Higher Structures & Embedding Tensor (Yunhe Sheng, Rong Tang, C. Zhu)

KS

$\boxed{\text{Embedding Tensor}} \xrightarrow{F} \text{Leibniz alg} \xrightarrow{G} L_\infty\text{-alg}$

supergravity $\xrightarrow{\text{Kotov-Strobl}}$ tensor hierarchies $\longrightarrow$ higher Gauge Theory

- cohomology
  - KS $\boxed{\text{controlling alg}}$ ⇝ technique
    - deformation
    - centrale extension
- homotopy
  - M.C. element in
  - ET_∞
  - KS Stable under ∞ ??
    - Yes $ET_\infty \xrightarrow{F} Leib_\infty \xrightarrow{G} L_\infty$
    - ?? $ET_\infty \xrightarrow{F_\infty} Leib_\infty \xrightarrow{G_\infty} L_\infty$

Embedding Tensor : $T : V \to \mathfrak{g}$, $\mathfrak{g} \curvearrowright V$ s.t.

$$[a, T(v)] = T(a \cdot v), \quad \forall\, a \in \mathrm{Im}\,T,\ v \in V.$$

Average Operator

$$\begin{array}{ccccc} \text{s.lie2} & & & & \\ \wr & & & & \\ ET & \xrightarrow{F} & \text{Leib} & \xrightarrow{G} & L_\infty \text{ alg} \\ T & \longmapsto & (V, [\cdot,\cdot]_T) & & \end{array}$$

$[u,v]_T = \rho(Tu)v.$

PBW

$T(V) \cong U(\mathrm{Lie}\,V) \cong S(\mathrm{Lie}\,V)$

Cartier-Milnor-Moore   Loday-Pirashvili

Zinbiel $\to$	dendriform $\to$	prelie
Comm $\to$	Ass $\to$	lie
perm $\to$	di $\to$	Leibniz

$\uparrow$ split $\rightsquigarrow$ Rota Baxter Operator

$\downarrow$ duplication $\rightsquigarrow$ Averager Operator

$\boxed{\text{controlling alg}}$

homology   homotopy

Example 1 (Lie) $\mathfrak{g}$: a vect. space $\left(\bigoplus_k \mathrm{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g}), [-,-]_{NR}\right)$ g.l.a.

① $\mu \in \mathrm{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$, s.t. $[\pi, \pi]_N = 0$ (M.C. elet)

② μ defines lie bracket on $\mathfrak{g}$

- $\left(\mathrm{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g}),\ d_\mu := [\mu, -]_{NR}\right)$ $\rightsquigarrow$ CE complex of $\mathfrak{g}$

$\rightsquigarrow$ cohomology of $\mathfrak{g}$

Example 2 (Leibniz) $\mathfrak{g}$: a vect space $\left(\bigoplus_k \mathrm{Hom}(\otimes^k \mathfrak{g}, \mathfrak{g}), [-,-]_B\right)$ g. l. a.

$$[P, Q]_B = P \bar{\circ} Q - (-1)^{pq} Q \bar{\circ} P, \qquad P \bar{\circ} Q = \sum_{k=1}^{|P|+1} P \circ_k Q,$$

$$(P \circ_k Q)(x_1, \dots, x_{|P|+|Q|+1}) = \sum_{\sigma \in Sh(k-1, |Q|)} (-1)^{(k-1)|Q|} (-1)^{\sigma} P(x_{\sigma(1)}, \dots, x_{\sigma(k-1)}, Q(x_{\sigma(k)}, \dots, x_{\sigma(k+|Q|-1)}, x_{k+|Q|}), x_{k+|Q|+1}, \dots, x_{|P|+|Q|+1})$$

Then $\Omega \in \mathrm{Hom}(\otimes^2 \mathfrak{g}, \mathfrak{g})$, &

$$[\Omega, \Omega]_B(x_1, x_2, x_3) = 2\left(\Omega(\Omega(x_1, x_2), x_3) - \Omega(x_1, \Omega(x_2, x_3)) + \Omega(x_2, \Omega(x_1, x_3))\right)$$

Thus Maurer-Cartan of $\Leftrightarrow$ Leibniz on $\mathfrak{g}$.

