Cumulants, Hausdorff Series, and Quasisymmetric Functions

T. Hasebe, F. Lehner, J.-C. Novelli, J.-Y. Thibon

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Classical cumulants

Let $m_n = m_n(X) = \mathbf{E}\, X^n$ be the **moments** of a random variable X.

The **cumulants** are characterized by the following properties

(K1) **Additivity**: If X and Y are independent random variables, then

$$\kappa_n(X+Y) = \kappa_n(X) + \kappa_n(Y).$$

(K2) **Homogeneity**: For any scalar λ the n-th cumulant is n-homogeneous:

$$\kappa_n(\lambda X) = \lambda^n \kappa_n(X).$$

(K3) **Universality**: There exist universal polynomials P_n in $n-1$ variables without constant term such that

$$m_n(X) = \kappa_n(X) + P_n(\kappa_1(X), \kappa_2(X), ..., \kappa_{n-1}(X)).$$

Generating function

The exponential generating functions satisfy the identity

Definition.

$$\underline{\mathbb{E}}\, e^{tX} = \sum_{n=0}^{\infty} \frac{m_n}{n!}\, t^n = \exp \sum_{n=1}^{\infty} \frac{\kappa_n}{n!}\, t^n$$

Thiele (1889): "halvinvarianter",
Hausdorff (1901): "logarithmische Momente"

Symmetric functions

compare with **symmetric functions**

$$H_t(X) = \sum_{n=0}^{\infty} h_n(X) t^n = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(X)}{n} t^n \right)$$

$$h_n = \sum_{i_1 \leq i_2 \dots \leq i_n} x_{i_1} \cdots x_{i_n}$$

$$p_n = \sum_i x_i^n$$

Sym freely generated by h_n

Character

$$\chi_X(h_n) = \frac{m_n(X)}{n!}$$

$$\chi_X(p_n) = (n-1)!\,\kappa_n$$

Analogy goes further!

Coproducts

$$X \otimes 1 \qquad 1 \otimes X$$

$$\Delta f(X,Y) = f(X \cup Y) =: f(X+Y)$$
$$\delta f(X,Y) = f(X \times Y) =: f(XY)$$

$(Sym, \cdot, \Delta)$ is a Hopf algebra.

$$XY = \{x_i y_j \mid i, j \in \mathbb{N}\}$$

$$x_n(X+Y) = x_n(X) + x_n(Y) \qquad p_n(X+Y) = p_n(X) + p_n(Y)$$

$$m_n(X+Y) = \sum \binom{n}{k} m_k(X) m_{n-k}(Y) \qquad h_n(X+Y) = \sum h_k(X) h_{n-k}(Y)$$

$$\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n$$
$$\Delta(h_n) = \sum h_k \otimes h_{n-k}$$

Formalization of independence

noncomm p.s. $\mathcal{A}$ algebra with 1

Let $(\mathcal{A}, \varphi)$ be a ncps.

$\varphi : \mathcal{A} \to \mathbb{C}$ unital

X and Y are **independent** if

$$\varphi(XY) = \varphi(X)\varphi(Y)$$

or formally

$$(X, Y) \overset{d}{\simeq} (X \otimes 1, 1 \otimes Y)$$

in $(\underline{\mathcal{A} \otimes \mathcal{A}}, \underline{\varphi \otimes \varphi})$

(X, Y) are indep

$(X, Y) \overset{d}{\simeq} (X^{(1)}, Y^{(2)})$ where $(X^{(1)}, Y^{(1)})$

$(X^{(2)}, Y^{(2)})$

are iid copies of (X, Y)

Algebraic setup

For a given ncps $(\mathcal{A}, \varphi)$ let

$$\mathcal{U} = \mathcal{A}^{\otimes\infty}$$
$$\tilde{\varphi} = \tilde{\varphi}^{\otimes\infty}$$

and embed

$$X \mapsto X^{(i)} = I \otimes I \otimes \cdots I \otimes X \otimes I \otimes \cdots$$

