Dynamical zeta function and topology.

Higher structures and Field theory, ESI online

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Motivation.

Algebra	Topology	Dynamics	Field theory
dim (V)	Euler $\chi(V, d)$ $\sum(-1)^i \dim(V^i)$	zeroes of vector fields $\sum_{c\in Crit(V)}(-1)^{\text{ind}_V(c)}$	SUSY states $\dim(\mathcal{H}_0) - \dim(\mathcal{H}_1)$
trace(T)	Lefschetz $\mathcal{L}(T)$ $\sum_{i=0}^{\dim(M)}(-1)^i Tr(T\|_{H^i(M)})$	fixed points of maps $\sum_{x=T(x)} \text{ind}_T(x)$	Super trace of (T) $Str\,(T)$
determinants	Torsion τ	periodic orbits flows $\prod_{\gamma\in \text{ prime}} \det\,(Id - \rho(\gamma)\Delta(\gamma))^{(-1)^{\text{ind}(\gamma)}}$	Partition function of BF $\int_{\mathcal{L}} DADBe^{\int_M B\wedge d^{\nabla} A}$

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Last column related to work of Hadfield–Kandel–Schiavina and lecture notes of Mnev, Fried conjecture from BV viewpoint.

Geometric context.

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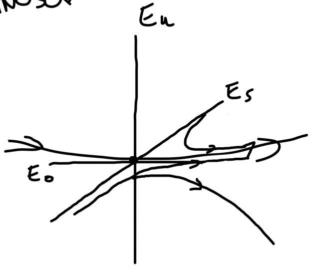
2. X Reeb field, $\theta(X) = 1$. X **Anosov** i.e. $TM = E_s \oplus E_u \oplus \langle X \rangle$, (E_s, E_u) called stable, unstable bundles $\exists C, \lambda > 0$ s.t. $\forall t \geqslant 0$:

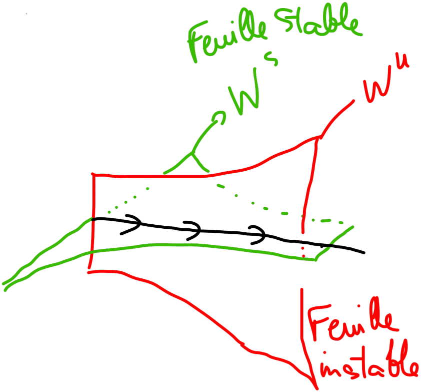
$$\|de^{tX}(v)\| \leqslant Ce^{-\lambda t}\|v\|, \forall v \in E_s, \ \ \|de^{-tX}(v)\| \leqslant Ce^{-\lambda t}\|v\|, \forall v \in E_u.$$

   Ex : X generator of the geodesic flow for metric g of negative curvature.
3. A flat bundle (E, ∇) and $\rho : \pi_1(M) \mapsto GL_n(\mathbb{C}) =$ monodromy of ∇.

ANOSOV



E_0 : neutre, E_s stable, E_u instable



Main object : twisted Ruelle zeta

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Dirichlet L-function, $\chi : \mathbb{N} \mapsto \mathbb{S}^1$ character, functions of (s,χ) :

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Using (X, χ), $\chi \in Hom(\pi_1(M), \mathbb{C}^*)$. We can form the twisted Ruelle zeta function (dynamical L functions)

$$\boxed{\zeta_{X,\chi}(s) = \prod_{\gamma\in\mathcal{P}} \left(1-\chi(\gamma)e^{-s\ell(\gamma)}\right)}$$

