Laplacians on graphs associated with self-similar group actions

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Spectral graph theory wants to understand how the spectra of various operators defined on (functions on) the graph are related to the geometry of the graph.

A graph $\Gamma = (V, E) \quad \rightarrow \quad$ the adjacency matrix A,

the Markov operator M = transition matrix of the simple random walk on the graph = the normalized adjacency matrix,

the discrete laplacian $\Delta = Deg - A$ or $\Delta = I - M$.

Important classes of examples:

- **Cayley graphs of finitely generated groups**, $Cay(G, S)$,

- **Schreier graphs** $Sch(G, H, S)$ with respect to a subgroup $H < G$,

- **lattices, self-similar graphs...**

We will understand M as an operator acting on the space $l^2(V(\Gamma))$.

The Markov operator on G with respect to S can be understood as an element of the group algebra

$$M(= M_S) \;=\; \frac{1}{|S|}\sum_{s\in S} s \quad \in \mathbb{C}[G]$$

and we can consider its images in different representations. The most classical one is the left regular representation $\pi : G \to l^2(G)$ so $M : l^2(G) \circlearrowleft$

But also quasi-resular representations of type $\pi_H : G \to l^2(G/H)$ where $H < G$.

The operator M becomes respectively the Markov operator of the simple random walk on the graph $Cay(G, S)$ or on $Sch(G, H, S)$.

$$Mf(g) = \frac{1}{|S|}\sum_{s\in S} f(gs), \text{ for } f \in l^2(Vert(\Gamma)), g \in Vert(\Gamma).$$

$$spec(M) \subseteq [-1, 1]$$

Q.1: Can one hear the shape of a (Cayley) graph? No.

For example, the spectrum of $\mathbb{Z}^d$ with standard generators is $[-1, 1]$ for all $d \geq 1$. *The same is true for any bipartite Cayey graph of a torsion free amenable group.*

On the spectrum of M, we have the projection-valued **spectral measure** μ and the associated measures $\mu_v, v \in V(\Gamma)$, with $\mu_v(\lambda) =< \mu(\lambda)\delta_v, \delta_v >$, whose n-th moments are the probabilities of return to v after n steps of the simple random walk on Γ.

Spectral Theorem: Spectrum + spectral measure μ determine the operator up to unitary equivalence. Among finite graphs, there are examples of non-isomorphic strongly-regular graphs with parameters (n, k, m, l) that are Cayley graphs.

Q.1': Find isospectral families of graphs with equivalent spectral measures.

Spectrum of $\mathbb{Z}^d$ with standard generators is $[-1, 1]$ for all $d \geq 1$.

The same is true for any bipartite Cayey graph of a torsion free amenable group.

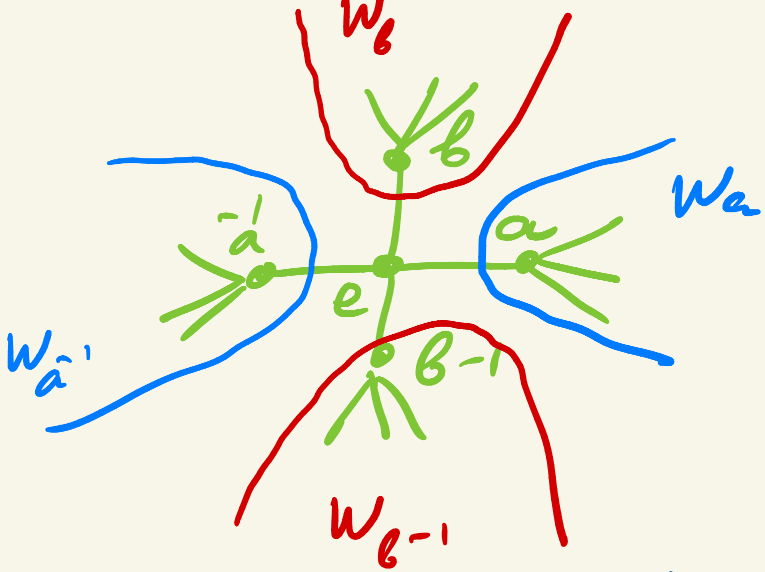
- The spectrum is symmetric iff G is bipartitie.

- Kesten's Criterion: G is amenable if and only if $1 \in \text{spec}(M(G, S))$ for some (equivalently, for every) finite symmetric generating set S.

- The absence of non-trivial idempotents in $C_r^*(G)$ of a torsion free group G (Kadison-Kaplansky Conjecture, true for amenable groups) implies that the spectrum is connected.