Similarly, X and Y are **free**, **Boolean independent** etc., if $(X, Y) \stackrel{d}{\simeq} (X^{(1)}, Y^{(2)})$ where $\mathcal{U} = \mathcal{A}^{*\infty}$ free product, etc.

$$X^{(1)} = X \otimes I \otimes I \cdots$$

$$X^{(2)} = I \otimes X \otimes I \cdots$$

action of σ_∞ :

$$\tilde\varphi\left(X_1^{(1)} X_2^{(3)} X_3^{(1)} X_4^{(2)} X_5^{(3)} \right)$$

$$= \tilde\varphi\left(X_1^{(5)} X_2^{(4)} X_3^{(5)} X_4^{(7)} X_5^{(4)} \right)$$

$$=: \varphi_\pi (X_1, X_2 \ldots X_5)$$

$\pi =$ 

Lattice reformulation of independence

Definition. Subalgebras $\mathcal{A}_j$ are **independent** if

$$\varphi_{\pi}(X_1, X_2, \ldots, X_n) = \varphi_{\pi \wedge \eta}(X_1, X_2, \ldots, X_n)$$

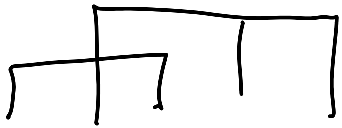
whenever η is a partition of X_i such that the X_i from each block come from one of the subalgebras and subalgebras for different blocks are different.

e.g. $\{X_1, X_3\} \perp\!\!\!\perp \{X_2, X_4, X_5\}$

$$\varphi_{\pi}(X_1, X_2, X_3, X_4, X_5) = \varphi\left(X_1^{(1)} X_2^{(1)} X_3^{(1)} X_4^{(2)} X_5^{(2)}\right)$$

$$= \varphi\left(X_1^{(1)} X_2^{(3)} X_3^{(1)} X_4^{(2)} X_5^{(2)}\right)$$

$= \varphi$  $(x_1 \dots x_5)$

 $=$  $\wedge$ 

$\pi \wedge \eta$ $\qquad\qquad$ π $\qquad\qquad$ η

lattice of set partitions

General setting:

$$(\mathcal{A}, \varphi) \hookrightarrow (\mathcal{U}, \hat{\varphi})$$

$$x \longmapsto x^{(i)}$$

$$\pi \wedge \eta = \{ B \cap C \mid B \in \pi, \ C \in \sigma \}$$

$\pi \le \sigma$ if every block of π is contained in block of σ

$\tilde{\varphi}$ is invariant under permutations:

$$\tilde{\varphi}\left(X_1^{(i_1)} \ldots X_n^{(i_n)}\right) = \tilde{\varphi}\left(X_1^{(\sigma(i_1))} \ldots X_n^{(\sigma(i_n))}\right)$$

$$\forall \sigma \in \mathfrak{S}_\infty$$

Cumulants

Rota's dot operation

$$N.X = X^{(1)} + X^{(2)} + \cdots + X^{(N)}$$

$$\tilde{\varphi}((N.X_1)(N.X_2)\cdots(N.X_n)) = N \cdot K_n(X_1, X_2, \ldots, X_n) + \omega(N^2)$$

$$\tilde{\varphi}(N.X) = \tilde{\varphi}(X^{(1)} + \cdots + X^{(N)}) = N \cdot \varphi(X)$$

$$\tilde{\varphi}((N.X)(N.Y)) = \tilde{\varphi}((X^{(1)} + \cdots + X^{(N)})(Y^{(1)} + \cdots Y^{(N)}))$$

$$= \sum_{i,j} \tilde{\varphi}(X^{(i)} Y^{(j)})$$

$$= \sum_{i=j} \hat{\varphi}(X^{(i)} Y^{(i)}) + \sum_{i \neq j} \hat{\varphi}(X^{(i)} Y^{(j)})$$

$$= N \varphi_{\sqcap}(X, Y) + N(N-1)\, \varphi_{11}(X, Y)$$

$$\pi = \sqcap$$

$$= N \underbrace{\left(\varphi(XY) - \varphi_{11}(X,Y)\right)}_{K_2} + N^2 \varphi_{11}(X,Y), \qquad \varphi_{11}(X,Y) = \varphi(X)\varphi(Y)$$

