \frac{V(t_1,t_2)^{1/3}}{\left(\sum_A |x_A|^2\right)^2\left(\sum_A |y_A|^2\right)^2}\Bigg\{$$
$$\Big[-0.3 - 0.07\log(0.38|\psi|^2)\log^2(t_{12})\Big]\sum_{a<b}|x_a|^2|x_b|^2|y_a|^2|y_b|^2$$
$$+ (0.38 + 0.037\log|\psi|^2)(0.18 + \log t_{12})\sum_{\substack{a\neq b,c\\ b<c}}|x_a|^4|y_b|^2|y_c|^2 + (t_{12}\to t_{21},\ x\leftrightarrow y)$$
$$+\Big[\log(|\psi|^2 + 0.85\Re\psi)(0.064 - 0.002\log t_{12}) + 0.32\log t_{12} - 0.1\Big]\sum_{a\neq b}|x_a|^4|y_a|^2|y_b|^2$$
$$+ (t_{12}\to t_{21},\ x\leftrightarrow y)$$
$$+\Big[0.08 + \big(0.024 + 0.014e^{-0.564(\Re\psi)^4}\big)\log^2(t_{12})\Big]\sum_{a\neq b}|x_a|^4|y_b|^4$$
$$+\Big[(0.07 - 0.001(|\psi|^2 - 0.28\Im\psi))(1.5 - \log|\psi|^2 + \log^2(t_{12}))\Big]\sum_a |x_a|^4|y_a|^4$$
$$-\Big[0.22 + 0.02\log((\Im\psi)^2 + 3.71(\Re\psi)^2)\big(e^{-0.46\Re\psi + 0.13(\Re\psi)^2} + \log^2(t_{12})\big)\Big]\sum_{a,b,c\,\text{distinct}}|x_a|^2|y_a|^2|x_b|^2|y_c|^2\Bigg\}.$$

- **At no point we enforced** that our Ansatz must reflect the $G$**-symmetry**
- However, **the fit gives a symmetric** $\alpha_{IJ}$ (# of independent coefficients drops from 1296 to 8) => the coefficients group together according to the $G$**-symmetry**
- **This serves as a non-trivial consistency check of the fit procedure and truncation order**

# $\alpha_{IJ}$ coefficients as functions of $(\psi, t_1, t_2)$

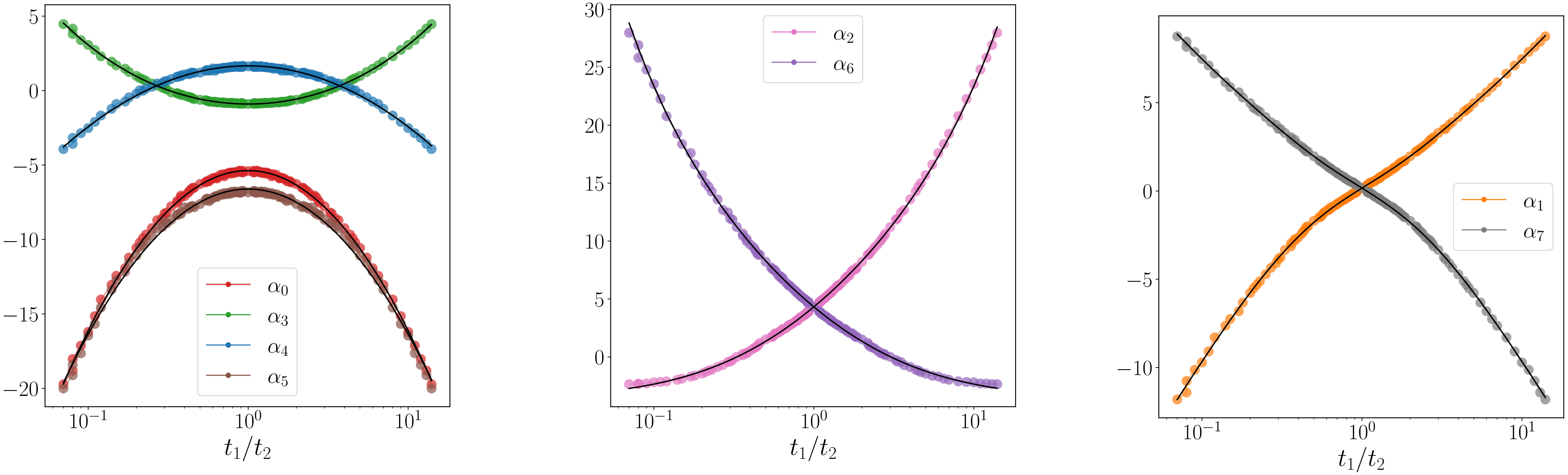



- The **symmetry** is also **respected in the moduli-dependent sector**.