Let G be a countable group. A **paradoxical decomposition** (PD) is $G = A_1 \sqcup \ldots \sqcup A_m \sqcup B_1 \sqcup \ldots \sqcup B_n$ st $\exists\, g_1, \ldots, g_m, h_1, \ldots, h_n \in G$ with $G = \bigcup_{i=1}^{m} g_i \cdot A_i = \bigcup_{j=1}^{n} h_j \cdot B_j$

Ex. $F_2 = F(a, b)$



$b^{-1} \cdot W_b \sqcup W_{b^{-1}} = F_2$

$a^{-1} \cdot W_a \sqcup W_{a^{-1}} = F_2$

Corollary: Hausdorff-Banach-Tarski Paradox: $B^3 \subset \mathbb{R}^3$ is paradoxical under the action of $Isom(\mathbb{R}^3)$.

Def: G amenable if it admits a finitely additive invariant measure $\mu : \mathcal{P}(G) \to [0,1]$ s.t. $\mu(G) = 1$.

Ex. Finite gps: $\mu(A) = \frac{|A|}{|G|}$

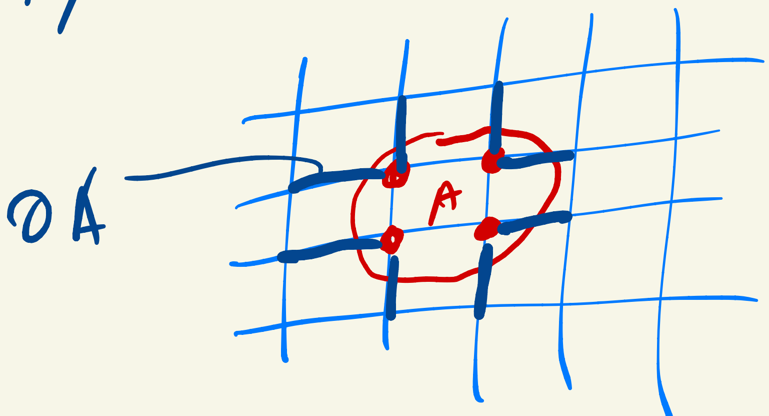
Abelian gps.

G amenable $\Longrightarrow$ G has no (PD) $\Longleftrightarrow$ Følner condition

$\Longleftarrow$ Tarski's Thm

Følner condition = subexponential isoperimetric inequality

$\Gamma = Cay(G, S)$, $S \to$ a finite gen-set



$$i(\Gamma) = \inf_{\substack{A \subset G \\ A \text{ finite}}} \frac{|\partial A|}{|A|}$$

G amenable $\iff$ $i(\Gamma) = 0$ $\forall\, \Gamma$ Cayley graph of G

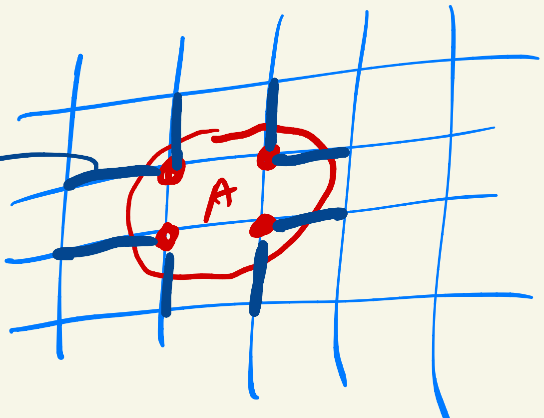
Kesten's Theorem. $i(\Gamma) = 0 \iff \rho(\Gamma) = 1$ ← subexponential decay of return probabilities

$$\rho(\Gamma) = \limsup_{n\to\infty} \sqrt[n]{p^{(n)}(e,e)}$$

Følner condition = subexponential isoperimetric inequality

$\Gamma = Cay(G, S)$ → S: a finite gen-set

∂A



$$i(\Gamma) = \inf_{\substack{A \subset G \\ A \text{ finite}}} \frac{|\partial A|}{|A|}$$

Q.2: What compact subsets of $[-1, 1]$ can be realized as the spectrum of M? What can the spectral measure type be on a Cayley or Schreier graph?

In general, the spectral measure has three components: pure-point, absolutely continuous w.r.t. the Lebsgue measure and continuous singular w.r.t. the Lebesgue measure.

In Cayley graphs: there are examples with the absolutely continuous spectrum on an interval plus maybe finitely or infinitely many isolated points (free products of finite groups, Kuhn; Cartwright-Soardi; lamplighter, Grigorchuk-Simanek).

The only known examples of Cayley graphs without a.c. part in the spectral measure are Cayley graphs of the lamplighter groups which are Diestel-Leader graphs $DL(k, k) = Cay(\mathcal{L}_k, S_k)$. This was first shown by Grigorchuk-Zuk (see also Lehner, Neuhauser, Woess): the spectrum is $[-1, 1]$ but the spectral measure is pure point.

