Hypersurface deformation structures and space-time models

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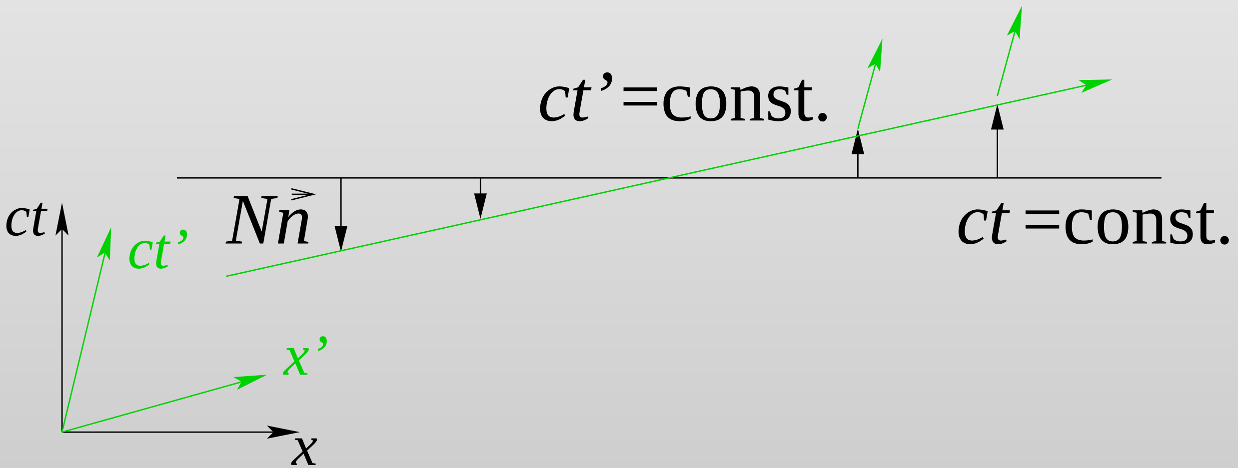
Special relativity

Poincaré group $\mathbb{R}^4 \rtimes \mathrm{O}(3,1)$ determines space-time structure.

Lorentz boost:
$ct' = \gamma ct + \alpha x$ and $x' = \alpha ct + \gamma x$ such that $\gamma^2 - \alpha^2 = 1$.

Transformations in coordinate system:

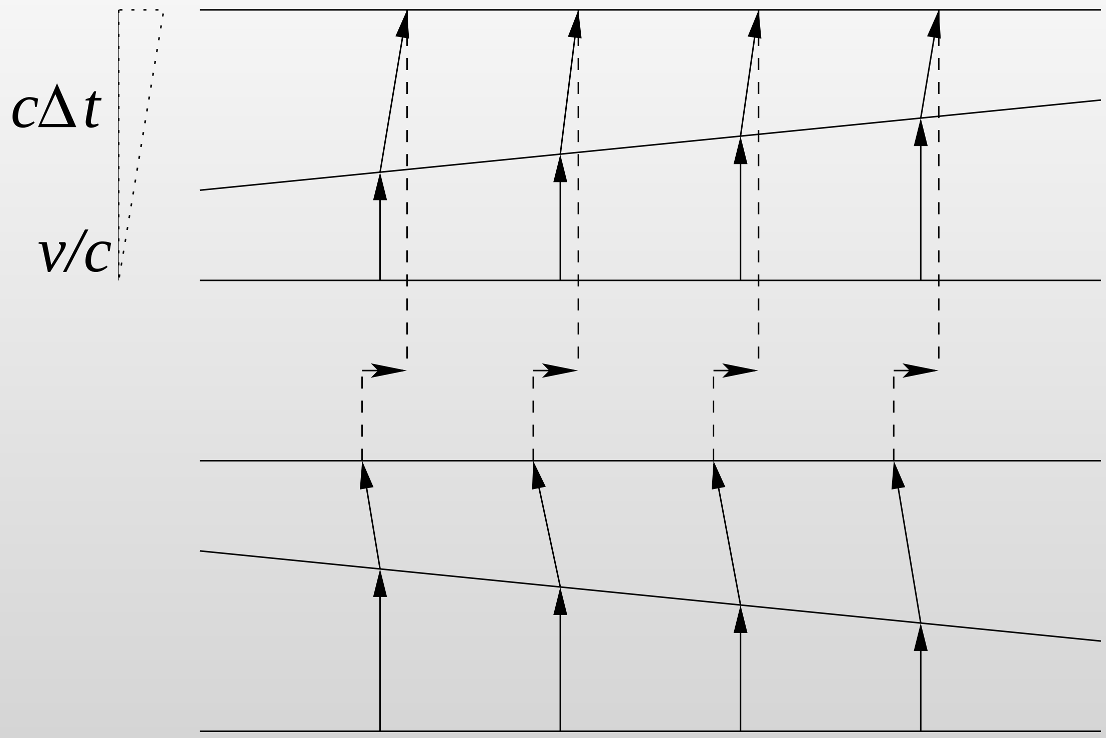


Poincaré transformations as linear deformations of spatial slice:

$$N(\vec{x}) = c\Delta t + (\vec{v}/c)\cdot\vec{x} \quad , \quad \vec{w}(\vec{x}) = \Delta\vec{x} + \mathbf{R}\vec{x}$$

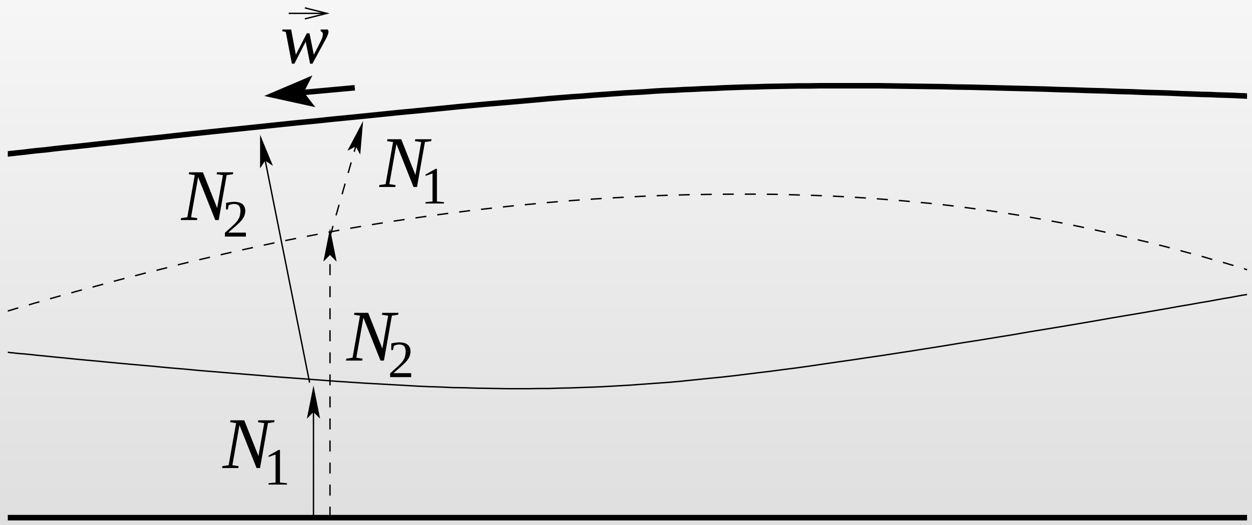
with $(\Delta t, \Delta\vec{x}) \in \mathbb{R}^4$, $\vec{v} \in \mathbb{R}^3$, $\mathbf{R} \in \mathrm{O}(3)$.

Commutators



Normal deformations by $N_1(x) = vx/c$ (Lorentz boost) and $N_2(x) = c\Delta t - vx/c$ (reverse Lorentz boost and waiting Δt) commute up to spatial displacement $\Delta x = v\Delta t$.

