Presymplectic AKSZ form of Einstein gravity

Maxim Grigoriev

Lebedev Physical Institute
&
Institute for Theoretical and Mathematical Physics, Lomonosov MSU

Based on:

M.G., Kotov, 2020; M.G. 2016; Alkalaev, M.G. 2014

ESI program Higher Structures and Field Theory
September 16, 2020

Motivation

- Theories of fundamental interactions (Gravity, YM, Strings, M-Theory, Higher-spin theories . . .) are gauge theories.

- Batalin-(Fradkin-) Vilkovisky (BV/BFV) approach (and its generalizations) gives a proper framework for gauge theories. *Batalin, (Fradkin), Vilkovisky, 1981 . . .* .

- For topological models BV can be brought to a very concise and geometrical *Alexandrov-Kontsevich-Schwartz- Zaboronsky* (AKSZ) form. Reveals underlying structures. Unifies Lagrangian (BV) and Hamiltonian (BFV) formalism.

- AKSZ formulation of a local theory leads to the infinite-dimensional target space. Parent approach *Barnich, M.G, 2010, M.G. 2012*

- An interesting alternative: AKSZ-like sigma model with finite-dimensional presymplectic target gives an elegant construction of frame-like Lagrangians *Alkalaev, M.G.*. However, its BV interpretation remains unclear.

AKSZ construction

Alexandrov, Kontsevich, Schwartz, Zaboronsky, 1994

$\mathfrak{m}$ - supermanifold (target space) with coordinates ψ^A:
Ghost degree − gh()
(odd)symplectic structure $\omega^{\mathfrak{m}}$, gh($\omega^{\mathfrak{m}}$) $= n - 1$ and hence
(odd)Poisson bracket $\{\cdot\,,\,\cdot\}$, gh($\{\cdot\,,\,\cdot\}$) $= -n + 1$
"BRST potential" $\mathcal{H}_{\mathfrak{m}}(\Psi)$, gh($\mathcal{H}_{\mathfrak{m}}$) $= n$, master equation $\{\mathcal{H}_{\mathfrak{m}}, \mathcal{H}_{\mathfrak{m}}\} = 0$
(QP structure: $q = \{\cdot\,, \mathcal{H}_{\mathfrak{m}}\}$ and $P = \{\cdot\,,\,\cdot\}$)

$\mathcal{X}$ - supermanifold (source space)
Ghost degree gh( )
d_X − odd vector field, $d_X^2 = 0$, gh(d_X) $= 1$
Typically, $\mathcal{X} = T[1]X$, coordinates $x^\mu, \theta^\mu \equiv dx^\mu$, $d_X = \theta^\mu \frac{\partial}{\partial x^\mu}$, $\mu = 0, \ldots n - 1$

Supermanifold of supermaps: $\widehat{\sigma} : T[1]X \to \mathfrak{m}$. Coordinates (fields): $\psi^A(x,\theta) := \widehat{\sigma}^*(\psi^A)$. BV master action

$$S^{BV}[\widehat{\sigma}] = \int_{T[1]X} (\widehat{\sigma}^*(\chi)(d_X) - \widehat{\sigma}^*(\mathcal{H}_{\mathfrak{m}}))\,, \qquad \text{gh}(S^{BV}) = 0$$

χ is the potential: $\omega^{\mathfrak{m}} = d\chi$. In components:

$$S_{BV} = \int d^n x d^n\theta \left[\chi_A(\psi(x,\theta)) d_X \psi^A(x,\theta) - \mathcal{H}_{\mathfrak{m}}(\psi(x,\theta))\right]$$

BV symplectic structure:

$$\omega^{BV} = \int_{T[1]X} \widehat{\sigma}^*(\omega^{\mathfrak{m}}_{AB}) \delta\psi^A(x,\theta) \wedge \delta\psi^B(x,\theta)$$

BV antibracket:

$$\Big(F, G\Big) = \int_{T[1]X} \frac{\delta^R F}{\delta\psi^A(x,\theta)} \omega^{AB}_{\mathfrak{m}} \frac{\delta G}{\delta\psi^B(x,\theta)}\,, \qquad \text{gh}\Big(,\Big) = 1$$

Master equation:

$$\Big(S^{BV}, S^{BV}\Big) = 0 \qquad \text{modulo boundary terms}$$

BRST differential:

$$s = \int d^n x d^n \theta (d_X \psi^A(x,\theta) - q^A(\psi(x,\theta)) \frac{\delta}{\delta \psi^A(x,\theta)} \,.$$

Natural lift of q and d_X to the space of supermaps.