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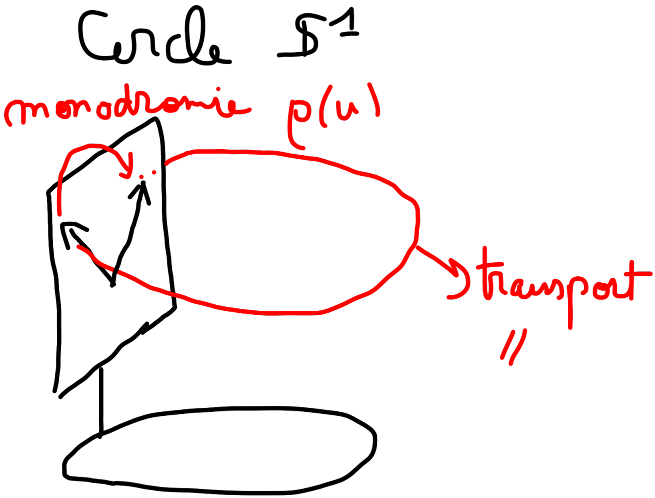
Example

On $\mathbb{S}^1$ *of length* ℓ, *flow* ∂_θ, *u generator of* $\pi_1(M)$, **monodromy** $\rho(u)\in\mathbb{C}^*$,

$\boxed{\zeta_{X,\rho}(s) = (1-\rho(u)e^{-s\ell})}$.


Circle S^1
monodromie ρ(u)
transport
//

Some questions on $\zeta_{X,\rho}$.

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Both problems deeply related.

Abstract abelian BF theory of chain complexes.

- $(C^\bullet, d)$ cochain complex, $\mathbb{Z}$–graded vector space.
- Fields $\mathcal{F} = C^\bullet[-1] \oplus (C^\bullet)^*[-2]$. $(A, B) \in \mathcal{F}$ where A cochain, B chain.
- Action functional : $S(A, B) = \langle B, dA \rangle = \sum_{j=0}^{n-1} \langle B^{j+1}, dA^j \rangle$.
- Chain contraction K on $C^\bullet$ satisfies the chain homotopy equation

$$Id - \Pi = dK + Kd \tag{1}$$

  defines Hodge decomposition $Im(K) + Im(d) + Im(\Pi)$ where $Im(K)$ called **coexact**.
- Then Lagrangian (BV version of gauge fixing) in $\mathcal{F}$

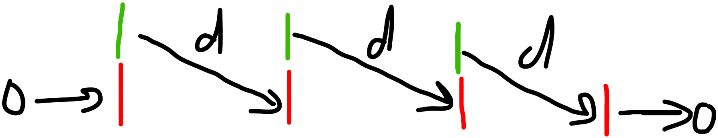
$$\mathbb{L}_K = C^\bullet_{coex}[-1] \oplus (C^\bullet_{coex})^*[-2] \subset \mathcal{F}$$

$(C^\bullet, d)$ acyclic,

$$\tau(C^\bullet, d) = \int_{\mathcal{L}} e^{iS(A,B)} \mu \tag{2}$$

where **half–density** μ complicated normalization. Gaussian integral.

Acyclicité image



- KER d = Im d
- COKER d

Geometric implementation.

Follows Mnev, Cattaneo–Mnev–Reshetikhin.

	Combinatorial	Continuum
Spacetime	CW-complex $(C^\bullet, d)$	Riemannian Manifold (M, g)
Fields	$C^\bullet[-1] \oplus (C^\bullet)^*[-2]$	$\Omega^{-\bullet}(M, E)[1] \oplus \Omega^{-\bullet}(M, E)[n-2]$
Propagator K	$[d, K] = Id$ chain contraction	$K = d^{\nabla *}\Delta^{-1}$ Hodge $[d^\nabla, K] = Id$
Lagrangian	$\mathbb{L}_K$	$\mathbb{L}_\Delta$
Partition fun. $\int_{\mathbb{L}} e^{iS(A,B)}\mu$	$\tau(C^\bullet, d) = \tau_R$ CMR, Reidemeister torsion	$\tau(M, d^\nabla) = \prod_{j=0}^n \det_\zeta(\Delta_{\Omega^j})^{\frac{j}{2}(-1)^{j+1}}$ Schwarz, Ray–Singer

Cheeger–Müller : $\tau_R = \tau_{RS}$ hence both approach coincide.

