Kinematic Varieties I
Massless Particles

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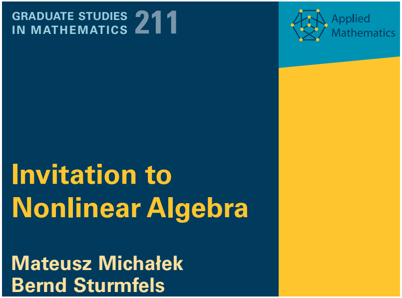

$\langle i\, j \rangle$

$\langle i\, j\, k \rangle$

$[\, i\, j\,]$

???

Mini-Course at ESI Vienna, within
Amplitudes and Algebraic Geometry
February 16, 2026

Sources

This lecture is based on two recent articles:

S. Rajan, BSt, S. Sverrisdóttir:
Kinematic varieties for massless particles,
Le Matematiche, 2025.

Y. El Maazouz, A. Pfister, BSt:
Spinor-helicity varieties,
SIAM Journal on Applied Algebra and Geometry, 2025.

Initial motivation: understand the mathematics behind Smita's Bachelor thesis (Physics at Brown U), and answer a question in

A. Pokraka, S. Rajan, L. Ren, A. Volovich, W. Zhao:
Five-dimensional spinor helicity for all masses and spins,
Journal of High Energy Physics, 2025.

Particles in d-dimensional spacetime

Spacetime is $\mathbb{R}^d$ or $\mathbb{C}^d$, with the Lorentzian inner product

$$x \cdot y \;=\; x_1 y_1 - x_2 y_2 - \cdots - x_n y_n.$$

The *Lorentz group* $\mathrm{SO}(1, d-1)$ consists of $d \times d$ matrices g such that $\det(g) = 1$ and $(gx) \cdot (gy) = x \cdot y$ for all $x, y \in \mathbb{C}^d$.

A configuration of n particles is given by momentum vectors

$$p_i \;=\; (p_{i1}, p_{i2}, \ldots, p_{id}) \;\in\; \mathbb{C}^d.$$

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$$p_i \; = \; (p_{i1}, p_{i2}, \ldots, p_{id}) \; \in \; \mathbb{C}^d.$$

Assume that each particle is *massless*, i.e. $p_i \cdot p_i = 0$:

$$p_{i1}^2 \; - \; p_{i2}^2 \; - \; p_{i3}^2 \; - \; \cdots \; - \; p_{id}^2 \; = \; 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Also assume *momentum conservation* $\sum_{i=1}^n p_i = 0$:

$$p_{1j} + p_{2j} + \cdots + p_{nj} \; = \; 0 \quad \text{for} \quad j = 1, 2, \ldots, d.$$

Ideals, varieties and algorithms

Let $I_{d,n} \subset \mathbb{C}[p]$ be the ideal generated by the n quadrics for massless and the d linear forms for momentum conservation.

Here $\mathbb{C}[p]$ is the polynomial ring in nd variables p_{ij}.

Example ($n = d = 3$)

Three particles on the icecream cone. Let's try it in `Macaulay2`:

i1 : R = QQ[p11,p12,p13, p21,p22,p23, p31,p32,p33];

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i2 : I = ideal($p11+p21+p31$, $p12+p22+p32$, $p13+p23+p33$, $p11^2-p12^2-p13^2$, $p21^2-p22^2-p23^2$, $p31^2-p32^2-p33^2$);

i3 : codim I, degree I

o3 = (6, 8)

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i3 : codim I, degree I

o3 = (6, 8)

i4 : isPrime I, isPrimary I

o4 = (false, true)

i5 : radical I33

o5 = ideal(. . . , $p23*p31 - p21*p33$, $p22*p31 - p21*p32$, . . .)

Prime time

Theorem

$I_{d,n}$ is prime and a complete intersection, provided $\max(n, d) \geq 4$.

Proof: technical commutative algebra

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How about using a parametric representation of the variety $V(I_n)$?

One idea is to express the variables in the first row and column in terms of the entries of the $(n-1) \times (d-1)$ matrix $p' = (p_{ij})_{i,j\geq 2}$.

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Remark (Bad News)

The elimination ideal $I_{d,n} \cap \mathbb{C}[p']$ is principal. Its generator is a large polynomial of degree 2^{n-1}. This hypersurface is a notable obstruction to any easy parametrization. This does not exist.

Example: for $n = 4, d = 5$, the polynomial has 4671 terms of degree 8.