# $\alpha_{IJ}$ coefficients at fixed $\psi$

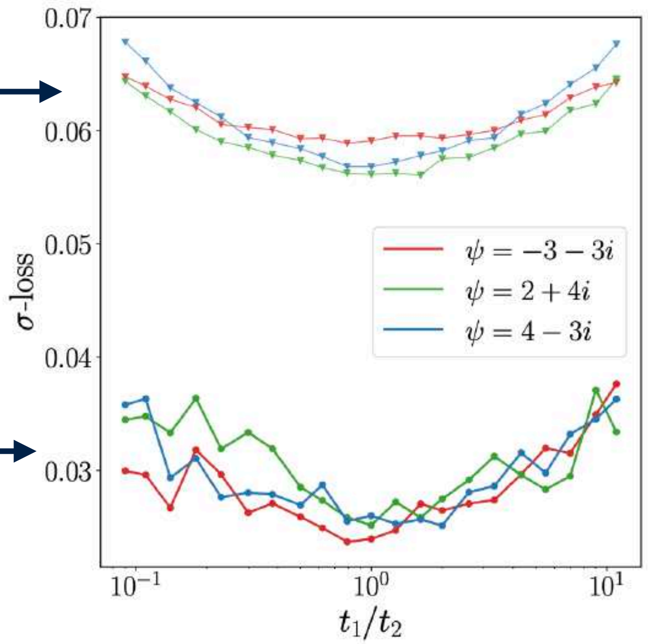


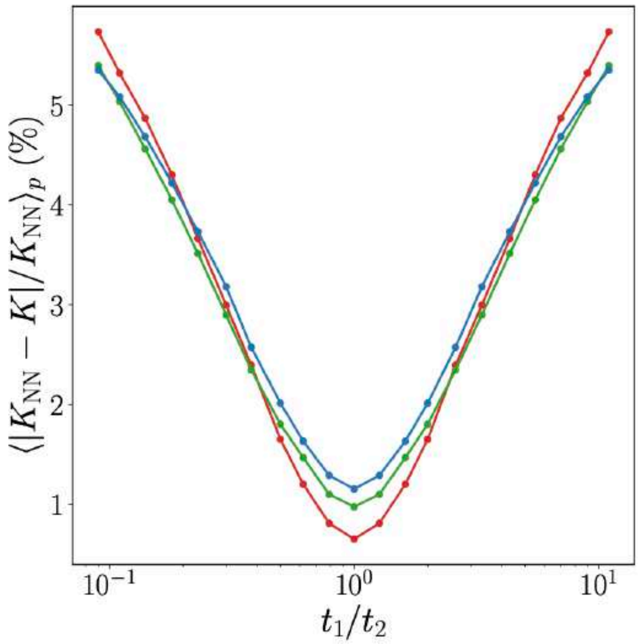

$G$-symmetry $\rightarrow$ $\alpha_1\left(\frac{t_1}{t_2}\right) = \alpha_7\left(\frac{t_2}{t_1}\right)$ , $\alpha_2\left(\frac{t_1}{t_2}\right) = \alpha_6\left(\frac{t_2}{t_1}\right)$ , $\alpha_i\left(\frac{t_1}{t_2}\right) = \alpha_i\left(\frac{t_2}{t_1}\right)$ for $i = 0,3,4,5$ .

# Accuracy

Analytic results

NN numerical results




$$\sigma_{\mathrm{loss}} = \frac{1}{N} \left\| 1 - \frac{\det g_{\mathrm{pr}}}{\kappa\, \Omega \wedge \overline{\Omega}} \right\|_1 ,$$

**Small values of $k_1$ and $k_2$** already yield analytic expressions that **agree with the underlying numerically data at percent-level accuracy**

Can we entirely skip the NN step?

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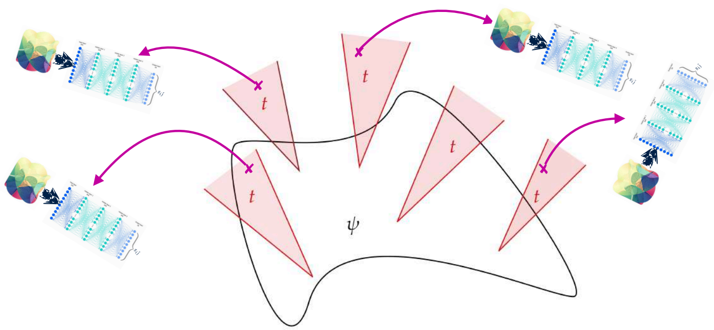