More generally, in regular graphs:

there are examples of absolutely continuous spectrum on a union of infinitely many intervals (Aizenmann-Schenker: lines with decorations);

of the pure point spectrum on a countable set of points accumulating on a Cantor set of Lebesgue measure 0 (Malozemov-Teplyaev: Sierpinski triangle),

of spectra with a non-trivial singular continuous component (Simon, Breuer: trees with growing degrees)

Q. 2': Can one get such exotic spectra in Schreier graphs?
Q. 3: How does spectral type depend on the generating set?

Bartholdi-Grigorchuk (2000): Schreier graphs of some self-similar groups with the spectrum a union of two intervals, a Cantor set.

Valette-Beguin-Zuk (1997): the spectrum of Heisenberg group.

Our Results: Grigorchuk-N-Perez IMRN'22 + in progress; Grigorchuk-Lenz-N Math.Ann.'18, Adv.Math.'22 (+ Sell)

Uncountable families of pairwise non quasi-isometric isospectral Cayley graphs;

Uncountable families of pairwise non-isomorphic Schreier graphs with unitary equivalent laplacians (and hence isospectral in a strong sense, i.e., such that the spectral measures also coincide);

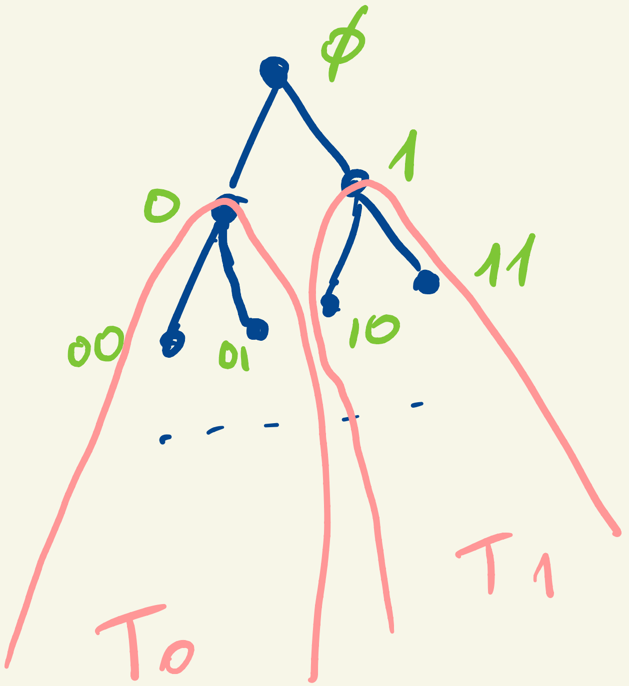
Cayley graphs with spectrum a union of two intervals;

Schreier graphs with pure point spectral measure, spectral measure with non-trivial singular continuous component.

Examples of group actions such that the corresponding Schreier graphs have the spectral measure absolutely continuous w.r.t. Lebesgue for one generating set and singular continuous on a Cantor set of Lebesgue measure 0 for another generating set.

Self-similar groups.

$T = T_d$ = d-regular rooted tree. $T_d \simeq \{0, \dots, d-1\}^*$



$Aut(T)$.

$g \in Aut\, T \Rightarrow$ $g = (g_0, g_1, \dots, g_{d-1})\, \sigma_g$

$g_0 = g|_{T_0} \in Aut\, T$

$g_i = g|_{T_i} \in Aut\, T$

$\sigma_g \in Sym(d)$: the action of g on the 1st level of the tree.

$G < Aut(T)$ is <u>self-similar</u> if

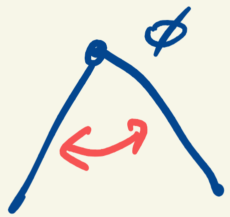
$\forall g \in G$, $g_0, \dots, g_{d-1} \in G$.

Examples:

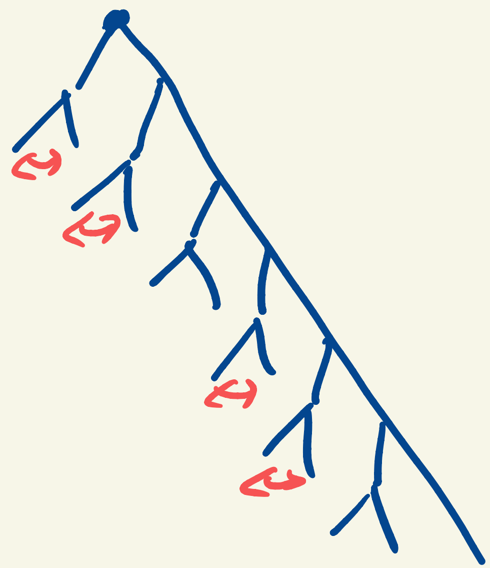
1. Grigorchuk group of intermediate growth.