General relativity



Hypersurface-deformation commutators:

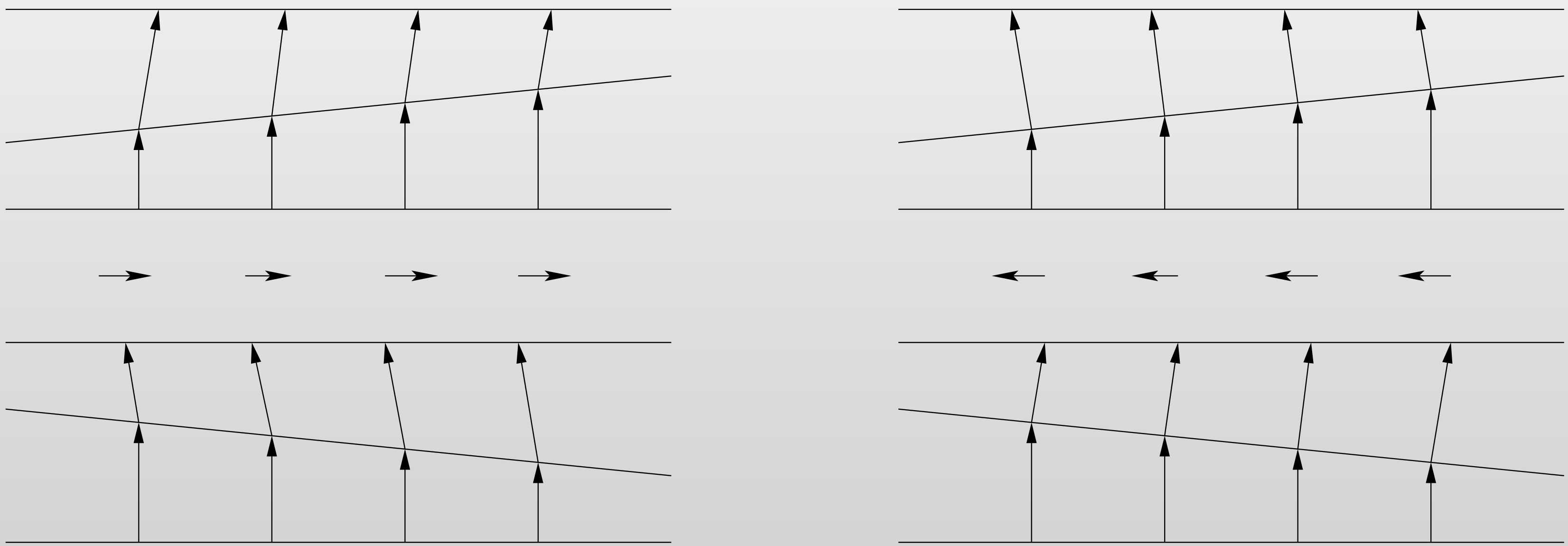
$$
\begin{aligned}
[S(\vec{w}_1), S(\vec{w}_2)] &= S(\mathcal{L}_{\vec{w}_1}\vec{w}_1) \\
[T(N), S(\vec{w})] &= -T(\mathcal{L}_{\vec{w}}N) \\
[T(N_1), T(N_2)] &= S(N_1\nabla N_2 - N_2\nabla N_1)
\end{aligned}
$$

∇N requires metric: $\nabla N = q^{ab}\partial_a N$ (structure functions).

"Bracket of bundle sections over space of spatial metrics q."

Example

Compare Minkowski space-time with Euclidean space:



Minkowski　　　　Euclid

$$[T(N_1), T(N_2)] = \pm S(N_1 \nabla N_2 - N_2 \nabla N_1)$$

Symmetries

Dirac 1958: Canonical formulation of general relativity.

Brackets realized by phase-space functions $S(\vec{w})(q,K)$ and $T(N)(q,K)$. Depend on spatial metric q and extrinsic curvature K for given $\vec{w}(q,K)$ and $N(q,K)$.