Mandelstam invariants

Physical properties of our n particles are invariant under the group $G = \mathrm{O}(1, d-1)$. The ring of G-invariants in $\mathbb{C}[p]$ is generated by the *Mandelstam invariants* $s_{ij} = p_i \cdot p_j$. Consider the invariant ring

$$(\mathbb{C}[p]/I_{d,n})^G \;=\; \mathbb{C}[S]/M_{d,n}.$$

The *Mandelstam variety* is the GIT quotient

$$V(M_{d,n}) \;=\; \mathrm{Spec}\big(\,(\mathbb{C}[p]/I_{d,n})^G\,\big) \;=\; V(I_{d,n})/\!/G.$$

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Theorem

Let $n \geq 2$ and $d \geq 4$. The prime ideal $M_{d,n}$ equals

$$\langle\, s_{11}, s_{22}, \ldots, s_{nn} \,\rangle + \langle\, \textstyle\sum_{j=1}^{n} s_{ij} \;\; \textit{for } i = 1, \ldots, n \,\rangle$$
$$+ \langle\, (d{+}1) \times (d{+}1) \textit{ minors of the symmetric matrix } (s_{ij}) \,\rangle$$

The dimension of the Mandelstam variety is

$$\dim(V(M_{d,n})) \;=\; nd - n - d - \binom{d}{2} \;=\; \dim(V(I_{d,n})) - \dim(G).$$

Clifford algebras and spinors

Physicists use the following symbols:

$$\langle\, i\, j\, \rangle \qquad [\, i\, j\,] \qquad \langle\, i\, j\, k\, \rangle$$

What are these brackets? Which relations do they satisfy?

Kinematic data for n particles are expressed in terms of spinors:

H. Elvang and Y. Huang: *Scattering Amplitudes in Gauge Theory and Gravity*, Cambridge University Press, 2015.

This encoding rests on the Clifford algebra $\mathrm{Cl}(1, d-1)$:

M. Rausch de Traubenberg: *Clifford algebras in physics*, Adv. Appl. Clifford Algebr., 2009.

Mathematicians appreciate Bourbaki:

C. Chevalley: *The Algebraic Theory of Spinors and Clifford Algebras*: Collected Works of Claude Chevalley, Volume 2, Springer Verlag, 1996.

Dirac matrices

For us, spinors are vectors of length 2^k where $k = \lfloor d/2 \rfloor$. We recursively define $2^k \times 2^k$ matrices $\Gamma_1, \Gamma_2, \ldots, \Gamma_d$. For $d = 2$,

$$\Gamma_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For larger $d = 2k$, take tensor products with Pauli matrices:

$$\Gamma_i = \Gamma_{k-1,i} \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for } 1 \leq i \leq 2k-2,$$

$$\Gamma_{2k-1} = \mathrm{Id}_{2^{k-1}} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \Gamma_{2k} = \mathrm{Id}_{2^{k-1}} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

For $d = 2k+1$ odd, set $\Gamma_{2k+1} = -i^{k-1} \cdot \Gamma_1 \Gamma_2 \cdots \Gamma_{2k-1}$.

Proposition

The Dirac matrices satisfy the Clifford algebra relations:

$$\Gamma_1^2 = -2\,\mathrm{Id}_{2^k}, \ \Gamma_j^2 = 2\,\mathrm{Id}_{2^k} \ \textit{for } j \geq 2$$
$$\text{and } \Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 0_{2^k} \ \textit{for } i \neq j.$$

$Cl(1, d-1)$

One matrix for one particle

The *momentum space Dirac matrix* is the linear combination

$$P \;=\; -p_1\Gamma_1 + p_2\Gamma_2 + p_3\Gamma_3 + \cdots + p_d\Gamma_d.$$

Example $(d = 4, 5, 6)$

$$P = \begin{bmatrix} 0 & 0 & p_1 - p_2 & p_3 - ip_4 \\ 0 & 0 & p_3 + ip_4 & p_1 + p_2 \\ -p_1 - p_2 & p_3 - ip_4 & 0 & 0 \\ p_3 + ip_4 & -p_1 + p_2 & 0 & 0 \end{bmatrix},$$

$$P \;=\; \begin{bmatrix} p_5 & 0 & p_1 - p_2 & p_3 - ip_4 \\ 0 & p_5 & p_3 + ip_4 & p_1 + p_2 \\ -p_1 - p_2 & p_3 - ip_4 & -p_5 & 0 \\ p_3 + ip_4 & -p_1 + p_2 & 0 & -p_5 \end{bmatrix}.$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & -p_1 + p_2 & 0 & -p_3 + ip_4 & p_5 - ip_6 \\ 0 & 0 & 0 & 0 & 0 & -p_1 + p_2 & p_5 + ip_6 & p_3 + ip_4 \\ 0 & 0 & 0 & 0 & -p_3 - ip_4 & p_5 - ip_6 & -p_1 - p_2 & 0 \\ 0 & 0 & 0 & 0 & p_5 + ip_6 & p_3 - ip_4 & 0 & -p_1 - p_2 \\ p_1 + p_2 & 0 & -p_3 + ip_4 & p_5 - ip_6 & 0 & 0 & 0 & 0 \\ 0 & p_1 + p_2 & p_5 + ip_6 & p_3 + ip_4 & 0 & 0 & 0 & 0 \\ -p_3 - ip_4 & p_5 - ip_6 & p_1 - p_2 & 0 & 0 & 0 & 0 & 0 \\ p_5 + ip_6 & p_3 - ip_4 & 0 & p_1 - p_2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Spin representation

Corollary

The relations of the Clifford algebra $\mathrm{Cl}(1, d-1)$ *imply*

$$\begin{aligned} P^2 &= (-p_1^2 + p_2^2 + \cdots + p_d^2)\,\mathrm{Id}_{2^k}, \\ \det(P) &= (p_1^2 - p_2^2 - \cdots - p_d^2)^{2^{k-1}}. \end{aligned}$$

For massless particles, the momentum space Dirac matrix P *is nilpotent and its rank equals half of its size, i.e.* $\mathrm{rank}(P) = 2^{k-1}$.

