1 Constructing Normal Toric Varieties

What is a toric variety? We offer three answers: good, better, best.

Concise Account. A one-sentence version for algebraic geometers.

Definition. A *(normal) toric variety* is an irreducible (normal) variety X such that the algebraic torus $(\mathbb{C}^\times)^d$ is a Zariski open subset of X, and the action of $(\mathbb{C}^\times)^d$ on itself extends to an action on X.

Informally, a toric variety consists of $(\mathbb{C}^\times)^d$ plus some "extra stuff".

Although technically correct, this perspective is almost useless. It obfuscates the raison d'être, power, and ubiquity of these spaces.

Combinatorial Viewpoint. Toric geometry provides a rich dictionary between polyhedral combinatorics and algebraic geometry.

Pair of Lattices. The *lattice* N is a free abelian group of finite rank. When N has rank d, choosing a $\mathbb{Z}$-basis gives $N \cong \mathbb{Z}^d$. The *dual lattice* is $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and the bilinear map $\langle \bullet, \bullet \rangle \colon M \times N \to \mathbb{Z}$ is the canonical pairing. The algebraic torus is $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong (\mathbb{C}^\times)^d$. Extending scalars from $\mathbb{Z}$ to $\mathbb{R}$ yields a dual pair of $\mathbb{R}$-vector spaces: $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$.

Each $\mathbf{v} \in N$ gives a homomorphism $\lambda^{\mathbf{v}} \colon \mathbb{C}^\times \to T$ defined by $\lambda^{\mathbf{v}}(t) = \mathbf{v} \otimes t$, so N is the lattice of 1-parameter subgroups of T. Similarly, each $\mathbf{u} \in M$ gives a homomorphism $\chi^{\mathbf{u}} \colon T \to \mathbb{C}^\times$ defined by $\chi^{\mathbf{u}}(\sum_i \mathbf{v}_i \otimes t_i) = \prod_i t_i^{\langle \mathbf{u}, \mathbf{v}_i \rangle}$, so M is the character group of T.

Polyhedral Definitions. A *(finitely-generated rational convex) cone* in $N_{\mathbb{R}}$ is a subset of the form

$$\begin{aligned} \sigma &= \operatorname{cone}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m) \\ &:= \{ c_0\mathbf{v}_0 + c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m \mid \mathbf{v}_i \in N \text{ and } c_i \in \mathbb{R} \text{ where } c_i \geqslant 0 \}. \end{aligned}$$

The dimension of σ is the dimension of the smallest linear subspace containing σ. A *face* of σ is a subcone generated by an appropriate subset of minimal generators for σ.

Dually, a cone σ is the finite intersection of closed half-spaces $H_{\mathbf{u}}^+ := \{\mathbf{v} \in N_{\mathbb{R}} \mid \mathbf{u} \in M \text{ and } \langle \mathbf{u}, \mathbf{v} \rangle \geqslant 0\}$, and a face of σ is the intersection of σ with a supporting hyperplane.

A *(strongly convex rational) fan* in $N_{\mathbb{R}}$ is a family Σ of nonempty cones such that

- every nonempty face of a cone in Σ is also a cone in Σ,
- the intersection of any two cones in Σ is a face of both, and
- $\{\mathbf{0}\} \in \Sigma$ (and thus a face of every cone in Σ).

The set of k-dimensional cones in Σ is denoted by $\Sigma(k)$.

Topologically, a fan is a special type of CW-complex or polyhedral complex.

Describing a fan Σ is equivalent to listing $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-1} \in N$ that generate $\Sigma(1)$ and the subsets of $\{0, 1, \dots, n-1\}$ that determine the maximal cones in Σ.

Theorem. *Each fan Σ in $N_{\mathbf{R}}$ determines a normal toric variety $X := X_\Sigma$, and the fan can be recovered from the T_N-action on X.*

More formally, the category of separated normal toric varieties endowed with an algebraic torus action corresponds to the category of strongly convex rational fans.

Proof Outline. The toric variety X is defined by gluing affine charts. Each $\sigma \in \Sigma$ gives an open affine variety $U_\sigma := \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ where $\mathbb{C}[\sigma^\vee \cap M]$ is the semigroup algebra of the dual cone. The ring $\mathbb{C}[\sigma^\vee \cap M]$ is isomorphic to the quotient of the polynomial ring $\mathbb{C}[z_0, z_1, \dots, z_m]$ by the kernel of the $\mathbb{C}$-algebra map $z_i \mapsto \chi^{\mathbf{u}_i}$ where $\sigma^\vee = \operatorname{cone}(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_m)$ and $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_m$ generate $\sigma^\vee \cap M$. A face τ of the cone σ gives inclusions $\mathbb{C}[\sigma^\vee \cap M] \subseteq \mathbb{C}[\tau^\vee \cap M]$ and $U_\tau \hookrightarrow U_\sigma$. The variety X is obtained by identifying U_σ and $U_{\sigma'}$ along their common open subset $U_{\sigma \cap \sigma'}$.