Thm $K_n(x_1 \ldots x_n) = \sum_{\pi \in P(n)} \varphi_\pi(x_1 \ldots x_n)\, \mu(\pi, 1)$

Partitioned cumulants and Möbius inversion

$$\varphi_\pi(N.X_1, N.X_2, \ldots, N.X_n) = N^{|\pi|} K_\pi(X_1, X_2, \ldots, X_n) + \omega(N^{|\pi|+1})$$

Theorem.

$$K_\pi(X_1, X_2, \ldots, X_n) = \sum_{\sigma \leqslant \pi} \varphi_\sigma(X_1, X_2, \ldots, X_n)\, \mu(\sigma, \pi)$$

Mixed cumulants

Theorem. *Independence $\iff$ mixed cumulants vanish.*

i.e. $K_\pi(X_1 \dots X_n) = 0$

where in same block of π

there are independent r.v.

$K_n(X_1 \dots X_n) = 0$

if $X_1 \dots X_n$ can be split into

two independent subsets

i.e. if $\exists$ partition η of $X_1 \dots X_n$ into indep subsets

s.t. $\pi \nleq \eta$

$\varphi_\pi = \varphi_{\pi \wedge \eta}$ $\qquad$ $K_\pi(X_\eta) = 0$

indep $\qquad$ if $\pi \not\leq \eta$

$$\Leftarrow \quad \varphi_\pi = \sum_{\sigma \leq \pi} K_\sigma = \sum_{\substack{\sigma \leq \pi \\ \sigma \leq \eta}} K_\sigma = \sum_{\sigma \leq \pi \wedge \eta} K_\sigma = \varphi_{\pi \wedge \eta}$$

$\Rightarrow$ Weisner's lemma

Weisner's Lemma (1935)

P a lattice, $a, b, c \in P$, then

$$\sum_{\substack{x \in P \\ x \wedge a = b}} \mu(x, c) = \begin{cases} \mu(b, c) & a \geqslant c \\ 0 & \text{otherwise} \end{cases}$$

$$K_\pi = \sum_{\sigma \leq \pi} \varphi_\sigma \, \mu(\sigma, \pi)$$

$$= \sum_{\sigma \leq \pi} \varphi_{\sigma \wedge \eta} \, \mu(\sigma, \pi)$$

$$= \sum_{\tau} \Bigg(\underbrace{\sum_{\substack{\sigma \leq \pi \\ \sigma \wedge \eta = \tau}} \mu(\sigma, \pi)}_{=0} \Bigg) \cdot \varphi_\tau$$

because $\pi \not\leq \eta$

NC symmetric functions

$X = \{X_1, X_2, \ldots\}$ noncommutative alphabet.
WSym is the algebra generated by the monomial symmetric functions

$$\mathbf{m}_\pi = \sum_{\ker \underline{i} = \pi} X_{i_1} X_{i_2} \ldots X_{i_n}$$

$$\mathbf{m}_{\sqcap\!\sqcap} = \sum_{i \neq j} X_i X_j X_i X_j$$

$\ker \underline{i} \hat{=}$ Equiv rel

$k \sim l$ if $i_k = i_l$

NC power sums

$$
\begin{aligned}
\boldsymbol{\phi}_\pi &= \sum_{\ker \underline{i} \geqslant \pi} X_{i_1} X_{i_2} \dots X_{i_n} \\
&= \sum_{\sigma \geqslant \pi} \mathbf{m}_\sigma \\
\mathbf{m}_\pi &= \sum_{\sigma \geqslant \pi} \mu(\pi, \sigma) \boldsymbol{\phi}_\sigma
\end{aligned}
$$

$\hookrightarrow$ wrong order of cumulants

$$
\phi_{\sqcap\!\sqcap} = \sum_{i,j} X_i X_j X_i X_j
$$

Calculation rules

$$\mathbf{m}_\pi \mathbf{m}_\rho = \sum_{\sigma \wedge (\hat{1}_m | \hat{1}_n) = \pi | \rho} \mathbf{m}_\sigma$$

$$\boldsymbol{\phi}_\pi \boldsymbol{\phi}_\sigma = \boldsymbol{\phi}_{\pi|\sigma}$$

Coproduct

As before the coproduct

Commuting copies of X

$$\Delta F(X, Y) = \Delta F(X + Y)$$

is cocommutative.