$$Gr = \langle a, b, c, d \rangle < \mathrm{Aut}(T_2).$$

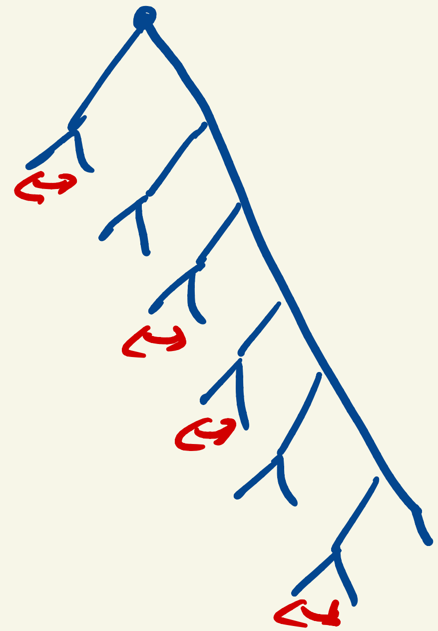
$a = (1,1)\sigma$



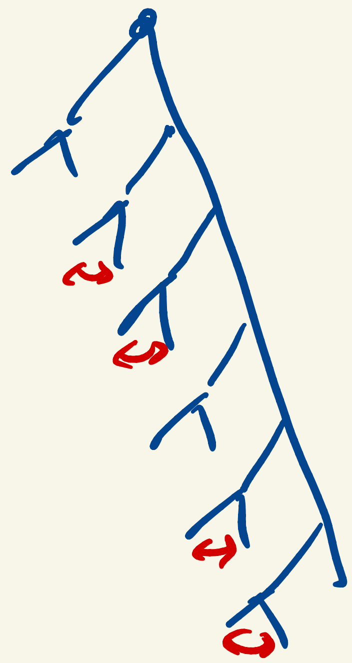
$b = (a, c)$



$c = (a, d)$



$d = (1, b)$



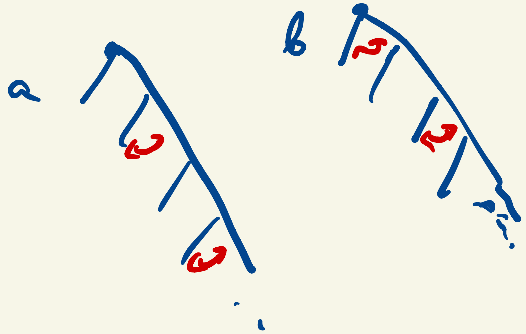
An uncountable family of groups $G_\omega = \langle a, b_\omega, c_\omega, d_\omega \rangle$.