Dynamical fields: those of vanishing ghost degree

$$\psi^A(x,\theta) = \overset{0}{\psi}{}^A(x) + \overset{1}{\psi}{}^A_\mu(x)\theta^\mu + \ldots \qquad \mathrm{gh}(\overset{k}{\psi}{}^A_{\mu_1\ldots\mu_k}) = \mathrm{gh}(\psi^A) - k$$

If $\mathrm{gh}(\psi^A) = k$ with $k \geqslant 0$ then $\overset{k}{\psi}{}^A_{\mu_1\ldots\mu_k}(x)$ is dynamical.

AKSZ equations of motion:

$$\omega^{\mathfrak{m}}_{AB}(\psi(x,\theta))(d_X\psi^A - q^A) = 0\,, \quad \Rightarrow \quad (d_X\psi^A(x,\theta) - q^A(\Psi(x,\theta))) = 0$$

provided $\omega^{\mathfrak{m}}_{AB}$ is invertible.

More invariantly, if $\psi^A(x,\theta) = \sigma^*(\psi^A)$ the equations of motion read as:

$$d_X\sigma^*(\psi^A) = \sigma^*(q\psi^A) \qquad \Leftrightarrow \qquad d_X \circ \sigma^* = \sigma^* \circ Q$$

so that σ^* is a morphism of respective complexes. Pure gauge solutions are trivial morphisms, i.e. σ^* of the form

$$\sigma^* = d_X \circ \xi^* + \xi^* \circ q$$

Features:

- "Knows" BV at the very deep level. Field theory encoded in a finite-dimensional graded manifold. In particular, local BRST cohomology is isomorphic to the target space ones *Barnich, M.G. (2009)*

- Unifies BV and BFV. For $X = \Sigma \times \mathbb{R}^1$ taking $T[1]\Sigma$ as a source gives BFV-AKSZ sigma model. *M.G., Damgaard, 2000, M.G. Barnich 2003*. Further developments: *Cattaneo, Mnev, Reshtikhin 2012, M.G, Bekaert 2013, Ikeda Strobl 2019,. . .* . Gives a natural framework to study gauge theories with (asymptotic) boundaries

- Admits nonlagrangian (at the level of equations of motion) version, giving a BV extension of free differential algebras (with constraints)*Sullivan; Fre, D'Auria; Vasiliev*

Presymplectic AKSZ

Alkalaev, M.G., 2013, implicitly in AKSZ

$\mathfrak{m}$ - supermanifold (target space) with coordinates ψ^A:

Ghost degree – gh()

(odd) 2-form $\omega^{\mathfrak{m}}$, gh$(\omega^{\mathfrak{m}}) = n - 1$

Homological vector field q, $q^2 = 0$, gh$(q) = 1$ satisfying

$$L_q \omega^{\mathfrak{m}} = 0\,, \qquad d\omega^{\mathfrak{m}} = 0$$

It follows there is function $\mathcal{H}$, gh$(\mathcal{H}) = n$ satisfying

$$\imath_q \omega^{\mathfrak{m}} = d\mathcal{H}$$

for some $\mathcal{H}, \text{gh}(\mathcal{H}) = n$. Note that $i_q i_q \omega^{\mathfrak{g}[1]} = 0 = q\mathcal{H}$

Source: $\mathcal{X} = T[1]X$

Ghost degree gh( )

$d_X = \theta^\mu \frac{\partial}{\partial x^\mu}$ – odd vector field, $d_X^2 = 0$, gh$(d_X) = 1$