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The Dirac representation of $\mathrm{Cl}(1, d-1)$ gives rise to the spin representation of the Lie algebra $\mathfrak{so}(1, d-1)$. The commutators

$$\Sigma_{jk} \;=\; \frac{1}{4}[\,\Gamma_j, \Gamma_k]$$

satisfy same relations as the generators of $\mathfrak{so}(1, d-1)$.

The spin representation of $\mathrm{SO}(1, d-1)$ is the action of the matrix exponentials $\exp(\Sigma_{jk})$ on spinor space $\mathbb{C}^{2^k}$.

Charge conjugation matrix

An equivariant linear map from the spin representation of $\mathfrak{so}(1, d-1)$ to its dual is represented by a $2^k \times 2^k$ matrix C:

$$
\begin{aligned}
CP &= -P^T C && \text{if } d = 2k \text{ is even,} \\
CP &= (-1)^k P^T C && \text{if } d = 2k+1 \text{ is odd.}
\end{aligned}
$$

Example ($d = 4, 5, 6$)

$$
C = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

Proposition (Symmetries)

1. *C is symmetric for $k \equiv 0, 3 \bmod 4$, otherwise skew symmetric.*
2. *C is block diagonal for $k \equiv 0 \bmod 2$, else anti-block diagonal.*
3. *the $2^{k-1} \times 2^{k-1}$ blocks of C are skew symmetric when $k = 2, 3 \bmod 4$; otherwise the blocks are symmetric.*

Bra and ket

Our goal: model interactions among n massless particles
$p_i = (p_{i1}, \ldots, p_{id})$. The tuple $(p_1, \ldots, p_n)$ lies in $V(I_{d,n}) \subset \mathbb{C}^{nd}$.
The *momentum space Dirac matrix* for the ith particle is

$$P_i \;=\; -p_{i1}\Gamma_1 + p_{i2}\Gamma_2 + p_{i3}\Gamma_3 + \cdots + p_{id}\Gamma_d.$$

Matrix has size 2^k and rank 2^{k-1}. Clifford relations imply

$$P_i P_j + P_j P_i \;=\; 2 p_i \cdot p_j \,\mathrm{Id}_{2^k} \;=\; 2 s_{ij} \,\mathrm{Id}_{2^k}.$$

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We parameterize the column space of P_i using a vector
$z_i \;=\; (\, z_{i,1},\, z_{i,2},\, \ldots\, ,\, z_{i,2^{k-2}}\,,\, 0,\, 0,\, \ldots\, ,\, 0\,,\, z_{i,2^{k-2}+1},\, \ldots\, , z_{i,2^{k-1}}\,)^T.$

Use Dirac's ket-notation for vectors in this column space:

$$|\, i\, \rangle \;=\; P_i\, z_i.$$

Use the bra-notation $\langle\, i\,|$ for the row vector $|\, i\,\rangle^T$. The spinors $|\, i\,\rangle$ and $\langle\, i\,|$ depend on $d + 2^{k-1}$ parameters. They represent particle i.

Spinor brackets

The *spinor brackets* of order two and three are

$$\langle\, ij \,\rangle \;=\; \langle\, i \,|\, C \,|\, j \,\rangle \quad \text{and} \quad \langle\, ij\, k \,\rangle \;=\; \langle\, i \,|\, CP_j \,|\, k \,\rangle.$$

Here $i, j, k \in \{1, 2, \ldots, n\}$. The ℓ*-th order spinor brackets* are

$$\langle\, i_1 i_2 \,\cdots\, i_\ell \,\rangle \;=\; \langle\, i_1 \,|\, CP_{i_2} \cdots P_{i_{\ell-1}} \,|\, i_\ell \,\rangle.$$

Spinor brackets are Lorentz-invariant elements in the ring

$$R_{d,n} \;=\; \mathbb{C}[p, z]/I_{d,n},$$

which is generated by nd parameters p_{ij} and $n2^{k-1}$ parameters z_{ij}.