- cohomology theory for embedding tensors

① controlling alg for $(\mathfrak{g}, \curvearrowright V)$ Lie Rep

$$\mathrm{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}(\mathfrak{g} \otimes V, V)$$

$$\left(\bigoplus_{k=0}^{\infty} \mathrm{Hom}(\wedge^{k+1} \mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}(\wedge^k \mathfrak{g} \otimes V, V), [\ ,\ ]_B\right) \ni (\mu, \rho)$$

$$\hookrightarrow$$

$$L = \mathrm{Hom}(\otimes^k(\mathfrak{g} \oplus V), (\mathfrak{g} \oplus V))$$

$[(\mu, \rho), (\mu, \rho)]_B = 0$

$\Leftrightarrow$ μ Lie bra

ρ rep.

② controlling alg for embedding tensor

$$\left(\bigoplus_{k=1}^{\infty} \mathrm{Hom}(\otimes^k V, \mathfrak{g}), [\![\ ,\ ]\!]\right)$$

M.C. $T \in \mathrm{Hom}(V, \mathfrak{g})$.

$$[\![T, T]\!](v_1, v_2) = 2\left([Tv_1, Tv_2] - T(\rho(Tv_1)v_2)\right)$$

Yvette K.S. derived bracket

① d.g.l.a. $(L, [\cdot,\cdot]_B, d_{\mu+\rho})$

② abelian sub. $L' \subset L$

③ derived bracket: $[\![-,-]\!] \triangleq [d-,-]_B$ is a lie on sub. (if closed)

③ cochain cx for a cohomology theory of ET is then

$$T \overset{\Phi}{\rightsquigarrow} \Omega \qquad \partial_T \theta = \pm [\![T, \theta]\!]$$

$$\begin{tikzcd}
\theta \arrow[r, "0"] \arrow[d, dashed] & \mathfrak{g} \arrow[r, "\partial_T"] \arrow[d, dashed] & \mathrm{Hom}(V, \mathfrak{g}) \arrow[r, "\partial_T"] \arrow[d, dashed] & \mathrm{Hom}(\otimes^2 V, \mathfrak{g}) \arrow[r, "\partial_T"] \arrow[d, dashed] & \cdots \\
V \arrow[r, "\partial_\Omega"] & \mathrm{Hom}(V, V) \arrow[r, "\partial_\Omega"] & \mathrm{Hom}(\otimes^2 V, V) \arrow[r, "\partial_\Omega"] & \mathrm{Hom}(\otimes^3 V, V) \arrow[r, "\partial_\Omega"] & \cdots
\end{tikzcd}$$

$$\Phi(f)(v_1, \dots, v_{k+1}) = \rho(f(v_1, \dots, v_k))(v_{k+1})$$

- Cohomology theory for $(T, \mathfrak{g}, V)$ ⇝ lie leibniz pair.

∵ for deformation & centrale extension, we'll change all three of them.

$\left(C(T) \oplus C(\mathfrak{g}, V),\ \text{new brackets} \right)$

Voronov higher

Yvette K.S. derived bracket

① d.g.l.a. $(L, [\cdot,\cdot]_B, \cancel{d_{\mu+\rho}})$ Δ, s.t. $[\Delta, \Delta] = 0$

② abelian sub, $L' \subset L$ & $L \xrightarrow{P} L'$,

③ derived bracket : $[\![\cdot, \cdot]\!] \stackrel{\Delta}{=} \cancel{[d -, -]_B}$ $\quad l_k = P[\underbrace{\ ,[\ \cdot\ [\Delta, -]\cdots]}_{k}$

is a lie on sub. (if closed)

Cochain cx,

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C^n(\mathfrak{g}, V) & \xrightarrow{\partial_{\mu,\rho}} & C^{n+1}(\mathfrak{g}, V) & \xrightarrow{\partial_{\mu,\rho}} & C^{n+2}(\mathfrak{g}, V) & \xrightarrow{\partial_{\mu,\rho}} & \cdots \\
& & \searrow^{\Omega_T} & & \searrow^{\Omega_T} & & \searrow^{\Omega_T} & \\
\cdots \longrightarrow & C^n(T) & \xrightarrow{\partial_T} & C^{n+1}(T) & \xrightarrow{\partial_T} & C^{n+2}(T) & \longrightarrow & \cdots
\end{array}$$