This data defines BRST-like differential:

$$s = \int d^n x d^n \theta (d_X \psi^A(x,\theta) - q^A(\psi(x,\theta)) \frac{\delta}{\delta \psi^A(x,\theta)}$$

BV-like action:

$$S^{BV}[\hat{\sigma}] = \int_{T[1]X} (\hat{\sigma}^*(\chi)(d_X) - \hat{\sigma}^*(\mathcal{H}))$$

which reduced to $S[\sigma]$ if one sets to zero fileds of nonvanishing degree and BV-like presymplectic structure:

$$\omega^{BV} = \int_{T[1]X} \hat{\sigma}^*(\omega^{\mathfrak{m}}_{AB}) \delta\psi^A(x,\theta) \wedge \delta\psi^B(x,\theta)$$

Compatibility (in place of master equation):

$\omega^{BV}(s,s) = 0\,,$ $\qquad i_s \omega^{BV} = \delta S^{BV}$ $\qquad$ modulo boundary terms.

– presymplectic AKSZ sigma model

Is it also topological? What this has to do with the usual BV?

Presymplectic AKSZ form of Cartan-Weyl action

Alkalaev, MG 2013

Take as $\mathfrak{m}$ a graded manifold $\mathfrak{g}[1]$ with $\mathfrak{g}$ being $(A)dS$ or Poincare algebra in n dimensions. Standard coordinates ξ^a, ρ^{ab} correspond to the translation (transvections) and the Lorentz rotation generators. The Lie algebra structure on $\mathfrak{g}$ defines a Q-structure on $\mathfrak{g}[1]$:

$$q\xi^a = \rho^a{}_b \xi^b\,, \qquad q\rho^{ab} = \rho^a{}_c \rho^{cb} + \lambda \xi^a \xi^b,$$

q-invariant presymplectic structure:

$$\omega^{\mathfrak{g}[1]} = \mathcal{V}_{abc}(\xi) d\xi^a d\rho^{bc}\,, \quad \mathcal{V}_{a_1\ldots a_k}(\xi) = \frac{1}{(n-k)!}\epsilon_{a_1\ldots a_k b_1\ldots b_{n-k}} \xi^{b_1} \ldots \xi^{b_{n-k}}$$

Presymplectic potential and BRST potential:

$$\chi = \mathcal{V}_{ab}(\xi) d\rho^{ab}\,, \qquad \mathcal{H} = \mathcal{V}_{ab}(\xi)(\rho^a{}_c \rho^{cb} + \lambda \xi^a \xi^b)$$

A map σ is parameterized by:

$$\sigma^*(\xi^a) = e^a_\mu(x)\theta^\mu\,, \qquad \sigma^*(\rho^{ab}) = \omega^{ab}_\mu(x)\theta^\mu\,,$$

interpreted as frame field and Lorentz connection. Restrict to configuations such that $e^a_\mu(x)$ is invertible.

The action of the AKSZ model:

$$S[\sigma] = \int_{T[1]X} (\sigma^*(\chi)(d_X) - \sigma^*(\mathcal{H})) = \int_X \mathcal{V}_{ab}(e)(d_X\omega^{ab} + \omega^a{}_c\omega^{cb} + \lambda e^a e^b)$$

Familiar Cartan-Weyl action.

What is a BV interpretation of $S^{BV}(\hat{\sigma})$?

Presymplectic BV

Usual BV (M, s, ω)

$$i_s\omega = dS\,, \quad d\omega = 0\,,$$

ω-nondegenerate.

Take ω possibly degenerate but regular. Kernel distribution $K \subset TM$ of ω is involutive and

$$YS = 0, \qquad L_Y\omega = 0 \quad \forall Y \in K$$

On the space of leaves N (in general, exists only locally) one gets:

$$i_{s_N}\omega^N = dS_N\,, \quad d\omega_N = 0\,,$$

ω_N is invertible, i.e. standard BV setting is recovered.