Hadfield–Kandel–Schiavina's idea.

Instead of using Hodge theory (the **metric** g) for gauge fixing, use the vector field X (**dynamics**). Key idea :

$$\underbrace{[d^\nabla, \iota_X] = \mathcal{L}_X^\nabla}_{\text{Lie–Cartan}} \text{ versus } \underbrace{[d^\nabla, d^{\nabla *}] = \Delta}_{\text{Hodge–de Rham}}.$$

Define Lagrangian (similar to axial gauge fixing)

$$\mathbb{L}_X = Im(\iota_X) \cap \Omega^{-\bullet}(M, E)[1] \oplus Im(\iota_X) \cap \Omega^{-\bullet}(M, E)[n-2].$$

In fact $Im(\iota_X) = \ker(\iota_X)$ since Koszul complex

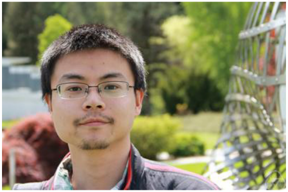
$$\xrightarrow{\iota_X} \Omega^\bullet(M, E) \xrightarrow{\iota_X} \Omega^{\bullet-1}(M, E) \xrightarrow{\iota_X}$$

is acyclic.
Formally or **definition** (HKS) :

$$\int_{\mathbb{L}_X \subset \Omega^1(M,E) \oplus \Omega^{n-2}(M,E)} e^{iS(A,B)} \mu = \underbrace{\prod_{j=0}^{n} \det{}^\flat (\mathcal{L}_X|_{\Omega^j})^{j(-1)^j}}_{\text{flat determinants instead of } \zeta} = \underbrace{\prod_\gamma \det(Id - \rho(\gamma))^{(-1)^d}}_{\text{Ruelle zeta}}.$$

Fried

Relate $|\zeta_{X,\rho}(0)|$ to $\tau_R(\rho)$.
Reformulation (HKS) : is the BV integral gauge fixing invariant if we go from $\mathbb{L}_\Delta$ (analytic torsion) to $\mathbb{L}_X$ (Ruelle zeta) ? Why is there a difficulty ? Because in ∞–dimension, BV Stokes Theorem might no longer hold true !



Theorem

1. *when* $M = S^*\mathcal{M}$ *for* **hyperbolic** $\mathcal{M}$, ρ **unitary**, *then Fried(1986) showed*

$$\tau_R(\rho) = |\zeta_{X,\rho}(0)|^{(-1)^{d-1}}. \quad (3)$$

2. *D–Guillarmou–Rivière–Shen, if for some flat connection* ∇ *and Anosov* X_0, *we have* $\ker(X_0) = \{0\}$ *then*

$$\zeta_{X,\rho} = \zeta_{X_0,\rho} \quad (4)$$

for all X *near* X_0. *In particular, the Fried conjecture holds true for* X *Anosov in* $3d$ *if* $b_1(M) > 0$ *and in* $5d$ *near geodesic flows of hyperbolic manifolds.*

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Fields	$\Omega^{-\bullet}(M,E)[1] \oplus \Omega^{-\bullet}(M,E)[n-2]$	idem
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Two problems.

- What if ρ acyclic but $\ker(X) \neq \{0\}$? $\zeta_{X,\rho}(0)$ might vanish or poles : ill–defined.
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Key observation : if X is **contact Anosov**, **canonical involution** Γ on $C(0)$.

Yann Chaubet.