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Example. For $d = 3$ we have

$$\begin{aligned}
\langle ij \rangle &= \begin{bmatrix} z_{i1} & 0 \end{bmatrix} \begin{bmatrix} p_{i3} & p_{i1}{+}p_{i2} \\ -p_{i1}{+}p_{i2} & -p_{i3} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p_{j3} & -p_{j1}{+}p_{j2} \\ p_{j1}{+}p_{j2} & -p_{j3} \end{bmatrix} \begin{bmatrix} z_{j1} \\ 0 \end{bmatrix} \\
&= -p_{i1}p_{j3}z_{i1}z_{j1} - p_{i2}p_{j3}z_{i1}z_{j1} + p_{i3}p_{j1}z_{i1}z_{j1} + p_{i3}p_{j2}z_{i1}z_{j1}, \\
\langle ijk \rangle &= p_{i1}p_{j1}p_{k1}z_{i1}z_{k1} + p_{i1}p_{j1}p_{k2}z_{i1}z_{k1} - p_{i1}p_{j2}p_{k1}z_{i1}z_{k1} - p_{i1}p_{j2}p_{k2}z_{i1}z_{k1} \\
&\quad - p_{i1}p_{j3}p_{k3}z_{i1}z_{k1} + p_{i2}p_{j1}p_{k1}z_{i1}z_{k1} + p_{i2}p_{j1}p_{k2}z_{i1}z_{k1} - p_{i2}p_{j2}p_{k1}z_{i1}z_{k1} \\
&\quad - p_{i2}p_{j2}p_{k2}z_{i1}z_{k1} - p_{i2}p_{j3}p_{k3}z_{i1}z_{k1} + p_{i3}p_{j1}p_{k3}z_{i1}z_{k1} + p_{i3}p_{j2}p_{k3}z_{i1}z_{k1} \\
&\quad - p_{i3}p_{j3}p_{k1}z_{i1}z_{k1} - p_{i3}p_{j3}p_{k2}z_{i1}z_{k1}.
\end{aligned}$$

Matrices of spinor brackets

Multiply matrices of formats $n \times 2^k$, $2^k \times 2^k$ and $2^k \times n$ to define

$$S \;:=\; \big(\langle\, ij \,\rangle\big)_{1 \le i,j \le n} \;=\; \big(|\,1\,\rangle, \ldots, |\,n\,\rangle\big)^T \cdot C \cdot \big(|\,1\,\rangle, \ldots, |\,n\,\rangle\big).$$

$$T_j \;:=\; \big(\langle\, ijk \,\rangle\big)_{1 \le i,k \le n} \;=\; \big(|\,1\,\rangle, \ldots, |\,n\,\rangle\big)^T \cdot C \cdot P_j \cdot \big(|\,1\,\rangle, \ldots, |\,n\,\rangle\big).$$

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Theorem

The $n \times n$ matrix S has rank $\le 2^k$ with zeros on the diagonal. If $k \equiv 0,3 \bmod 4$ then S is symmetric; otherwise skew symmetric:

$$\langle\, i\, i\,\rangle \;=\; 0 \quad \text{and} \quad \langle\, i\, j\,\rangle \;=\; \pm\langle\, j\, i\,\rangle \qquad \textit{for}\;\; 1 \le i,j \le n.$$

The matrix T_j has rank $\le 2^{k-1}$ with zeros row and column j. If $d \equiv 1,2,3,4 \bmod 8$ then T_j is symmetric; else skew symmetric:

$$\langle\, j\, j\, k\,\rangle \;=\; \langle\, i\, j\, j\,\rangle \;=\; 0 \;\text{ and }\; \langle\, i\, j\, k\,\rangle \;=\; \pm\langle\, k\, j\, i\,\rangle \quad \textit{for}\;\; 1 \le i,j,k \le n.$$

The sum of the matrices T_j is zero: $T_1 + T_2 + \cdots + T_n = 0.$

Example: Four particles for flatlanders

For $d = 3, k = 1, n = 4$, there are six order two spinor brackets:

$$S = \begin{bmatrix} 0 & \langle 12\rangle & \langle 13\rangle & \langle 14\rangle \\ -\langle 12\rangle & 0 & \langle 23\rangle & \langle 24\rangle \\ -\langle 13\rangle & -\langle 23\rangle & 0 & \langle 34\rangle \\ -\langle 14\rangle & -\langle 24\rangle & -\langle 34\rangle & 0 \end{bmatrix}.$$

The 24 spinor brackets of order three are the entries of

$$T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \langle 212\rangle & \langle 213\rangle & \langle 214\rangle \\ 0 & \langle 213\rangle & \langle 313\rangle & \langle 314\rangle \\ 0 & \langle 214\rangle & \langle 314\rangle & \langle 414\rangle \end{bmatrix}, \quad T_2 = \begin{bmatrix} \langle 121\rangle & 0 & \langle 123\rangle & \langle 124\rangle \\ 0 & 0 & 0 & 0 \\ \langle 123\rangle & 0 & \langle 323\rangle & \langle 324\rangle \\ \langle 124\rangle & 0 & \langle 324\rangle & \langle 424\rangle \end{bmatrix},$$

$$T_3 = \begin{bmatrix} \langle 131\rangle & \langle 132\rangle & 0 & \langle 134\rangle \\ \langle 132\rangle & \langle 232\rangle & 0 & \langle 234\rangle \\ 0 & 0 & 0 & 0 \\ \langle 134\rangle & \langle 234\rangle & 0 & \langle 434\rangle \end{bmatrix}, \quad T_4 = \begin{bmatrix} \langle 141\rangle & \langle 142\rangle & \langle 143\rangle & 0 \\ \langle 142\rangle & \langle 242\rangle & \langle 243\rangle & 0 \\ \langle 143\rangle & \langle 243\rangle & \langle 343\rangle & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example: Four particles for flatlanders