Moreover, there exists a bijection between the cones in the fan Σ and the T_N-orbits in X such that

- the dimension of the orbit corresponding to σ is $\dim N_{\mathbb{R}} - \dim \sigma$,
- U_σ is the union of the orbits corresponding to faces of σ, and
- the orbit corresponding to a cone σ lies in the closure of the orbit corresponding to the cone τ if and only if τ is a face of σ. □

The adjective "normal" means that each domain $\mathbb{C}[\sigma^\vee \cap M]$ is integrally closed in its fraction field. Equivalently, every complex point has an arbitrarily small neighborhood U such that U minus the singular locus of X is connected.

Proposition. *Let X be the toric variety determined by the fan Σ.*

- *X is compact if and only if Σ is complete: $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma = N_{\mathbb{R}}$.*
- *X is smooth if and only if each $\sigma \in \Sigma$ is generated by part of a $\mathbb{Z}$-basis for the lattice N.*
- *X is an orbifold (has at worst finite abelian quotient singularities) if and only if each $\sigma \in \Sigma$ is simplicial (its minimal generators are linearly independent in $N_{\mathbb{R}}$).*
- *X is the product of a nontrivial torus and a smaller toric variety if and only if the ray generators $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ do not span $N_{\mathbb{R}}$.*

Global Outlook. To capitalize on techniques from computational commutative algebra, we need another point of view.

Assume that the normal toric variety X has no torus factors and set $n := |\Sigma(1)|$. There is a short exact sequence

$$0 \longrightarrow M \cong \mathbb{Z}^d \xrightarrow{\left[\begin{smallmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_{n-1} \end{smallmatrix}\right]} \mathbb{Z}^n := \operatorname{Div}_T(X) \xrightarrow{[\mathbf{w}_0 \ \mathbf{w}_1 \ \cdots \ \mathbf{w}_{n-1}]} \operatorname{Cl}(X) \longrightarrow 0\,.$$

Applying the $\operatorname{Hom}_{\mathbb{Z}}(\bullet, \mathbb{C}^\times)$ gives

$$1 \longrightarrow G := \operatorname{Hom}_{\mathbb{Z}}\big(\operatorname{Cl}(X), \mathbb{C}^\times\big) \longrightarrow (\mathbb{C}^\times)^n \longrightarrow T_N \longrightarrow 1\,.$$

The *Cox ring* or *total coordinate ring* of X is the polynomial ring $S := \mathbb{C}[x_0, x_1, \dots, x_{n-1}]$. For each $\sigma \in \Sigma$, the monomial

$$x^{\hat{\sigma}} := \prod_{j \notin \sigma(1)} x_j$$

is the product of the variables corresponding to rays not in σ. The *irrelevant ideal* of X is the monomial ideal $B := \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma \rangle$.

Theorem. *Let X be a normal toric variety without torus factors.*

- *X is the good categorical quotient $\big(\mathbb{C}^n \setminus \mathrm{V}(B)\big) /\!\!/ G$.*
- *X is the geometric quotient $\big(\mathbb{C}^n \setminus \mathrm{V}(B)\big) \big/ G$ if and only if Σ is simplicial.*

This quotient construction may also be viewed as a symplectic reduction.