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Definition (Dynamical torsion)

Inspired by Braverman–Kappeler and Hutchings thesis,

$$\boxed{\tau(X,\rho) = \underbrace{\tau_\Gamma(X)}_{\textit{torsion of } \ker} \times \underbrace{\lim_{s\to 0^+} s^{-m}\zeta_X(s,\rho)}_{\textit{renormalized zeta}}.} \quad (5)$$

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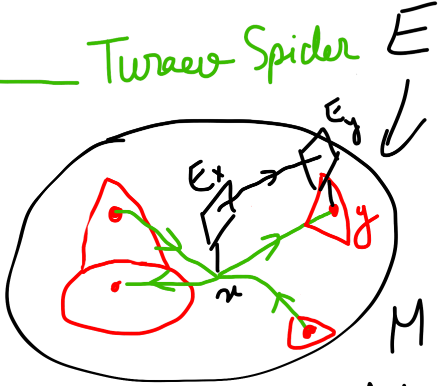
In field theory terminology, similar to Wilsonian philosophy and BV fiber integral
Torsion = Torsion of low energy fields $\times$ torsion of high energy fields

Turaev Spider



f Morse

- Crit(f)

Δ cellules instables

Euler structures.

Definition

Spider fixes ambiguity of trivialization of (E, ∇) *over cells* $=$ *Euler structures* $Eul(M)$*.*
$H_1(M, \mathbb{Z})$ *acts freely and transitively on* $Eul(M)$*.*

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Choice Euler structure $\mathfrak{e} \in \mathbb{Z}$, $u \in \mathbb{C}^* \setminus \{1\} \mapsto \tau_\mathfrak{e}(u) = u^{\mathfrak{e}+1}(u-1)^{-1}$ **holomorphic**.
Observe that $u \in \mathbb{S}^1 \setminus \{1\}$ *is acyclic unitary,* $|\tau_\mathfrak{e}(u)| = |(1-u)^{-1}| = \tau_R(u)$ *hence* $\tau_\mathfrak{e}$ **extends and refines** τ_R.

Main Theorem.

$\mathcal{A}$ = space of Anosov vector fields on mfd M, open by structural stability.
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constant C *does not depend on* X,ρ *both sides* **holomorphic functions** *of* $\rho\in Rep_0$.

Questions.

- What is the BF theory interpretation of Turaev's torsion ?
- Can I interpret the fact that if two Anosov vector fields (X_1, X_2) define different Euler structures then $\tau(X_1, \rho) \neq \tau(X_2, \rho)$ as some **failure** of the BV Stokes Theorem in ∞-dim ?
- Analogy with the framing anomalies for Chern–Simons theory ?

Two point function = Poincaré series.

M compact with variable negative curvature, (q_1, q_2) pair of points on M. Define the series

$$\boxed{\eta(q_1, q_2; z) = \sum_{\gamma} e^{-\ell(\gamma)z}} \tag{6}$$

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Poincaré series studied by Margulis (phd 1970), Pollicott, Sharp, Paternain, Mañé, Paulin–Parkkonen and many others

Theorem (Margulis 1970)

Counting function

$$N_T = |\{\gamma | \ell(\gamma) \leqslant T\}| \simeq Ce^{h_{top}T} \implies$$

η **holomorphic** *on* $Re(z) > h_{top} =$ **topological entropy** *of the geodesic flow.*

Fried's philosophy : Poincaré series at $z = 0$

Do you remember

$$\boxed{1 + 1 + \cdots + 1 + \cdots = \zeta(0) = -\frac{1}{2} \in \mathbb{Q}.} \tag{8}$$

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The function $z \mapsto \eta(\gamma_1, \gamma_2; z)$ *has* **analytic continuation** *to* $\mathbb{C}$. *If* M **surface**, *then*

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For γ_1, γ_2 closed homologically trivial geodesic :

$$\boxed{\eta(\gamma_1, \gamma_2; 0) = 1 + \cdots + 1 + \cdots = \frac{\chi(\Omega_1)\chi(\Omega_2)}{\chi(M)} - \chi(\Omega_1 \cap \Omega_2) + \frac{1}{2}\chi(\gamma_1 \cap \gamma_2) \in \mathbb{Q}} \tag{10}$$

where Ω_1, Ω_2 **surfaces bounding** *γ_1, γ_2.*

Thanks for the invitation and for listening !