Examples: a) $f(z) = z^2 - 1$

Basilica

$B = \langle a, b \rangle$. $\quad a = (1, b)$ $\quad b = (1, a)\sigma$

$B \curvearrowright T_2$



b) $f(z) = z^3\left(-\frac{3}{2} + i\frac{\sqrt{3}}{2}\right) + 1$

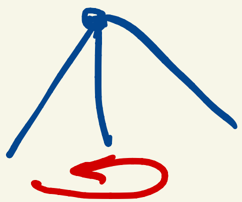
Gupta-Fabrykowsky group

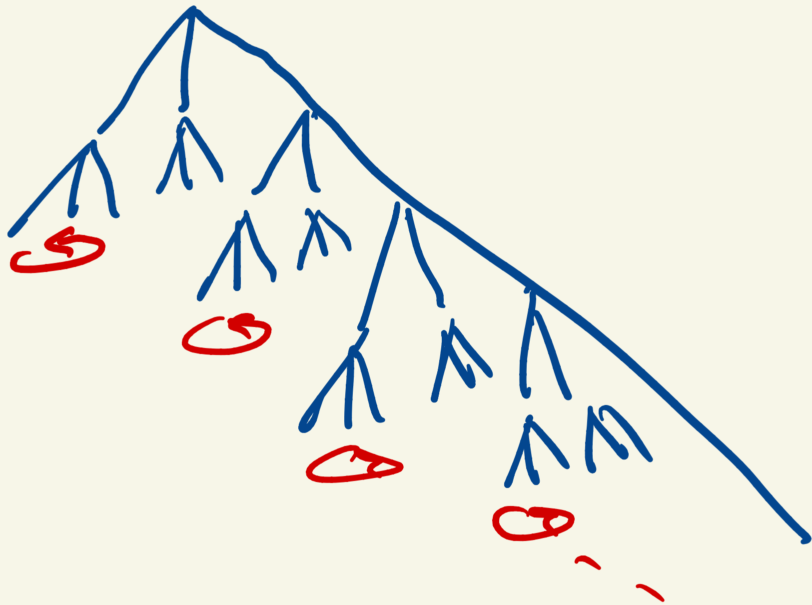
$b = (a, 1, b)$

$GF = \langle a, b \rangle$ $\quad a = (1, 1, 1)\sigma$

$\sigma = (012)$

$GF \curvearrowright T_3$



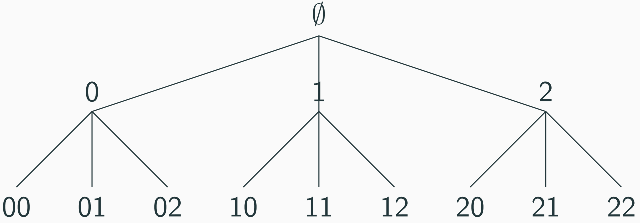


Spinal groups. (A variation of a construction of Bartholdi and Sunic, '00)

Let $d \geq 2$ be an integer, and let T_d be the d-regular infinite rooted tree.

If $X = \{0, 1, \ldots, d-1\}$, then T_d can be identified with X^*.

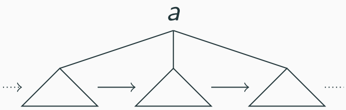
$G \leq \mathrm{Aut}(T_d)$, transitive on each level of the tree. By continuity, the action naturally extends to an action of G on ∂T_d by homeomorphisms. The boundary ∂T_d can be identified with $X^{\mathbb{N}}$.

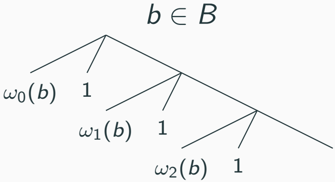


Let $d \geq 2$, $m \geq 1$.

Consider $A = \mathbb{Z}/d\mathbb{Z} = \langle a \rangle$ and $B = (\mathbb{Z}/d\mathbb{Z})^m$. Define an alphabet $\Omega_{d,m} = \mathrm{Epi}(B, A)$.

For each $\omega = \omega_0\omega_1 \cdots \in \Omega_{d,m}^{\mathbb{N}}$ define a group $G_\omega = \langle A, B \rangle \leq \mathrm{Aut}(T_d)$ with the generating set $S = A \cup B \setminus \{1\}$.





Examples. $d = 2$, $m = 1$: the infinite dihedral group;

$d = 2$, $m = 2$: the uncountable family of Grigorchuk's groups;

$d = 3$: $m = 1$: An uncountable family of groups including the Fabrykowski-Gupta group corresponding to the constant sequence

Schreier graphs.

(G, S). $G \curvearrowright A$ (a set) transitively $\rightsquigarrow Sch(A, G, S)$.

Vert $= A$; Edges $= \{(a, s.a) \mid a \in A, s \in S\}$.

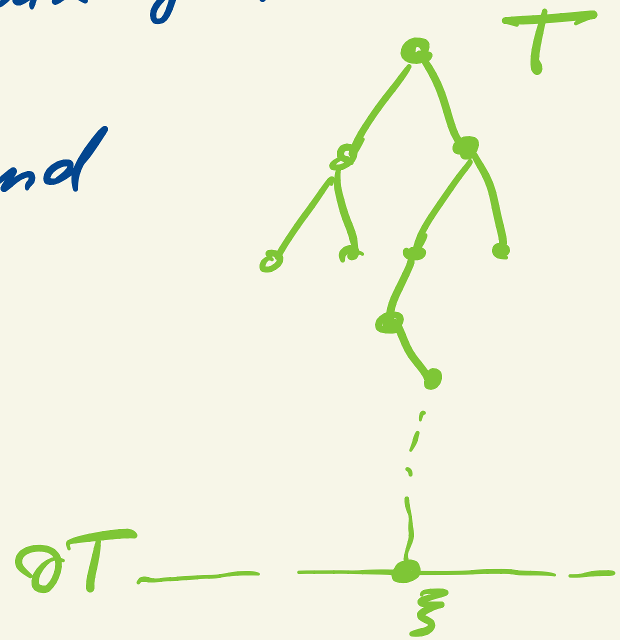
Suppose $G < Aut(T_d)$; G preserves the levels L_n

If G is transitive on each level, then we get a family of finite graphs $\{\Gamma_n\}_n$

$\Gamma_n = Sch(L_n, G, S)$; and

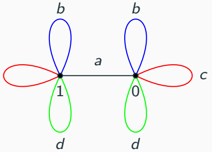
their limits $\{\Gamma_\xi\}_{\xi \in \partial T}$:

$\Gamma_\xi = Sch(G.\xi, G, S)$, $\xi \in \partial T$.