Invariance under hypersurface deformations implies general covariance:

$$\mathcal{L}_\xi f(q,K) = X_{T(N\xi^t)(q,K)+S(\vec{\xi}+\vec{w}\xi^t)(q,K)} f(q,K)$$

in space-time with line element

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + ||\mathrm{d}\vec{x} + \vec{w}\mathrm{d}t||_q^2$$

provided constraints are satisfied (on-shell):
$S(\vec{w})(q,K) = 0$ and $T(N)(q,K) = 0$ for all $\vec{w}$ and N.

Dynamics

Hojman, Kuchař, Teitelboim 1974–76:

Procedure to determine field equations for q and K covariant under canonical hypersurface deformations.

- → At second derivative order, found to be unique up to constants (Newton's and cosmological).
- → Equivalent to Einstein's equation for general relativity:

$$\mathcal{L}_t f(q,K) = X_{T(N)(q,K)+S(\vec{w})(q,K)} f(q,K)$$

if (q,K) induced on spatial slice of space-time solution.

- → Constraints correspond to tt-component of Einstein's equation ($T(N)(q,K)=0$) and $t\vec{x}$ components ($S(\vec{w})(q,K)=0$), respectively.

Classification questions

→ Off-shell meaning of hypersurface deformations?

Geometrical picture uses Hamiltonian vector fields of $S(\vec{w})(q, K)$ and $T(N)(q, K)$.
Does not require constraint equations.

→ Solutions to Einstein's equation are extrema of the Einstein–Hilbert action.

Generally covariant metric theories are extrema of higher-curvature actions.

All higher-curvature actions have generators of hypersurface-deformation brackets in canonical form.
[Deruelle, Sasaki, Sendouda, Yamauchi 2009]

Are there hypersurface-deformation invariant theories that are not of higher-curvature form?

Example

Scalar field $\phi(x)$, momentum $p(x)$, one spatial dimension.

$$T(N) = \int \mathrm{d}x N \left(f(p) - \frac{1}{4}(\phi')^2 - \frac{1}{2}\phi\phi'' \right) \quad , \quad S(w) = \int \mathrm{d}x w \phi p'$$

Spatial diffeomorphisms:

$$\delta_w \phi = \{\phi, S(w)\} = -(w\phi)' \quad , \quad \delta_w p = \{p, S(w)\} = -wp'$$

T-bracket:

$$\{T(N), T(M)\} = S(\beta(p)(N'M - NM'))$$

with $\beta(p) = \frac{1}{2}\mathrm{d}^2 f/\mathrm{d}p^2$.