The 30 brackets define the 4-diml kinematic variety in $\mathbb{P}^5 \times \mathbb{P}^{23}$.
The ideal is generated by 10 linear forms in $T_1 + T_2 + T_3 + T_4$ plus $54 = 1 + 24 + 29$ quadrics. The Plücker quadric

$$\langle 12\rangle\langle 34\rangle - \langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle \quad = \quad \text{Pfaffian}(S),$$

ensures that S has rank two. The 24 binomial quadrics

$$\langle i\, j\, k\rangle\langle l\, j\, m\rangle - \langle i\, j\, m\rangle\langle l\, j\, k\rangle.$$

are 2×2 minors of the slices T_j. Finally, 29 bilinear relations like

$$\langle 12\rangle\langle 324\rangle - \langle 34\rangle\langle 142\rangle \quad \text{and} \quad \langle 12\rangle\langle 243\rangle - \langle 13\rangle\langle 242\rangle + \langle 23\rangle\langle 142\rangle$$

ensure that the $4 \times 4 \times 5$ tensor ST has rank two. They are in the radical of the 3×3 minors of the 4×20 matrix (S, T_1, T_2, T_3, T_4).

Varieties in matrix space

The $\binom{n}{2}$ order two spinor brackets $\langle ij \rangle$ form an $n \times n$ matrix S which is either symmetric or skew symmetric. The set of all such matrices defines the *kinematic variety* $\mathcal{K}^{(2)}_{d,n}$ in $\mathbb{P}^{\binom{n}{2}-1}$.

Theorem

For $d = 3$, the ideal of $\mathcal{K}^{(2)}_{3,n}$ is given by 4×4-Pfaffians of a skew $n \times n$ matrix, so it is the Grassmannian $\mathrm{Gr}(2, n)$. For $d = 4, 5$, we get 6×6-Pfaffians, so $\mathcal{K}^{(2)}_{d,n}$ is the secant variety of $\mathrm{Gr}(2, n)$.

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The $\binom{n}{2}$ order two spinor brackets $\langle ij \rangle$ form an $n \times n$ matrix S which is either symmetric or skew symmetric. The set of all such matrices defines the *kinematic variety* $\mathcal{K}^{(2)}_{d,n}$ in $\mathbb{P}^{\binom{n}{2}-1}$.

Theorem

For $d = 3$, the ideal of $\mathcal{K}^{(2)}_{3,n}$ is given by 4×4-Pfaffians of a skew $n \times n$ matrix, so it is the Grassmannian $\mathrm{Gr}(2, n)$. For $d = 4, 5$, we get 6×6-Pfaffians, so $\mathcal{K}^{(2)}_{d,n}$ is the secant variety of $\mathrm{Gr}(2, n)$.

Conjecture

For $d = 6, 7, 8, 9$, the kinematic variety $\mathcal{K}^{(2)}_{d,n}$ consists of all symmetric $n \times n$ matrices with zero diagonal and rank $\leq 2^{\lfloor d/2 \rfloor}$.

For d even, spin representation splits into two irreducibles.

Use separate brackets $\langle ij \rangle$ and $[ij]$ for each block.

Coming up soon: $d = 4$.

Varieties in tensor space

Write $\mathcal{K}_{d,n}^{(3)}$ for the kinematic variety of $n \times n \times (n+1)$ tensors ST.
The $n \times n$ slices $S, T_1, \ldots, T_n$ are symmetric or skew symmetric, depending on residue classes of $k = \lfloor d/2 \rfloor$ mod 4 and d mod 8.

The ideal of $\mathcal{K}_{d,n}^{(3)}$ is $\mathbb{Z}^2$-graded.

The variety lives in $\mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{K-1}$, where

- $K = n \cdot \binom{n}{2}$ when slices T_j are symmetric $(d \equiv 1, 2, 3, 4 \bmod 8)$,
- $K = n \cdot \binom{n-1}{2}$ when slices T_j are skew symmetric.

Varieties in tensor space

Write $\mathcal{K}^{(3)}_{d,n}$ for the kinematic variety of $n \times n \times (n+1)$ tensors ST.
The $n \times n$ slices $S, T_1, \ldots, T_n$ are symmetric or skew symmetric, depending on residue classes of $k = \lfloor d/2 \rfloor$ mod 4 and d mod 8.

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- $K = n \cdot \binom{n-1}{2}$ when slices T_j are skew symmetric.

Conjecture (Flatlanders)

The variety $\mathcal{K}^{(3)}_{3,n}$ has dimension $3n - 8$. Its points are tensors ST of rank 2, where S is skew symmetric and the T_j are symmetric of rank ≤ 1, summing to 0, with zeros in j-th row/column.