Dual basis

The dual $\mathbf{WSym}^*$ is commutative. Define dual bases N^π and Φ^π by

$$\langle N^\pi, \mathbf{m}_\sigma \rangle = \delta_{\pi,\sigma}$$
$$\langle \Phi^\pi, \boldsymbol{\phi}_\sigma \rangle = \delta_{\pi,\sigma}$$

Then

$$N^\pi = \sum_{\sigma \leqslant \pi} \Phi^\sigma$$
$$\Phi^\pi = \sum_{\sigma \leqslant \pi} N^\sigma \mu(\sigma, \pi)$$

good order for cumulants

“Character”

Given a sequence $(X_i) \subseteq \mathcal{A}$, we define a linear map

$$\hat{\varphi} : \mathbf{WSym}^* :\to \mathbb{C}$$
$$\hat{\varphi}(N^\pi) = \varphi_\pi(X_1, X_2, \ldots, X_n)$$

Then cumulants are encoded by

$$\hat{\varphi}(\Phi^\pi) = K_\pi(X_1, X_2, \ldots, X_n)$$

Internal product

The internal product on **WSym*** is inherited from **WQSym*** (later) and takes the form

$$N^\pi * N^\sigma = N^{\pi \wedge \sigma}$$

and thus is an incarnation of the **Möbius algebra** of the partition lattice.

$$\delta f(X, Y) = f(X \cdot Y)$$

$$X \cdot Y = \{ X_i Y_j \mid i, j \in \mathbb{N} \}$$

with lex order

"internal product"

Möbius algebra: $\mathbb{Z}[\mathcal{P}(n)]$

product: $\pi \cdot \sigma := \pi \wedge \sigma$

Möbius idempotents

Weisner

$e_\pi := \sum_{\sigma \leqslant \pi} \sigma \mu(\sigma, \pi)$ are orthogonal idempotents in the Möbius algebra $\mathbb{Z}[\Pi_n]$ and thus

$$\Phi^\pi = \sum_{\sigma \leqslant \pi} N^\sigma \mu(\sigma, \pi)$$

are orthogonal idempotents in $\mathbf{WSym}^*$ with respect to the internal product.

Independence and mixed cumulants revisited

Whenever $\eta \in \Pi_n$ is a partition of X_i into mutually independent, then

$$\hat{\varphi}(N^{\pi}) = \hat{\varphi}(N^{\pi \wedge \eta}) = \hat{\varphi}(N^{\pi} * N^{\eta})$$

and thus

$$K_{\pi}(X_1, X_2, \ldots, X_n) = \hat{\varphi}(\Phi^{\pi}) = \hat{\varphi}(\Phi^{\pi} * N^{\eta}) = 0$$

because for $\pi \not\leqslant \eta$ we have

$$\Phi^{\pi} * N^{\eta} = 0$$

Spreadability and ordered set partitions

A **spreadability system** is an algebra $(\mathcal{U}, \tilde{\varphi})$ and a family of embeddings

$$\begin{aligned} \mathcal{A} &\to \mathcal{U} \\ X &\mapsto X^{(i)} \end{aligned}$$

such that

$$\tilde{\varphi}(X_1^{(i_1)} X_2^{(i_2)} \cdots X_n^{(i_n)}) = \tilde{\varphi}(X_1^{(h(i_1))} X_2^{(h(i_2))} \cdots X_n^{(h(i_n))})$$

for every *strictly increasing* map $h : \mathbb{N} \to \mathbb{N}$.