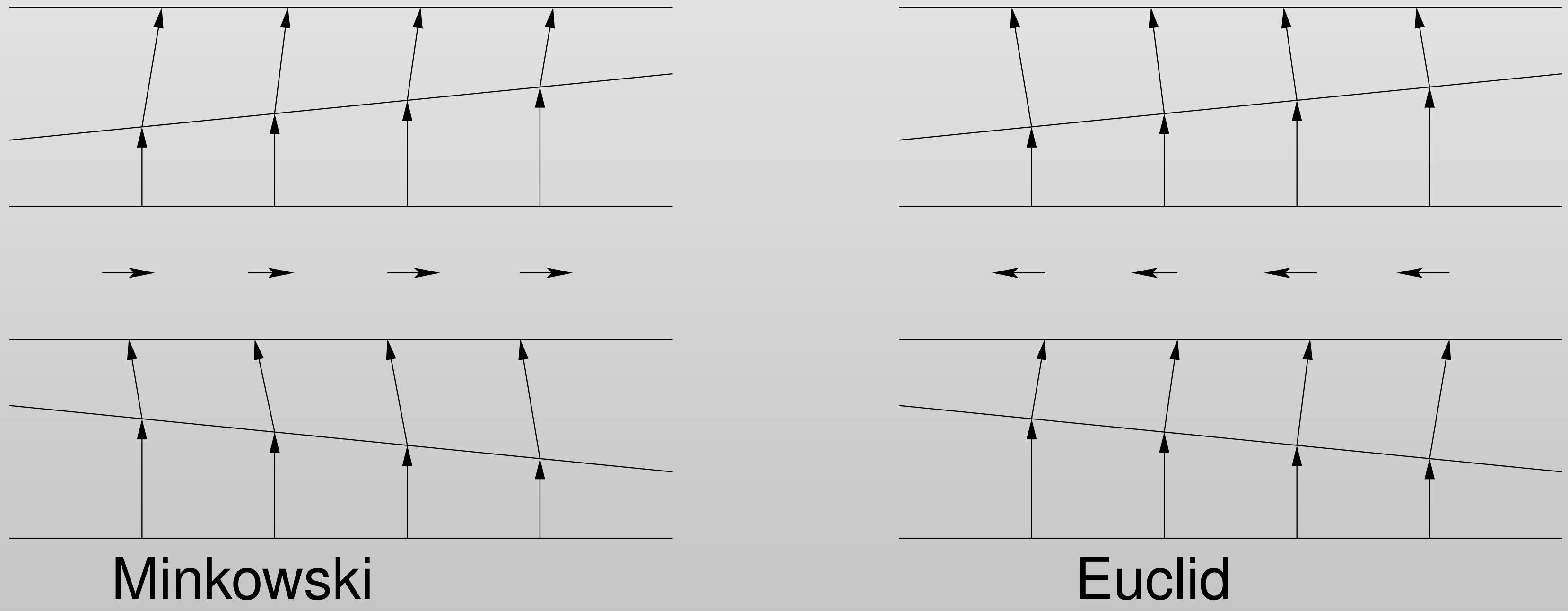
Lorentzian-type hypersurface deformations for $f(p) = p^2$.

Continuous signature change

If $f(p) = \sin^2(\delta p)/\delta^2$ (such that $f(p) \approx p^2$ for $\delta \ll 1$),

$$\beta(p) = \frac{1}{2}\mathrm{d}^2 f/\mathrm{d}p^2 = \cos(2\delta p)$$

not positive definite.



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Algebroids

$$\begin{aligned} [S(\vec{w}_1), S(\vec{w}_2)] &= S(\mathcal{L}_{\vec{w}_1}\vec{w}_2) \\ [T(N), S(\vec{w})] &= -T(\mathcal{L}_{\vec{w}}N) \\ [T(N_1), T(N_2)] &= S(N_1\nabla N_2 - N_2\nabla N_1) \end{aligned}$$

Brackets depend on spatial metric through ∇.
Suggests bundle over space of metrics.

Blohmann, Barbosa Fernandes, Weinstein 2010:

Construction of Lie algebroid with hypersurface deformation brackets when restricted to metric-independent N and $\vec{w}$.

Blohmann, Schiavina, Weinstein 2022:

Lie-Rinehart algebra or L_∞-algebroid for non-constant N and $\vec{w}$, also if constraints are not satisfied.

Evaluating canonical theories

Solution (q,K) of evolution equations $\mathcal{L}_t f(q,K) = X_{T(N)+S(\vec{w})} f(q,K)$ for given N and $\vec{w}$ determines line element

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + ||\mathrm{d}\vec{x} + \vec{w}\mathrm{d}t||_q^2$$

such that K is extrinsic curvature of $t = \mathrm{const}$ slice.

Coordinate transformations of (q,K) given by

$$\mathcal{L}_\xi f(q,K) = X_{T(N\xi^t)+S(\vec{\xi}+\vec{w}\xi^t)} f(q,K)$$

Commutator

$$\mathcal{L}_t\mathcal{L}_\xi - \mathcal{L}_\xi\mathcal{L}_t = \mathcal{L}_{[t,\xi]}$$

determines transformations of N and $\vec{w}$ compatible with dynamical solution.