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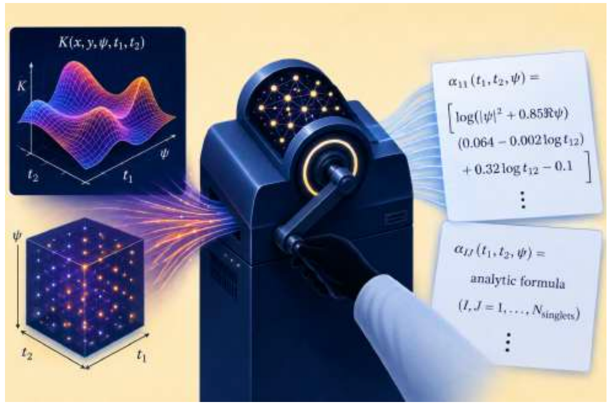
***To determine $\alpha_{IJ}$ as functions of the moduli***

5. ***Use Symbolic Regression to get analytical expressions for $\alpha_{IJ}$***




*[GPT generated with inspiration from Quanta Magazine]*

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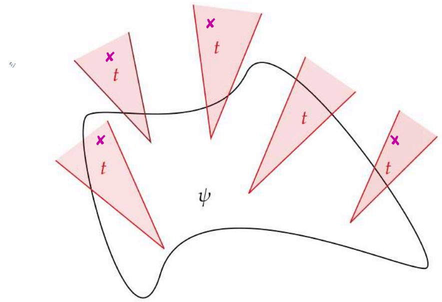


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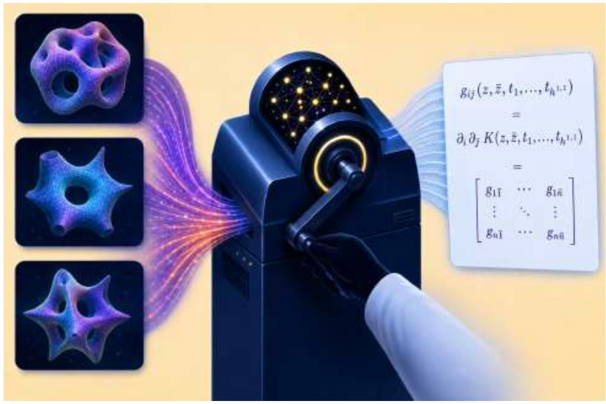
5. ***Use Symbolic Regression to directly get the approximate analytic metric***

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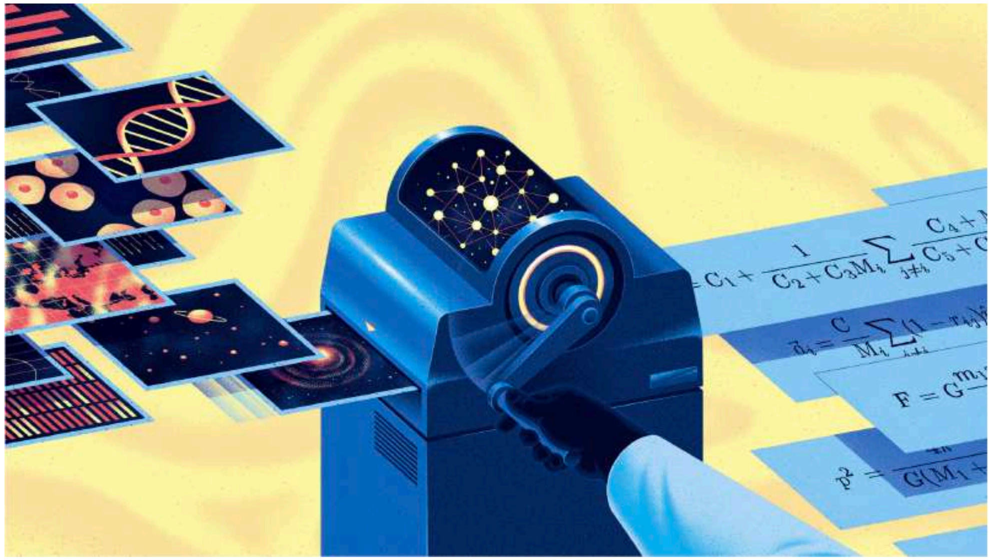
# How does Symbolic Regression work?

- **Symbolic regression** is a Machine Learning technique which aims to **retrieve symbolic expressions from numerical data**.

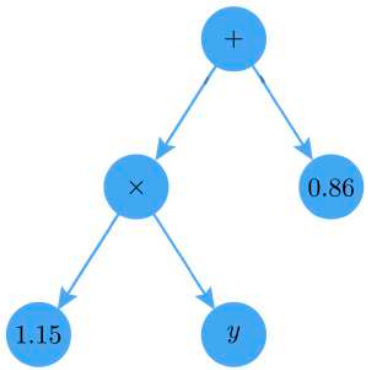
- It makes use of **Genetic Programming (GP)**: algorithms inspired by **evolutionary processes**.