Its ideal is generated by linear forms and quadrics: the entries of $T_1 + \cdots + T_n$, the 4×4 pfaffians of S, the 2×2 minors of the T_j, and bilinear Pfaffians in the radical of the 3×3 minors of the flattening $(S, T_1, \ldots, T_n)$.

Example: Five particles in five-dim'l spacetime

The variety $\mathcal{K}_{5,5}^{(3)} \subset \mathbb{P}^9 \times \mathbb{P}^{29}$ has dimension 13. Its ideal is generated by 10 linear forms, 25 quadrics, 15 cubics and 5 quartics. Each T_j is a skew symmetric with a zero row, so it contributes one Pfaffian $\langle ijk\rangle\langle \ell jm\rangle - \langle ij\ell\rangle\langle kjm\rangle + \langle ijm\rangle\langle kj\ell\rangle$.

The other 20 quadrics are bilinear, e.g. five 4×4 Pfaffians of

$$\begin{bmatrix} 0 & \langle 12\rangle & \langle 13\rangle & \langle 14\rangle & \langle 15\rangle \\ -\langle 12\rangle & 0 & \langle 213\rangle & \langle 214\rangle & \langle 215\rangle \\ -\langle 13\rangle & -\langle 213\rangle & 0 & \langle 314\rangle & \langle 315\rangle \\ -\langle 14\rangle & -\langle 214\rangle & -\langle 314\rangle & 0 & \langle 415\rangle \\ -\langle 15\rangle & -\langle 215\rangle & -\langle 315\rangle & -\langle 415\rangle & 0 \end{bmatrix}.$$

The 15 cubics ensure that $(S, T_1, T_2, T_3, T_4, T_5)$ has rank ≤ 4. One of them is $\langle 213\rangle\langle 123\rangle\langle 435\rangle - \langle 213\rangle\langle 325\rangle\langle 134\rangle + \langle 213\rangle\langle 324\rangle\langle 135\rangle + \langle 314\rangle\langle 123\rangle\langle 235\rangle - \langle 314\rangle\langle 325\rangle\langle 132\rangle - \langle 315\rangle\langle 123\rangle\langle 234\rangle + \langle 315\rangle\langle 324\rangle\langle 132\rangle$.

The 5 quartics are 4×4 minors of mixed slices like

$$\begin{bmatrix} 0 & \langle 12\rangle & \langle 13\rangle & \langle 14\rangle & \langle 15\rangle \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \langle 123\rangle & \langle 124\rangle & \langle 125\rangle \\ 0 & \langle 132\rangle & 0 & \langle 134\rangle & \langle 135\rangle \\ 0 & \langle 142\rangle & \langle 143\rangle & 0 & \langle 145\rangle \\ 0 & \langle 152\rangle & \langle 153\rangle & \langle 154\rangle & 0 \end{bmatrix}, \quad \begin{bmatrix} -\langle 12\rangle & 0 & \langle 23\rangle & \langle 24\rangle & \langle 25\rangle \\ 0 & 0 & \langle 213\rangle & \langle 214\rangle & \langle 215\rangle \\ 0 & 0 & 0 & 0 & 0 \\ -\langle 132\rangle & 0 & 0 & \langle 234\rangle & \langle 235\rangle \\ -\langle 142\rangle & 0 & \langle 243\rangle & 0 & \langle 245\rangle \\ -\langle 152\rangle & 0 & \langle 253\rangle & \langle 254\rangle & 0 \end{bmatrix}, \quad \text{etc}\ldots$$

Seen recently in SIAGA

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Spinor-Helicity Varieties*

Yassine El Maazouz†, Anaëlle Pfister‡, and Bernd Sturmfels‡

Abstract. The spinor-helicity formalism in particle physics gives rise to natural subvarieties in the product of two Grassmannians. These include two-step flag varieties for subspaces of complementary dimension. Taking Hadamard products leads to Mandelstam varieties. We study these varieties through the lens of combinatorics and commutative algebra, and we explore their tropicalization, positive geometry, and scattering correspondence.

Key words. Mandelstam invariants, scattering amplitudes, flag variety, Gröbner and Khovanskii bases

Spinors are easier when spacetime has dimension $d = 4$.

Let's **reboot** and start from scratch.

Introduction

Given two matrices λ and $\widetilde{\lambda}$ of format $k \times n$, consider the property that the $k \times k$ matrix $\lambda \cdot \widetilde{\lambda}^T$ has rank at most r where $0 \leq r \leq k \leq n$. We wish to express this property in terms of the $k \times k$ minors of the matrices λ and $\widetilde{\lambda}$. This situation arises in the study of *scattering amplitudes* in quantum field theory [3]. The special case when $k = 2$ and $r = 0$ is known as *spinor-helicity formalism*; for textbook basics see [3, Section 1.8] and [16, Section 2.2]. In physics, it is customary to write $\langle ij \rangle$ for the 2×2 minors of λ and $[ij]$ for the 2×2 minors of $\widetilde{\lambda}$, where $1 \leq i < j \leq n$, and these minors satisfy the *momentum conservation* relations.