Modified gravity

Physical evaluation uses Hamiltonian vector fields for general covariance and dynamics

Assumes Poisson bracket, not compatible with L_∞-structure.

Metric-dependent N and $\vec{w}$ suggest new gravitational theories: $T'(N,\vec{w}) = T(\alpha N + \vec{\beta}\cdot\vec{w})$ (same $S(\vec{w})$) may not result in the same space-time solutions if α, β depend on (q,K).

Not guaranteed to provide Riemannian space-time solutions:

- → Must preserve hypersurface-deformation brackets.
- → Must be able to derive

$$\mathcal{L}_\xi f(q,K) = X_{T'(N\xi^t)+S(\vec{\xi}+\vec{w}\xi^t)} f(q,K)$$

and corresponding transformation of $(\vec{w},N)$.

Example

Spherically symmetric gravity:

$$\mathrm{d}s^2 = -N(t,x)^2\mathrm{d}t^2 + \frac{e_2(t,x)^2}{e_1(t,x)}(\mathrm{d}x + w(t,x)\mathrm{d}t)^2 + e_1(t,x)\mathrm{d}\Omega^2$$

with $e_1(t,x) > 0$. Momenta k_1, k_2 of e_1, e_2.

$$S(w) = \int \mathrm{d}x w \left(e_2 \frac{\partial k_2}{\partial x} - \frac{\partial e_1}{\partial x} k_1 \right)$$

$$T(N) = \int \mathrm{d}x N \Big(-\tfrac{e_2 k_2^2}{2\sqrt{e_1}} - 2\sqrt{e_1} k_1 k_2 - \tfrac{e_2}{2\sqrt{e_1}} + \tfrac{1}{8\sqrt{e_1} e_2} \left(\tfrac{\partial e_1}{\partial x}\right)^2 - \tfrac{\sqrt{e_1}}{2e_2^2} \tfrac{\partial e_1}{\partial x} \tfrac{\partial e_2}{\partial x} + \tfrac{\sqrt{e_1}}{2e_2} \tfrac{\partial^2 e_2}{\partial x^2} \Big)$$

Implement linear combination of S and T by linear transformation $N' = \alpha N$, $w' = \beta N + w$.

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Results

[Erick Duque]

If α depends only on e_1 and k_2, it must be of the form

$$\alpha(e_1, k_2) = \mu(e_1)\sqrt{1 - s\lambda(e_1)^2 k_2^2}$$

where $s = \pm 1$. Then,

$$\beta(e_1, k_2) = \mu(e_1)\frac{\sqrt{e_1}}{2e_2^2}\frac{\partial e_1}{\partial x}\frac{s\lambda(e_1)^2 k_2}{\sqrt{1 - s\lambda(e_1)^2 k_2^2}}$$

These conditions guarantee that hypersurface-deformation brackets are obtained for $S(w)$ and $T'(N) = T(\alpha N) + S(\beta N)$ and solutions produce well-defined line elements with

$$q^{xx} = \mu(e_1)^2\left(1 + \frac{1}{4e_2^2}\frac{s\lambda(e_1)^2}{1 - s\lambda(e_1)^2 k_2^2}\left(\frac{\partial e_1}{\partial x}\right)^2\right)\frac{e_1}{e_2^2}$$

Some implications

If $s = 1$, k_2 bounded for $\beta(e_1, k_2) = \mu(e_1)\sqrt{1 - s\lambda(e_1)^2 k_2^2}$ real.
Expressions somewhat simplified by canonical transformation

$$k_2 = \frac{\sin(\delta k_2')}{\delta} \quad , \quad e_2 = \frac{e_2'}{\cos(\delta k_2')} \qquad \text{assuming } \lambda(e_1) = 1$$

In this form, modification had been found earlier and shown to imply black-hole solutions without singularity.

[Alonso-Bardaji, Brizuela 2021, 2022]

Relationship with L_∞-structure clarifies origin of modifications in phase-space dependent linear combination of $S(w)$ and $T(N)$, rather than canonical transformation.

Classification problem still open.