& Mayer-Vietoris

$$0 \to C^\bullet(T) \xrightarrow{\iota} C^\bullet(\mathfrak{g}, V, T) \xrightarrow{p} C^\bullet(\mathfrak{g}, V) \to 0$$

$$\rightsquigarrow \quad \cdots \to H^n(T) \longrightarrow H^n(\mathfrak{g}, V, T) \longrightarrow H^n(\mathfrak{g}, V) \longrightarrow H^{n+1}(T) \longrightarrow \cdots$$

- $\underline{\text{Deformation theory for } (\mathfrak{g}, V, T)}$ Lie Leibniz triple
  - $[-,-]_t = [-,-]_{\mathfrak{g}} + t\omega$, on $\mathfrak{g}[[t]]/(t^2) =: \mathfrak{g}_t = \mathfrak{g} \oplus t\mathfrak{g}$, $\omega \in \mathrm{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$
  - $\rho_t = \rho + t\varepsilon$, path of rep of $\mathfrak{g}_t$ on $V_t = V \oplus tV$
  - $T_t = T + t\mathcal{T}$, path of E.T. : $V_t \to \mathfrak{g}_t$. $(\omega, \varepsilon, \mathcal{T})$ deformation

$\underline{\text{Thm}}$ $H^2(\mathfrak{g}, V, T)$ classifies equivalent deformations.

- $\underline{\text{Centrale extension of } (\mathfrak{g}, V, T)}$

Thm. $H^2(\mathfrak{g}, V, T; \underbrace{\mathfrak{h}, W, S}_{\text{abelian}})$ classifies equivt c.ex of

$(\mathfrak{g}, V, T)$ by $(\mathfrak{h}, W, S)$.

homotopy theory

Voronov

① controlling alg for (embedding tensors) ET_∞ $\{l_i\}$

$$C(T_\circ) = \left(\bigoplus_{k=1}^{\infty} \mathrm{Hom}(\otimes^k V^\bullet, \mathfrak{g}^\bullet), [\![\,,\,]\!]\right)$$

M.C. $T_\bullet \in \mathrm{Hom}(TV^\bullet, \mathfrak{g}^\bullet)$ defines us an ET_∞.

Yvette K.S. derived bracket

① d.g.l.a. $(L, [\,,\,]_B, d_{\mu-\rho})$,

② abelian sub, $L' \subseteq L$ $\quad l_i = [\cdots[d[d,-]-]-]$

③ derived bracket: ~~$[\![\,,\,]\!] \overset{\Delta}{=} [d-,-]_B$~~

is a lie on sub. ~~(if closed)~~

② $ET_\infty \xrightarrow{F} Leib_\infty$

$(T_\circ, V^\bullet, \mathfrak{g}^\bullet) \longmapsto (V^\bullet, \theta_i)$ $\quad \theta_k(v_1, \dots, v_k) = \left(e^{[-,T_\cdot]_B} \sum (\mu_k + \rho_k)\right)(v_1, \dots, v_k)$

③ $Leib_\infty \xrightarrow{G} L_\infty$

$(V^\bullet; \theta_i)$ $\qquad (\mathcal{L}ie(V^\bullet), l_i)$

$\updownarrow$ $\qquad \bar{\Delta}(x) = 1 \otimes x + x \otimes 1$ $\qquad \updownarrow$

$(T(V^\bullet), d, \bar{\Delta})$

IIS Milnor-Moore

$(U(\mathcal{L}ie(V^\bullet)), d, \tilde{\Delta}) \underset{PBW}{\overset{\sim}{=\!=}} (S(\mathcal{L}ie(V^\bullet)), d, \bar{\Delta})$

Δ : codifferential cofree conilpotent coZinbiel

$\bar{\Delta}$: codiff - - - - coshuffle

$\parallel$

$\Delta + \tau_{12}\Delta$

④ Thm (Sheng-Tang-Zhu) F, G are both functors. Then $KS = G \circ F$ is still a functor $ET_\infty \to L_\infty$

⑤ Conjecture They are also ∞-functors. For this we need to know

weak equivalence (W.E.)

fibration

in each cat.

& we can show G preserves fibration

$\vdots$