Example 1.1 $(k = 2, n = 5, r = 0)$**.** We consider the two skew-symmetric 5×5 matrices

$$P = \begin{pmatrix} 0 & \langle 12\rangle & \langle 13\rangle & \langle 14\rangle & \langle 15\rangle \\ -\langle 12\rangle & 0 & \langle 23\rangle & \langle 24\rangle & \langle 25\rangle \\ -\langle 13\rangle & -\langle 23\rangle & 0 & \langle 34\rangle & \langle 35\rangle \\ -\langle 14\rangle & -\langle 24\rangle & -\langle 34\rangle & 0 & \langle 45\rangle \\ -\langle 15\rangle & -\langle 25\rangle & -\langle 35\rangle & -\langle 45\rangle & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & [12] & [13] & [14] & [15] \\ -[12] & 0 & [23] & [24] & [25] \\ -[13] & -[23] & 0 & [34] & [35] \\ -[14] & -[24] & -[34] & 0 & [45] \\ -[15] & -[25] & -[35] & -[45] & 0 \end{pmatrix}.$$

These matrices have rank two, meaning that the 4×4 Pfaffians vanish for both matrices:

$$\langle ij\rangle\langle kl\rangle - \langle ik\rangle\langle jl\rangle + \langle il\rangle\langle jk\rangle \;=\; [ij][kl] - [ik][jl] + [il][jk] \;=\; 0 \;\; \text{for} \;\; 1 \leq i < j < k < l \leq 5. \quad (1)$$

These *quadratic Plücker relations* are known as *Schouten identities* in physics [3, eqn (1.116)]. Momentum conservation [3, eqn (1.117)] stipulates that the product $P \cdot Q^T$ is the zero matrix:

$$\langle i1\rangle[1j] + \langle i2\rangle[2j] + \langle i3\rangle[3j] + \langle i4\rangle[4j] + \langle i5\rangle[5j] \;=\; 0 \quad \text{for} \;\; 1 \leq i, j \leq 5. \qquad (2)$$

In total, we have a system of $5+5+25 = 35$ quadratic equations in $\binom{5}{2} + \binom{5}{2} = 20$ unknowns. The equations (1) define a product of two Grassmannians $\mathrm{Gr}(2,5) \times \mathrm{Gr}(2,5) \subset \mathbb{P}^9 \times \mathbb{P}^9$.

Geometry

The *spinor-helicity variety* $\mathrm{SH}(k,n,r)$ is

$$\{\,(V,W)\in \mathrm{Gr}(k,n)\times \mathrm{Gr}(k,n)\;:\;\dim(V\cap W^{\perp})\geq k-r\,\},$$

where $W^{\perp}$ is the orthogonal complement in $\mathbb{C}^n$.

For $0\leq r\leq k$ and $2k\leq r+n$, the variety $\mathrm{SH}(k,n,r)$ is irreducible of dimension $2k(n-k)-(k-r)^2$ in $\mathbb{P}^{\binom{n}{k}-1}\times\mathbb{P}^{\binom{n}{k}-1}$.

Q1: What is $\mathrm{SH}(k,n,k)$?

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A1: The product of Grassmannians $\mathrm{Gr}(k,n)\times \mathrm{Gr}(k,n)$

Q2: What is $\mathrm{SH}(k,n,0)$?

Geometry

The *spinor-helicity variety* $\mathrm{SH}(k,n,r)$ is

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Q1: What is $\mathrm{SH}(k,n,k)$?

A1: The product of Grassmannians $\mathrm{Gr}(k,n)\times\mathrm{Gr}(k,n)$

Q2: What is $\mathrm{SH}(k,n,0)$?

A2: The two-step flag variety $\mathrm{Fl}(k,n-k;\mathbb{C}^n)$

Q3: Why do we allow $k\geq 3$?

Geometry

The *spinor-helicity variety* $\mathrm{SH}(k,n,r)$ is

$$\{\,(V,W) \in \mathrm{Gr}(k,n) \times \mathrm{Gr}(k,n)\ :\ \dim(V \cap W^{\perp}) \geq k-r\,\},$$

where $W^{\perp}$ is the orthogonal complement in $\mathbb{C}^n$.

For $0 \leq r \leq k$ and $2k \leq r+n$, the variety $\mathrm{SH}(k,n,r)$ is irreducible of dimension $2k(n-k)-(k-r)^2$ in $\mathbb{P}^{\binom{n}{k}-1} \times \mathbb{P}^{\binom{n}{k}-1}$.

Q1: What is $\mathrm{SH}(k,n,k)$?

A1: The product of Grassmannians $\mathrm{Gr}(k,n) \times \mathrm{Gr}(k,n)$

Q2: What is $\mathrm{SH}(k,n,0)$?

A2: The two-step flag variety $\mathrm{Fl}(k,n-k;\mathbb{C}^n)$

Q3: Why do we allow $k \geq 3$?