Slight generalization of the standard BV. No need to explicitly take the quotient explicitly. i.e. formal path integral

$$\int_L \exp \frac{i}{\hbar} S$$

where gauge-fixing submanifold L projects to a Lagrangian submanifold of N. In components: gauge fixing conditions $G_\alpha = 0$ are to be suplemented with $T_i = 0$ identifying N as a submanifold of M.

In field theory some care is to be taken. Safe situation:

$$M \to J_X(M) = Maps(X, M)$$

Then $J_X(N)$ is a usual BV. As we are going to see this happens for presymplectic AKSZ gravity

Supermanifold M

Presymplectic AKSZ for gravity: target $(\mathfrak{g}[1], q, \omega^{\mathfrak{g}[1]})$, source $(T[1]X, d_X)$. Consider $M = Smaps(T_x[1]X, \mathfrak{g}[1])$ so that (locally)

$$Smaps(T[1]X, \mathfrak{g}[1]) = Smaps(X, Smaps(T_x[1]X, \mathfrak{g}[1]))$$

M is a finite-dimensional presymplectic manifold. Coordinates ψ^A:

$$\hat{\sigma}^*(\xi^a) = \overset{0}{\xi}{}^a + e^a_\mu \theta^\mu + \overset{2}{\xi}{}^a_{\mu\nu}\theta^\mu\theta^\nu + \dots\,,$$
$$\hat{\sigma}^*(\rho^{ab}) = \overset{0}{\rho}{}^{ab} + \omega^{ab}_\mu \theta^\mu + \overset{2}{\rho}{}^{ab}_{\mu\nu}\theta^\mu\theta^\nu + \dots\,,$$

form-degree k components carry ghost degree $1-k$. Presymplectic structure:

$$\omega^M = \int d^n\theta\, \omega^{\mathfrak{g}[1]}_{AB}(\psi(\theta))d\psi^A(\theta) \wedge d\psi^B(\theta)\,,$$

Proposition: ω^M is regular provided e^a_μ is invertible.

Proof: consider submanifold $M_0 \subset M$:

$$\xi^a = 0, \qquad \overset{2}{\xi}{}^a_{\mu\nu} = 0, \qquad \dots, \qquad \overset{n}{\xi}{}^a_{\mu_1\dots\mu_n} = 0$$

Consider ω^M at a given $p \subset M_0$. Changing the basis in $T_x X$ one can assume $e^a_\mu = \delta^a_\mu$ and one gets (from now on $n = 4$):

$$\omega^M_p = de^b_c \wedge d\overset{2}{\rho}{}^c_b + d\xi^b \wedge d\overset{3}{\rho}_b + d\rho^{ab} \wedge d\overset{3}{\xi}_{ab} + d\omega^{ab}_c \wedge d\overset{2}{\xi}{}^c_{ab}$$

where $\overset{2}{\rho}{}^c_b$, $\overset{3}{\rho}_b$, $\overset{3}{\xi}_{ab}$, and $\overset{2}{\xi}{}^c_{ab}$ parameterize the following components:

$$\overset{2}{\rho}{}^{cd}_{bd}\,, \qquad \overset{3}{\rho}{}^{cd}_{bcd}\,, \qquad \overset{3}{\xi}{}^c_{abc}\,, \qquad \overset{2}{\xi}{}^c_{ab}\,.$$

Spectrum of coordinates required for minimal BV formulation of GR. The rank can't drop off M_0.

Consider the following distribution on $\mathfrak{g}[1]$:

$$X^4_a = \xi^{(4)}\frac{\partial}{\partial \xi^a}\,, \qquad X^3_{ab} = \xi^{(3)}_a\frac{\partial}{\partial \xi^b} + \xi^{(3)}_b\frac{\partial}{\partial \xi^a}$$

$$Y^4_{ab} = \xi^{(4)}\frac{\partial}{\partial \rho^{ab}}\,, \quad Y^3_{abc} = \xi^{(3)}_a\frac{\partial}{\partial \rho^{bc}} + \xi^{(3)}_b\frac{\partial}{\partial \rho^{ac}}\,, \quad Y^2_{abcd} = \xi^{(2)}_{ab}\frac{\partial}{\partial \rho^{cd}} + \dots\,,$$

where ... symmetrization in ac and bd and

$$\xi^{(k)}_{a_1 \dots a_{4-k}} = \frac{1}{k!}\epsilon_{a_1 \dots a_{4-k} a_{4-k+1} \dots a_4}\xi^{a_{4-k+1}} \dots \xi^{a_4}$$

. All these are in the kernel of $\omega^{\mathfrak{g}[1]}$ and commute to one another.