Computational Examples in ***Macaulay2***.

```
Macaulay2, version 1.26.06
Type "help" to see useful commands
i1 : needsPackage "NormalToricVarieties";
i2 : PP4 = normalToricVariety(
          {{-1,-1,-1,-1},{1,0,0,0},{0,1,0,0},{0,0,1,0},{0,0,0,1}},
          subsets(5,4))
o2 = PP4
o2 : NormalToricVariety
```

```
i3 : rays PP4
o3 = {{-1,-1,-1,-1}, {1,0,0,0}, {0,1,0,0}, {0,0,1,0}, {0,0,0,1}}
o3 : List
i4 : max PP4
o4 = {{0,1,2,3}, {0,1,2,4}, {0,1,3,4}, {0,2,3,4}, {1,2,3,4}}
o4 : List
i5 : isWellDefined PP4
o5 = true
i6 : PP4' = toricProjectiveSpace 4
o6 = PP4'
o6 : NormalToricVariety
i7 : rays PP4'
o7 = {{-1,-1,-1,-1}, {1,0,0,0}, {0,1,0,0}, {0,0,1,0}, {0,0,0,1}}
o7 : List
i8 : max PP4'
o8 = {{0,1,2,3}, {0,1,2,4}, {0,1,3,4}, {0,2,3,4}, {1,2,3,4}}
o8 : List
i9 : PP4 === PP4'
o9 = false
i10 : ring PP4
o10 = QQ[x ..x ]
          0   4
o10 : PolynomialRing
i11 : fan PP4 == fan PP4'
o11 = true
i12 : ideal PP4
o12 = ideal (x , x , x , x , x )
              4   3   2   1   0
o12 : Ideal of QQ[x ..x ]
                   0   4
i13 : degrees ring PP4
o13 = {{1}, {1}, {1}, {1}, {1}}
o13 : List
i14 : isSmooth PP4
o14 = true
i15 : X = weightedProjectiveSpace {1,1,2,2,2}
o15 = X
o15 : NormalToricVariety
i16 : rays X
o16 = {{-1,-2,-2,-2}, {1,0,0,0}, {0,1,0,0}, {0,0,1,0}, {0,0,0,1}}
o16 : List
i17 : max X
o17 = {{0,1,2,3}, {0,1,2,4}, {0,1,3,4}, {0,2,3,4}, {1,2,3,4}}
o17 : List
i18 : degrees ring X
o18 = {{1}, {1}, {2}, {2}, {2}}
o18 : List
i19 : isSmooth X
o19 = false
```

```
i20 : isSimplicial X
o20 = true
i21 : isProjective X
o21 = true
i22 : ideal X
o22 = ideal (x , x , x , x , x )
              4   3   2   1   0
o22 : Ideal of QQ[x ..x ]
                   0   4
i23 : orbits X
o23 = HashTable{0 => {{0,1,2,3},{0,1,2,4},{0,1,3,4},{0,2,3,4},{1,2,3,4}}       }
                1 => {{0,1,2},{0,1,3},{0,1,4},{0,2,3},{0,2,4},{0,3,4},...
                2 => {{0,1},{0,2},{0,3},{0,4},{1,2},{1,3},{1,4},{2,3},{2,4},...
                3 => {{0},{1},{2},{3},{4}}
                4 => {{}}
o23 : HashTable
i24 : Y = kleinschmidt(2,{1});
i25 : rays Y
o25 = {{-1, 0}, {1, 0}, {0, 1}, {1, -1}}
o25 : List
i26 : max Y
o26 = {{0, 2}, {0, 3}, {1, 2}, {1, 3}}
o26 : List
i27 : orbits Y
o27 = HashTable{0 => {{0, 2}, {0, 3}, {1, 2}, {1, 3}}}
                1 => {{0}, {1}, {2}, {3}}
                2 => {{}}
o27 : HashTable
i28 : dim Y
o28 = 2
i29 : isSmooth Y
o29 = true
i30 : S = ring Y
o30 = S
o30 : PolynomialRing
i31 : degrees S
o31 = {{0, 1}, {-1, 1}, {1, 0}, {1, 0}}
o31 : List
i32 : ideal Y
o32 = ideal (x x , x x , x x , x x )
              1 3   1 2   0 3   0 2
o32 : Ideal of S
i33 : max Y
o33 = {{0, 2}, {0, 3}, {1, 2}, {1, 3}}
o33 : List
i34 : PP2 = toricProjectiveSpace 2;
i35 : PP22 = PP2 ** PP2
o35 = PP22
o35 : NormalToricVariety
i36 : degrees ring PP22
o36 = {{1, 0}, {1, 0}, {1, 0}, {0, 1}, {0, 1}, {0, 1}}
o36 : List
```

```
i37 : matrix rays PP22
o37 = | -1 -1 0  0  |
      | 1  0  0  0  |
      | 0  1  0  0  |
      | 0  0  -1 -1 |
      | 0  0  1  0  |
      | 0  0  0  1  |
                 6       4
o37 : Matrix ZZ  <-- ZZ
i38 : U = normalToricVariety ({{4,-1,0},{0,1,0}},{{0,1}});
i39 : isWellDefined U
o39 = true
i40 : isDegenerate U
o40 = true
i41 : ring U
stdio:42:4:(3):[1]: -- not yet implemented for degenerate varieties
i42 : debugLevel = 1;
i43 : X = new MutableHashTable;
i44 : coneList = subsets(3,2)
o44 = {{0, 1}, {0, 2}, {1, 2}}
o44 : List
i45 : X#1 = normalToricVariety ({{-1,-1},{1,0},{0,1},{-1,0}}, coneList);
i46 : isWellDefined X#1
-- some ray does not appear in maximal cone
o46 = false
i47 : coneList' = {{0,1},{0,3},{1,2},{2,3},{3}};
i48 : X#2 = normalToricVariety ({{-1,0},{0,-1},{1,-1},{0,1}}, coneList');
i49 : isWellDefined X#2
-- some cone is not maximal
o49 = false
i50 : X#3 = normalToricVariety ({{-1,-1},{1,0},{0,1,1}},coneList);
i51 : isWellDefined X#3
-- expected `rays' to be a list of equal length lists
o51 = false
i54 : X#5 = normalToricVariety ({{1,0},{0,1},{-1,0}},{{0,1,2}});
i55 : isWellDefined X#5
-- not all maximal cones are strongly convex
o55 = false
i56 : X#6 = normalToricVariety ({{1,0,0},{0,1,0},{0,0,2}},{{0,1,2}});
i57 : isWellDefined X#6
-- the rays are not the primitive generators
o57 = false
i58 : X#7 = normalToricVariety ({{1,0},{0,1},{1,1}},{{0,1},{1,2}});
i59 : isWellDefined X#7
-- intersection of cones is not a cone
o59 = false
```