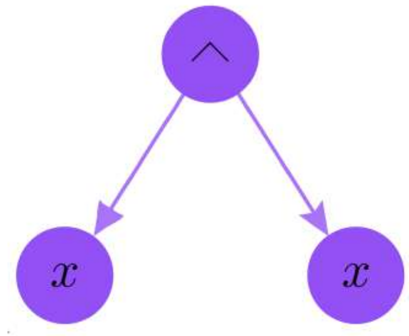
[Quanta Magazine]

# How does Symbolic Regression work?

- The algorithm constructs a **population of candidate symbolic expressions**
- Each **expression (individual) is internally represented as a tree** in the algorithm




→ $1.15y + 0.86$

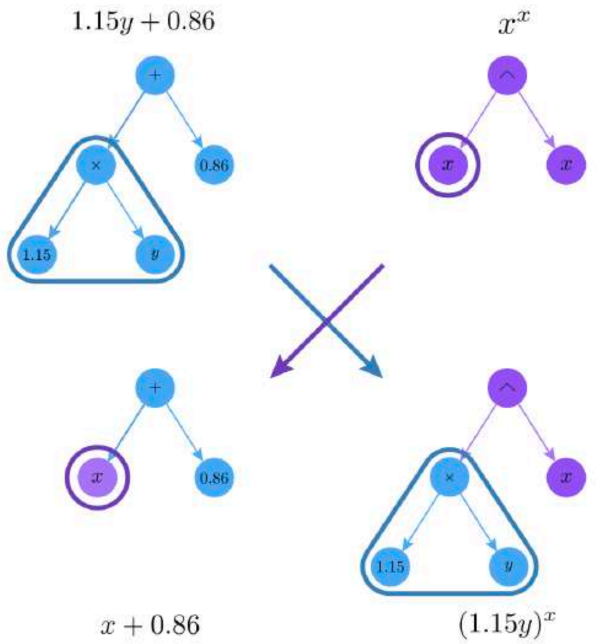



→ $x^x$

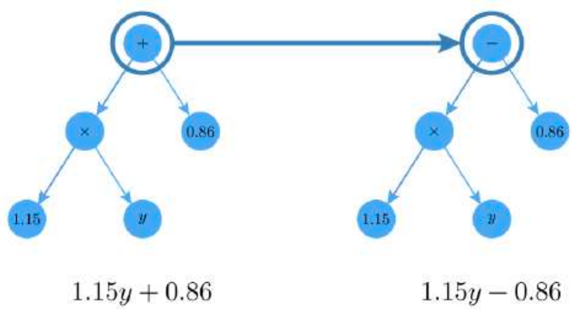
[Cramner, 2022]

# How does Symbolic Regression work?

- Expressions are **combined** (*cross-over between two individuals*) and **mutated**




*cross-over*




*mutation*

- The population evolves and **new generations** (new candidate symbolic expressions) **are created**

[Cramner, 2022]

# How does Symbolic Regression work?

- **Individuals are ranked based on how well they minimise a `loss'**, called **fitness function** in this context.

- The **fitness function** is usually **RMSE** and evaluated at eac

$$\text{Fitness} = -||\text{Data} - \text{Symbolic Exp}||^2$$

- If we want to **use Symbolic Regression to get approximate analytic expression of Ricci-flat metrics**, we would need to **encode a Ricci-flatness condition as a fitness function** and use the algorithm in an unsupervised fashion

$$\sigma_{\text{loss}} = \frac{1}{N}\left\|1 - \frac{\det g_{\text{pr}}}{\kappa\,\Omega \wedge \overline{\Omega}}\right\|_1,$$

*Work in progress…*

# Summary and outlook

- Used a **hybrid numerical-analytic approach to find an approximate Ricci-flat metric of a CY hypersurface**
- **Sampled ~2000 points in the full moduli space of a one-parameter family of bicubics.**
- **Used a NN to get the Ricci-flat Kähler potential at each point.**
- **Fitted the numerical data to an Ansatz obtaining numerical values of the moduli-dependent coefficients** $\alpha_{IJ}$ **.**
- **Used symbolic regression** on these coefficients to **get an approximate analytic Ricci-flat metric with full moduli dependence.**

- **Extend** the approach **to higher-dimensional moduli space.**
- **Bypass the neural network step** and use symbolic regression only. This requires encoding a Ricci-flatness condition directly into the symbolic regression loss.

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- **Used symbolic regression** on these coefficients to **get an approximate analytic Ricci-flat metric with full moduli dependence.**

- **Extend** the approach **to higher-dimensional moduli space.**
- **Bypass the neural network step** and use symbolic regression only. This requires encoding a Ricci-flatness condition directly into the symbolic regression loss.

*Thanks for your attention*