A3: CEGM theory is very beautiful

Prime Time

Theorem 2.7. *The prime ideal* $I_{k,n,r}$ *is minimally generated by quadratic forms. These quadrics are a Gröbner basis for the reverse lexicographic term order given by any linear extension of* $\mathcal{P}_{k,n,r}$*. The initial ideal of* $I_{k,n,r}$ *is generated by the incomparable pairs in* $\mathcal{P}_{k,n,r}$*.*

Prime Time

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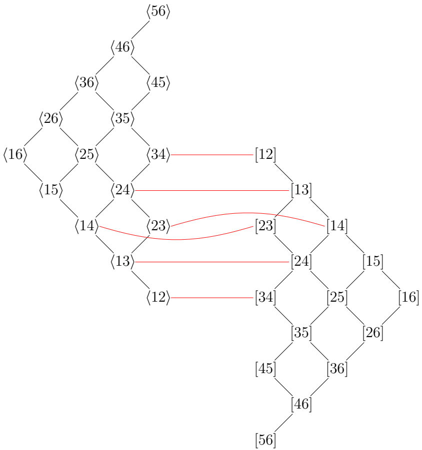


Figure 1: The poset $\mathcal{P}_{2,6,0}$ is created from $Y_{2,6}$ and $\widetilde{Y}_{2,6}$ by adding six covering relations.

Mandelstam Variety

The componentwise multiplication of two vectors is known as the Hadamard product. We consider the Hadamard product of two Plücker vectors. This gives rise to a rational map

$$s \; : \; \mathbb{P}^{\binom{n}{k}-1} \times \mathbb{P}^{\binom{n}{k}-1} \dashrightarrow \mathbb{P}^{\binom{n}{k}-1}. \tag{26}$$

Generalizing the case $k = 2$ in (6), the coordinates of s are called *Mandelstam invariants*:

$$s_{i_1 i_2 \cdots i_k} \;=\; \langle i_1 i_2 \cdots i_k \rangle [i_1 i_2 \cdots i_k]. \tag{27}$$

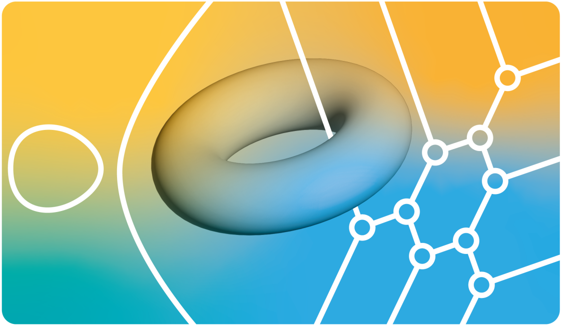
We define the *Mandelstam variety* $\mathrm{M}(k,n,r)$ to be the closure of the image of the spinor-helicity variety $\mathrm{SH}(k,n,r)$ under the Hadamard product map s. Thus, $\mathrm{M}(k,n,r)$ is an irreducible variety in $\mathbb{P}^{\binom{n}{k}-1}$. We write $\mathcal{I}(\mathrm{M}(k,n,r))$ for the homogeneous prime ideal of this variety. This comprises all polynomial relations among the Mandelstam invariants $s_{i_1 i_2 \cdots i_k}$.

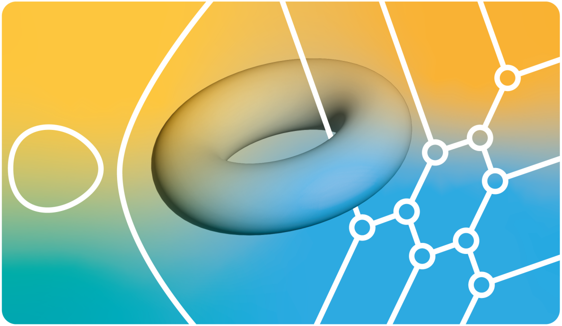
Proposition 4.1. *The linear span of the Mandelstam variety* $\mathrm{M}(k,n,r)$ *in* $\mathbb{P}^{\binom{n}{k}-1}$ *is the subspace* $\mathbb{P}^N$ *which is defined by the momentum conservation relations. Its dimension equals*

$$N \;=\; \binom{n}{k} - 1 - \binom{n}{k-r-1}.$$

For $k = 2, r = 0$ this is the variety $V(M_{4,n})$ on slide 6 of this lecture.

Complex, Real and Tropical Algebraic Geometry





5 Positivity and Tropicalization

In our last two sections, we set the stage for future research on spinor-helicity varieties, with a view towards tropical geometry, positive geometry, and applications to scattering amplitudes.

Bossinger, Drummond and Glew [7] studied the Gröbner fan and positive geometry of the variety $\mathrm{SH}(2,5,0)$ in Example 1.1 which they identified with the Grassmannian $\mathrm{Gr}(3,6)$. We shall examine this in a broader context. The following result explains their identification.

Proposition 5.1. *For any* $k \geq 1$*, the varieties* $\mathrm{SH}(k,2k{+}1,0)$ *and* $\mathrm{SH}(k{+}1,2k{+}1,1)$ *are isomorphic and their coordinate ring is isomorphic to that of the Grassmannian* $\mathrm{Gr}(k{+}1,2k{+}2)$.