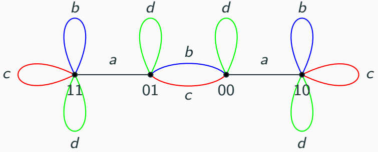


Schreier graphs of the Grigorchuk's group for the action on the levels of the tree. $\Gamma_n = Sch(G, H, S)$ with $H = Stab_G(x_1 \ldots x_n)$ where $x_1 \ldots x_n$ is any binary word of length n.

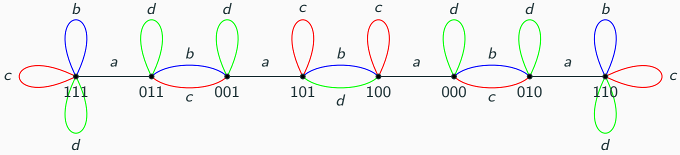
Γ_1



Γ_2



Γ_3



The choice of a generating set in G defines a map from the boundary of the tree to the space $\mathcal{G}_{*,S}$ of rooted, oriented, S-labeled graphs equipped with local convergence.

$$F : \partial T_d \to \mathcal{G}_{*,S} \quad \xi \mapsto (\Gamma_\xi, \xi)$$

To each point ξ of the boundary it associates its Schreier graph (Γ_ξ, ξ) where $\Gamma_\xi = Sch(G, Stab_G(\xi), S)$.

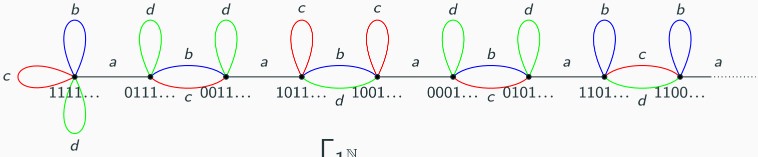
Proposition

- If $\xi = \xi_0\xi_1 \dots$, the sequence $(\Gamma_{\xi_0 \dots \xi_n}, \xi_n)$ converges to (Γ_ξ, ξ).
- For all G_ω except $d = 2$, $m = 1$, F is injective.
- For all G_ω, F is continuous on $\mathcal{R} = \partial T_d \setminus G_\omega \cdot (d-1)^{\mathbb{N}}$.

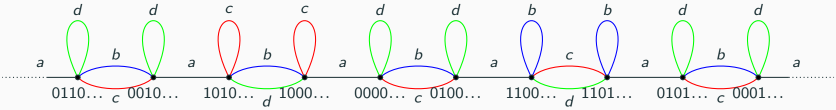
Schreier dynamical system: $G \curvearrowright (\overline{F(\partial T_d)} \setminus \{isolated\ points\}, F_*\nu)$, *where* ν *is the uniform measure on* ∂T_d.

Remark. Spectrum of Γ_ξ doesn't depend on ξ. But the spectral measure in general depends on ξ.

Infinite Schreier graphs of Grigorchuk's group for the action on the boundary of the tree:

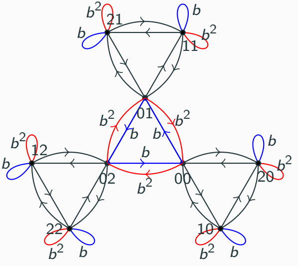


$\Gamma_{1^{\mathbb{N}}}$

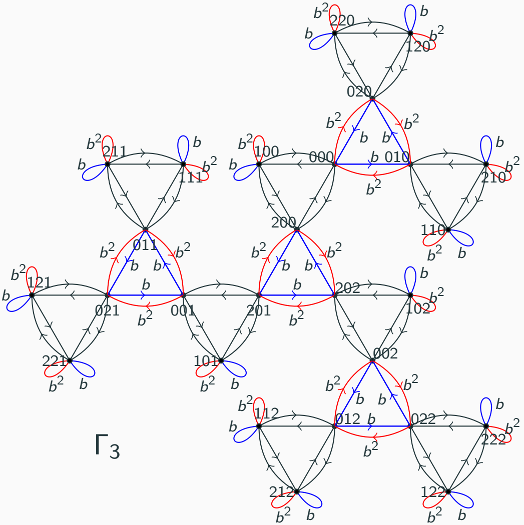


$\Gamma_{0^{\mathbb{N}}}$

Schreier graphs of the Fabrykowski-Gupta group for the action on the levels of the tree:



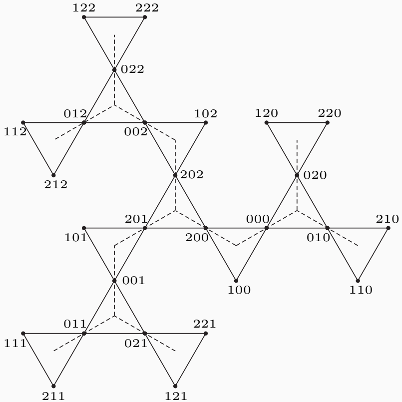
Γ_2

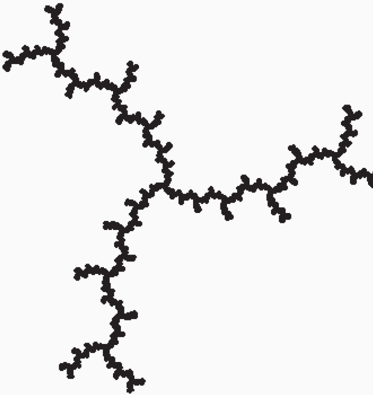


Γ_3

Scaling limit of finite Schreier graphs Γ_n

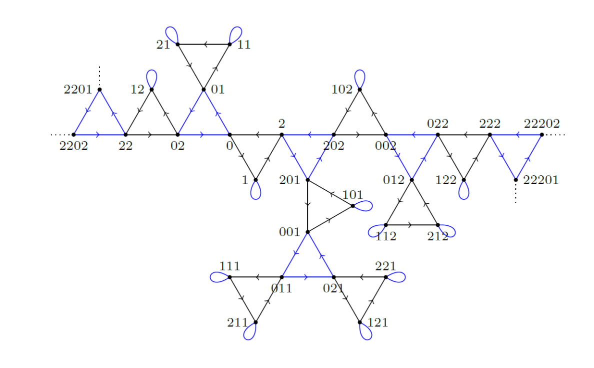
The limit space of the Fabrykowski-Gupta group is the Julia set $J(z^3(-\frac{3}{2} + i\frac{\sqrt{3}}{2}) + 1)$.





Infinite Schreier graphs of the Gupta-Fabrykowski group

$\left(d = 3,\ m = 1,\ \omega = \pi^{\mathbb{N}},\ G_\omega = \langle a, a^2, b, b^2 \rangle\right)$ for the action on the boundary of the tree:



Theorem 1. $d = 2$. Spectra of Schreier and Cayley graphs (Grigorchuk - Dudko, Grigorchuk - N. - Perez) For all $m \geq 2$ and $\omega \in \Omega_{d,m}^{\mathbb{N}}$, we have for G_ω:

$$\mathrm{spec}_{Cay}(M) = \mathrm{spec}_{Sch}(M) = \left[-\frac{1}{2^{m-1}}, 0\right] \cup \left[1 - \frac{1}{2^{m-1}}, 1\right].$$

Hence we obtain here a continuum of non quasi-isometric isospectral Cayley graphs.

Moreover, the spectrum is a union of two intervals.

Proposition. The spectral measure on the Schreier graph is a.c. w.r.t. the Lebesgue measure with the density

$$g(x) = \frac{\frac{1}{2} - \frac{1}{2^m} - x}{\frac{\pi}{2^m}\sqrt{1 - \left(2^m(\frac{1}{2} - \frac{1}{2^m} - x)^2 - \frac{4^{m-1}+1}{2^m}\right)^2}}.$$

What is the spectral measure on the Cayley graphs of G_ω?

Theorem 2: $d \geq 3$. **Spectra of infinite Schreier graphs**
(Grigorchuk-N.-Perez) Let $\Gamma = \Gamma_\xi$ be any infinite Schreier graph of any G_ω with $S = (A \cup B) \setminus \{1\}$ acting on ∂T_d. Then,

$$\mathrm{spec}(\Gamma) = \mathrm{spec}^0(\Gamma) \cup \mathrm{spec}^\infty(\Gamma),$$

where $\mathrm{spec}^\infty(\Gamma)$ is the Julia set of $x^2 - d(d-1)$ which is (for $d \geq 3$) a Cantor set of Lebesgue measure zero and

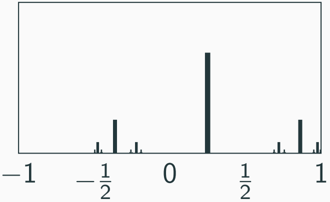
$\mathrm{spec}^0(\Gamma)$ is a countable set of points accumulating on spec^∞.

The spectral measure is pure point on the set $\mathrm{spec}^0(\Gamma)$.

Theorem 3: $d \geq 3$: spectral measures There exists an (explicitly described) measure one subset Y of ∂T_d , such that for every group G_ω and for every $\xi \in Y$ the operator $M(\Gamma_\xi)$ possesses a complete set of finitely supported eigenfunctions in $l^2(Vert(\Gamma_\xi))$. In particular, the spectral measure μ_v of $M(\Gamma_\xi)$ is discrete and concentrated on the set $\text{spec}^0(\Gamma)$.

The empirical spectral measure (the integrated density of states) can be explicitly computed.