quasi-symmetry

Spreadability

$$(X^{(1)}, Y^{(2)}, Z^{(3)}) \overset{d}{\simeq} (X^{(3)}, Y^{(7)}, Z^{(20)})$$

$$\tilde{\varphi}\left(x_1^{(1)} x_2^{(3)} x_3^{(1)} x_4^{(2)} x_5^{(3)}\right)$$

$$= \tilde{\varphi}\left(x_1^{(5)} x_2^{(7)} x_3^{(5)} x_4^{(6)} x_5^{(7)}\right)$$

1 2 3

ordered set partition = set composition

1 3 1 2 3 packed word

$$=: \varphi_{13123}(x_1, \dots, x_5)$$

Packed words

A **packed word** is a word $w = w_1 w_2 \ldots w_n$, with $w_i \in \mathbb{N}$ such that no letter is left out, i.e., if k occurs, then all $l < k$ occur as well.
Packed words encode ordered set partitions.
Any word can be arranged into a packed word

$$\mathrm{pack}(w) \simeq \ker w$$

i.e., if $b_1 < b_2 < \cdots < b_k$ are the letters occuring in w, then $\mathrm{pack}(w)$ is obtained by replacing each b_j by j.

Independence

X and Y are **independent** if

$$(X, Y) \overset{d}{\simeq} (X^{(1)}, Y^{(2)})$$

(but not necessarily $(X, Y) \overset{d}{\simeq} (X^{(2)}, Y^{(1)})$).
Partitioned moments ϕ_π or "packed moments" ϕ_u are analogously.

example: monotone indep

X indep of Y

$\not\Rightarrow$ Y indep of X

independence:

$$\varphi_\pi(X_1 \ldots X_n) = \varphi_{\pi \wedge \eta}(X_1 \ldots X_n) \qquad \text{remove } \emptyset$$

$$\pi \wedge \eta = (B_1 \cap C_1, B_1 \cap C_2, \ldots, B_1 \cap C_\ell, B_2 \cap C_1, \ldots)$$

$\longleftrightarrow$ Solomon-Tits algebra

$\rightsquigarrow$ mixed cumulants do not vanish !

Quasisymmetric functions

Let $X = X_1, X_2, \ldots$ be an (infinite) alphabet.
A **quasisymmetric function** is a formal power series

$$f(x_1, x_2, \ldots) = \sum x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

such that the coefficients are invariant under spreadings only:

$$[x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_n}^{\alpha_n}]f = [x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}]f$$

for any sequence $i_1 < i_2 < \cdots < i_n$.

NC Quasisymmetric functions

Let $\boldsymbol{X} = X_1, X_2, \ldots$ be an (infinite) noncommuting alphabet. The algebra **WQSym** of **noncommutative (word) quasisymmetric functions** is spanned by the "monomials"

$$\boldsymbol{M}_{\underline{u}} = \sum_{\mathrm{pack}(w)=u} X_w \qquad = \sum_{w}{}' X_w$$

packed word

che $w = u$

Again we define

$$\Delta(f) = f(\boldsymbol{X} \oplus \boldsymbol{Y})$$
$$\delta(f) = f(\boldsymbol{X} \times \boldsymbol{Y})$$

where $\boldsymbol{X} \oplus \boldsymbol{Y}$ is the ordered sum of alphabets and $\boldsymbol{X} \times \boldsymbol{Y}$ carries the lexicographic order.

Duality

WQSym is noncommutative and non-cocommutative.
Let $\boldsymbol{N}_u$ be the dual basis of $\boldsymbol{M}_u$

$$\langle \boldsymbol{N}_u, \boldsymbol{M}_v \rangle = \delta_{u,v}$$

We define as before

$$\hat{\varphi}(\boldsymbol{N}_u) = \varphi_\pi(X_1, X_2, \ldots, X_n)$$

where $\pi = \ker u$ (ordered kernel).