Consider natural prolongation to M

$$\widehat{X}\psi^A(\theta) = X^A(\psi(\theta))$$

At M_0, the distribution determined by $\widehat{X}$ and $\hat{Y}$ coincides with the $Ker(\omega^M)$. Because the dimension of the distribution can't drop off M_0 it holds everywhere. We've proved regularity!

Presymplectic BV for Cartan-Weyl

The above shows that (S^{BV}, s, ω^{BV}) gives presymplectic BV.

N can be relaized as an explicit submanifold, e.g. by zero locus of:

$$\overset{4}{\xi}{}^{a|}\,,\quad \overset{4}{\rho}{}^{ab|}\,,\quad \overset{3}{\xi}{}^{a|\mu}e^{b}_{\mu}+(ab)\,,\quad \overset{3}{\rho}{}^{ab|\mu}e^{c}_{\mu}+(ac)\,,\quad \overset{2}{\rho}{}^{ab|\mu\nu}e^{c}_{\mu}e^{d}_{\nu}+(ac)(bd)\,,$$

where e.g. $\overset{3}{\xi}{}^{a|\mu}$ stand for $\overset{3}{\xi}{}^{a}_{\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma}$.

In particular, coordinates $e^{a}_{\mu}, \omega^{\mu}_{ab}$ are not restricted so that S^{BV} restricts to the Cartan-Weyl action $S[e, \omega]$ when nonzero degree coordinates are set to zero. Together with:
(i) S^{BV}_{N} satisfies master equation on N
(ii) S^{BV}_{N} is proper (easy to prove by linearization).
this shows that we indeed arrived at a proper BV formulation of Einstein gravity.

Origin of the target space structures

Start with metric gravity: metric g^{ab}, diffeomorphism ghosts ξ^a and their canonically conjugate antifields g^*_{ab} and ξ^*_a. Underlying bundle $F \times X \to X$. The BV-BRST complex is given by local horizontal forms on the associated jet-bundle $J_X(F)$ equipped with the BRST differential s and the horizontal differential d_{h}.

The standard BV symplectic structure:

$$\omega^{sBV} = (dx)^n (d_{\mathsf{v}} g^{ab} \wedge d_{\mathsf{v}} g^*_{ab} + d_{\mathsf{v}} \xi^a \wedge d_{\mathsf{v}} \xi^*_a)\,, \quad \mathrm{gh}(\omega^{sBV}) = -1$$

The BRST differential is an evolutionary vector field satisfying

$$i_s \omega^{sBV} = d_{\mathsf{v}} L^{sBV} + d_{\mathsf{h}}(\cdot)\,,$$

where L^{sBV} is the integrand of the BV master action.

Complete ω^{sBV} to a cocycle of the total BRST differential $\widetilde{s} = d_{\mathsf{h}} + s$:

$$\omega^{tBV} = \omega^{sBV} + \omega^{sBV}_{n-1} + \dots \,, \qquad L_{\widetilde{s}}\omega^{tBV} = 0$$

ω^{sBV}_k of horizontal form degree k and $\mathrm{gh}(\omega^{sBV}_k) = n - 1 - k$. Related descent forms studied recently *Sharapov 2016*.

Equivalence of total BRST complexes: elimination of contractible pairs for $\widetilde{s} = d_{\mathsf{h}} + s$.

Parent formulation: AKSZ sigma model with the target being total BRST complex. *Barnich, M.G. 2010*

Elimination of contractible pairs corresponds to elimination of generalized auxiliary fields in the parent formulation i.e. leads to equivalent parent formulation.