For example, on the Fabrykowski-Gupta group Schreier graph:



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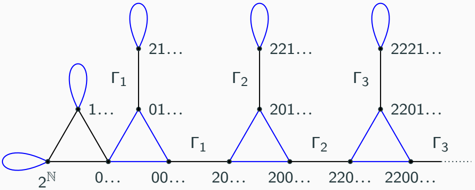
Theorem 4: $d \geq 3, m = 1$: spectral measures for limit graphs.

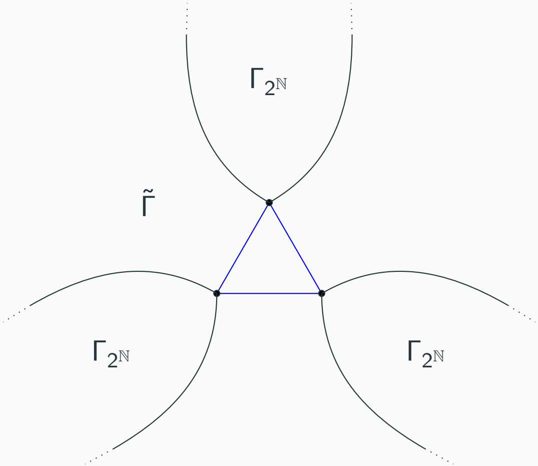
The spectrum of M on the graphs $\tilde{\Gamma}$ in $\overline{F(\partial T_d)} \setminus F(\partial T_d)$ coincides with the spectrum found in Theorem 3. The spectral measure μ_v for $\tilde{\Gamma}$ has additionally a non-trivial singular continuous component.

Methods: Approximation of infinite graphs by finite graphs + Schur complement for computing the joint spectra of infinite families finite graphs;

renormalization methods for dealing with the spectral measure (earlier work by Quint '09, Higuchi and Shirai '04)

$\Gamma_{2^{\mathbb{N}}}$





Q. 3. Changing the generating set

We already know that for $d = 2$, spinal groups with spinal generating sets give rise to Schreier graphs with absolutely continuous spectrum on a union of two disjoint intervals.

Proposition. (Follows from Grigorchuk, Lenz, N., '17; Grigorchuk, Lenz, N., Sell '19) For $d = 2$, $m \geq 2$ and any G_ω there exists a (minimal) generating set with the spectrum of any infinite Schreier graph Γ a Cantor set of Lebesgue measure zero. The spectral measure is purely singular continuous for almost every ω.

Q.: What spectrum can occur on Cayley graphs of groups G_ω for minimal generating sets?

Theorem. (GLN,'17; GLN + Sell,'19). $d = 2$. Let $M_\xi = \sum_{s \in S} p_s s$, $\sum_{s \in S} p_s = 1$, be a Markov operator on a Schreier graph Γ_ξ, $\xi \in \partial T_2$ of a spinal group. There exists a minimal subshift $(\mathcal{S}, \Sigma_\omega)$ over a finite alphabet such that for almost every $\xi \in \partial T_2$ there exists $\sigma \in \Sigma_\omega$ such that M_ξ is unitary equivalent to the Schroedinger operator H_σ acting on $l^2(\mathbb{Z})$ with $\alpha, \beta : \Sigma_\omega \to \mathbb{R}$ and, for every $u \in l^2(\mathbb{Z})$,

$$(H_\sigma u)(n) = \alpha(\mathcal{S}^n \sigma) u(n-1) + \alpha(\mathcal{S}^{n+1}\sigma) u(n+1) + \beta(\mathcal{S}^n \sigma) u(n)$$

The proof shows that the isotropic Markov operator, i.e., $p_s = 1/|S|$ for all $s \in S_\omega$, corresponds exactly to the Schroedinger operator with periodic functions α, β.

It is known that for subshifts of low complexity, the spectrum of such Schroedinger operators is an interval or a union of finitely many intervals with a.c. measure if α, β are periodic and, if not, it is a Cantor set of Lebesgue measure 0; ω - a.s. singular continuous.

Known sufficient conditions for this type of result, "Cantor spectrum of Lebesgue measure 0":

- linear repetitivity (ω - negligeble). A subshift $(\mathcal{S}, \Sigma)$ is called linearly repetitive (LR), if there exists a constant $C > 0$ such that any word $v \in Sub(\Sigma)$ occurs in any word $w \in Sub(\Sigma)$ of length at least $C|v|$. (Damanik - Lenz)

- Boshernitzan condition (ω a.s. condition). A subshift satisfies the Boshernitsan condition (B) if the same condition is satisfied for all v of length l_n, for a certain increasing sequence $\{l_n\}$. (Beckus, Pogorzelski)

In a joint work Grigorchuk - Lenz - N. - Sell we generalize these results and prove the Cantor spectrum of Lebesgue measure 0 theorem for simple Toeplitz subshifts. Together with the reduction theorem above this proves Cantor spectrum of Lebesgue measure 0 theorem for all groups G_ω.

Thank you!