Solomon-Tits algebra

Let η be an ordered partition of X_i into mutually independent subsets, then

$$\phi_\pi(X_1, X_2, \ldots, X_n) = \phi_{\pi \curlywedge \eta}(X_1, X_2, \ldots, X_n)$$

where

$$\pi \curlywedge \eta = (\pi_1 \cap \eta_1, \pi_1 \cap \eta_2, \ldots)$$

i.e., intersection $\pi_i \cap \eta_j$ in lexicographic order.
In terms of packed words this is the internal product on $\mathbf{WQSym}^*$ (induced by δ), which is isomorphic to the **Solomon-Tits algebra**.

Cumulants

Cumulants are defined as before

$$\tilde{\varphi}((N.X_1)(N.X_2)\cdots(N.X_n)) = N \cdot K_n(X_1, X_2, \ldots, X_n) + \omega(N^2)$$

$$\tilde{\varphi}((N \cdot X)(N \cdot Y)) = \sum_{i \neq j} \tilde{\varphi}(X^{(i)} Y^{(j)})$$

$$= \sum_{i>j} + \sum_{i=j} + \sum_{i<j} \tilde{\varphi}(X^{(i)} Y^{(j)})$$

$$= \frac{N(N-1)}{2} \varphi_{21}(X,Y) + N \varphi_{11}(X,Y) + \frac{N(N-1)}{2} \varphi_{12}(X,Y)$$

$$= N\Big(\underbrace{\boxed{\varphi(XY) - \tfrac{1}{2}\big(\varphi_{12}(X,Y) + \varphi_{21}(X,Y)\big)}}_{K_2} \Big)$$

$$K_n = \sum_{\pi \in \mathcal{P}(n)} \varphi_\pi \, \tilde{\mu}(\pi, 1)$$

Factorial Möbius inversion

Theorem.

$$\varphi_\pi(X_1, X_2, \dots, X_n) = \sum_{\sigma \leqslant \pi} \varphi_\pi(X_1, X_2, \dots, X_n)\, \tilde{\zeta}(\sigma, \pi)$$

$$K_\pi(X_1, X_2, \dots, X_n) = \sum_{\sigma \leqslant \pi} \varphi_\pi(X_1, X_2, \dots, X_n)\, \tilde{\mu}(\sigma, \pi)$$

where

$$\tilde{\zeta}(\sigma, \hat{1}) = \frac{1}{|\sigma|!}$$

$$\tilde{\mu}(\sigma, \hat{1}) = \frac{(-1)^{|\sigma|-1}}{|\sigma|} = \frac{\mu(\bar{\sigma}, \hat{1})}{|\sigma|!}$$

coeff of $\log(1+x) = \sum \frac{(-1)^{n-1}}{n} x^n$

Eulerian idempotents

If we set as before

$$\hat{\varphi}(\boldsymbol{N}_u) = \varphi_\pi(X_1, X_2, \ldots, X_n)$$

where $\pi = \ker u$, then

$$K_\pi(X_1, X_2, \ldots, X_n) = \hat{\varphi}(\boldsymbol{N}_u * E_n^{[r]})$$

where $r = |\pi|$ and $E_n^{[r]}$ is the so-called **Euler idempotent**.

Mixed cumulants

Theorem. *Whenever X_i can be partitioned into mutually independent subsets, say into $\eta \in \Pi_n$, then*

$$K_n(X_1, X_2, \ldots, X_n) = \sum_{\tau} K_\tau(X_1, X_2, \ldots, X_n)\, g(\tau, \eta)$$

*where $g(\tau, \eta)$ are the **Goldberg coefficients** appearing in the Campbell-Baker-Hausdorff series.*

$$e^X e^Y = e^Z$$

$$Z = \sum_{w \in \{X, Y\}^*} g_w \, w$$

CBH series

$$K_n(X_1+X_2) = K_n(X_1+X_2, \dots, X_1+X_2)$$

$$= \sum_{i_1 \dots i_n = 1}^{2} K_n(X_{i_1} \dots X_{i_n})$$

$$= K_n(X_1 \dots X_1)$$

$$+ K_n(X_2 \dots X_2)$$

$$= K_n(X_1) + K_n(X_2)$$