In the case of GR one can eliminate: antifields and their derivatives together with (derivatives) of equations of motion as well as some components of (derivatives) of metric and ghosts. Finally,

$$q\xi^a = \xi^a{}_c\,\xi^c\,, \qquad q\rho^{ab} = \rho^a{}_c\,\rho^{cb} + \lambda\xi^a\xi^b + \xi^c\xi^d W^{ab}_{cd}\,, \qquad \dots\,,$$

Minimal BRST complex for gravity. *Have been independently found by many authors. Mention R. Stora, F. Brandt*

Gives minimal AKSZ formulation (BV extension of unfolded formulation).

As a cocycle of $\widetilde{s}$, the total symplectic structure ω^{tBV} determines a respective cocycle of q in the minimal BRST complex:

$$\omega^{\mathcal{E}} = \epsilon_{abcd}\xi^a d\xi^b d\rho^{cd}\,.$$

The finite-dimensional $(\mathfrak{g}[1], q, \omega^{\mathfrak{g}[1]})$ is arrived at by factoring out W-coordinates.

BV extension of the intrinsic Lagrangian construction *M.G. 2016*

BFV phase space from presymplectic AKSZ

Given an AKSZ model with source $T[1](\Sigma \times \mathbb{R}^1)$, consider AKSZ model with source $T[1]\Sigma$. This will give BRST charge and the Poisson bracket of the associated BFV formulation *M.G. Damgaard 2000, Barnich, M.G. 2003*.

The same for presymplectic AKSZ gravity:
manifold $M_H = Smaps(T_x[1]X, \mathfrak{g}[1])$, presymplectic structure ω^{M_H} and Hamiltonian potential $\mathcal{H}_H$.

Submanifold M_H^0:

$$\xi^a = 0, \quad \overset{2}{\xi}{}^a_{ij} = 0\,, \ldots\,,, \quad \overset{n-1}{\xi}{}^a_{i_1 \ldots i_{n-1}} = 0$$

At a given point of M_H^0

$$\omega_p^{M_H} = de_j^i \wedge d\omega_i^{0j} + de_j^0 \wedge d\omega^j + d\xi^j \wedge d\overset{2}{\rho}_j + d\xi^0 \wedge d\overset{2}{\rho} + d\rho^{k0} \wedge d\overset{2}{\xi}_k + d\rho^{kj} \wedge$$

Correct spectrum for BFV phase space! where $\omega^j, \overset{2}{\rho}_j,\ \overset{2}{\rho},\ \overset{2}{\xi}_k$ parameterize the following components: $\overset{2}{\rho}{}^{cd}_{bd}, \overset{3}{\rho}{}^{cd}_{bcd}, \overset{3}{\xi}{}^{c}_{abc}, \overset{2}{\xi}{}^{c}_{ab}$.

However, M^H is not a regular symplectic manifold and a suitable symplectic quotient is not so easy to define.

Nevertheless there exists a symplectic submanifold $N_H \subset M_H$ which is a BFV phase space (more preciesley the version proposed recently by *Cattaneo, Caneppa, Schiavina 2020*)

How the complete BFV formulation is recovered on N_H requires further study.

Conclusions and outlook

- Presymplectic AKSZ form of gravity naturally encodes full-scale BV formulation through presymplectic BV

- Presymplectic BV can be employed just like usual BV. No need to take symplectic quotient explicitly

- Can be identified as a subsector of the parent formulation associated to the minimal BRST complex. The presymplectic structure originates as a homotopy transfer of the descent completion of the BV symplectic structure

- The approach extends to generic gauge theories. The difference is that for non diffeomorphims invariant systems one arrives at Q-bundle equipped with presymplectic structure rather than presymplectic AKSZ. Presymplectic extension of the parent formulation *Barnich M.G. 2010, M.G. 2012, Kotov M.G. 2019*

- Presymplectic AKSZ naturally reproduces the main structures of BFV formulation. Useful in studying boundary behaviour. Requires further study because the BFV-AKSZ presymplectic structure is not regular.