Conformally invariant differential operators on Heisenberg groups and minimal representations

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Outline

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Motivation: L^2-realizations for minimal representations

G: real reductive Lie group $\rightsquigarrow$ *construct* a convenient realization of the minimal representation

Definition (L^2-realization)

We call a unitary representation π of G on a Hilbert space $\mathcal{H}$ an L^2-*realization* if

- $\mathcal{H} = L^2(M, d\mu)$ for some manifold M with measure $d\mu$
(more general: L^2-sections of a vector bundle)
- the underlying Lie algebra representation $d\pi$ is given by differential operators on M

Examples

- The metaplectic representation of $G = \mathrm{Mp}(n, \mathbb{R})$ has an L^2-realization on $L^2(\mathbb{R}^n)$.
$\rightsquigarrow$ $d\pi$ by differential operators of order ≤ 2
- (S. Gelfand '80) The minimal representation of $G = G_{2(2)}$ on $L^2(\mathbb{R}^3)$
- (P. Torasso '83) A small representation of $G = \widetilde{\mathrm{SL}}(3, \mathbb{R})$ on $L^2(\mathbb{R}^2)$
- (Kobayashi–Ørsted '03) The minimal representation of $G = \mathrm{O}(p, q)$, $p + q \geq 6$ even, has an L^2-realization on $L^2(C, d\mu)$, where $C \subseteq \mathbb{R}^{p-1,q-1}$ isotropic cone

Motivation: L^2-realizations for minimal representations

Why L^2-realizations?

- Particularly suitable for studying branching problems:
  - Dual pair correspondences in minimal representations
  - L^2-spectral theory for differential operators available
  - Mackey theory applicable for subgroups acting "geometrically"
- Applications to number theory/automorphic representations:
  - Theta series for exceptional groups
  - Fourier coefficients of global automorphic forms
- Relations to special functions and classical analysis:
  - Hermite and Laguerre functions (metaplectic representation)
  - Bessel functions and Fourier transforms (minimal representation of $\mathrm{O}(p, q)$)

Outline

A construction by Kazhdan–Savin

- G connected split real reductive group of type D or E
- $P = MAN$ – parabolic subgroup of G with $N \simeq V \ltimes \mathbb{R}$ a Heisenberg group (use $A \simeq \mathbb{R}^\times$)
- ρ – irreducible unitary representation of N with non-trivial central character
  $\rightsquigarrow$ can be realized on $L^2(\Lambda)$, $\Lambda \subseteq V$ Lagrangian (Schrödinger model)
- **Claim:** ρ extends to $MN = M \ltimes N$ (by the metaplectic representation)
- $\pi = \mathrm{Ind}_{MN}^{MAN} \rho$ irreducible unitary representation of P on $L^2(\mathbb{R}^\times \times \Lambda)$
- Note: G is generated by P and some non-trivial Weyl group element w_0

Theorem (Kazhdan–Savin '90)

The representation π of P extends uniquely to an irreducible unitary representation $\pi_{\min}$ of G, the minimal representation, by

$$\pi_{\min}(w_0)f(\lambda, x_0, x') = \exp\left(i\frac{n(x')}{\lambda x_0}\right) f(x_0, \lambda, x').$$

Here: $x = (x_0, x') \in \Lambda = \mathbb{R} \times \mathcal{J}$ and $n : \mathcal{J} \to \mathbb{R}$ is a cubic polynomial (the Jordan determinant) on the cubic Jordan algebra $\mathcal{J}$.

A construction by Kazhdan–Savin – generalizations

Remarks

- This construction actually works in the more general context of simply laced split groups over local fields of characteristic $\neq 2$ (Kazhdan–Savin '90). It was later generalized to G_2 over a local field by G. Savin '93 and to all exceptional p-adic groups by K. Rumelhart '97.
- The realization obtained in this way coincides with the representation constructed previously by S. Gelfand '80 for $G = G_{2(2)}$, and it bears a striking resemblance with the realization of P. Torasso '83 for $G = \widetilde{SL}(3,\mathbb{R})$ and by H. Sabourin '96 for $G = \mathrm{Spin}(4,3)$ using different methods.

Questions

- Is there an explanation for the formula for $\pi(w_0)$?
  Is there an "intrinsic" construction of the representation π?
- Can the L^2-realization be generalized to other real groups?

A construction by Kazhdan–Savin – alternative approach

Observation

There is an explicit P-equivariant functional $L^2(\mathbb{R}^\times \times \Lambda)^\infty \to \mathbb{C}_\chi$, so by Frobenius reciprocity:

$$\{0\} \neq \operatorname{Hom}_P(\pi^\infty|_P, \chi) = \operatorname{Hom}_G(\pi^\infty, \operatorname{Ind}_P^G(\chi))$$

$\rightsquigarrow$ π^∞ is a subrepresentation of a degenerate principal series representation induced from P

Idea: Construct π as subrepresentation of a degenerate principal series
$\rightsquigarrow$ $\pi(w_0)$ "automatic" and the construction might generalize to other real groups

Difficulties

- How to describe the subrepresentation?
  $\rightsquigarrow$ image/kernel of an intertwining operator (difficult to write down a single function)
- What is the invariant inner product?
  $\rightsquigarrow$ invariant Hermitian form induced by the intertwining operator (singular integral kernel)

$\rightsquigarrow$ compare with other constructions of minimal representations inside degenerate principal series

Outline

Degenerate principal series for Siegel parabolic subgroups

- G – simply-connected real reductive group
- $P = MAN$ – maximal parabolic subgroup of G with N abelian (*Siegel parabolic subgroup*)
- $\pi_{\varepsilon,\lambda} = \mathrm{Ind}_P^G(\varepsilon \otimes e^\lambda \otimes \mathbf{1})$ – degenerate principal series representation associated with the characters $\varepsilon \in \widehat{M}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$
- Reducibility, composition series and unitarity of $\pi_{\varepsilon,\lambda}$ was studied in detail by Johnson '91, Sahi '92, '93, Ørsted–Zhang '95, Zhang '95, M.–Schwarz '14 by algebraic methods (K-types, Lie algebra action and standard intertwining operators)
  $\rightsquigarrow$ the/a minimal representation $\pi_{\min}$ occurs as subrepresentation of $\pi_{\varepsilon,\lambda}$ for some ε and λ

Problem

Describe the subrepresentation $\pi_{\min} \subseteq \pi_{\varepsilon,\lambda}$ and its invariant inner product.

Conformally invariant differential operators

- Restriction to $\overline{N} \subseteq G/P$ realizes $\pi_{\varepsilon,\lambda}$ on a subspace $\mathcal{S}(\overline{N}) \subseteq I_{\varepsilon,\lambda} \subseteq C^\infty(\overline{N})$
- Structure theory of $\overline{N} \simeq \overline{\mathfrak{n}}$ (Jordan algebra/triple) gives rise to a system of second order constant coefficient differential operators $P_j(\partial)$ which is *conformally invariant* for $\lambda = \lambda_{\min}$:

$$[d\pi_{\lambda,\varepsilon}(X), P_j(\partial)] = \sum_k C_{jk}(X) P_k(\partial).$$

Examples

- $G = \mathrm{Sp}(2n,\mathbb{R})$, $\overline{\mathfrak{n}} = \mathrm{Sym}(n,\mathbb{R})$, $P_j(\partial) = (2 \times 2 \text{ minor})(\partial)$.
- $G = \mathrm{O}(p+1,q+1)$, $\overline{\mathfrak{n}} = \mathbb{R}^{p+q}$, $P(\partial) = \partial_1^2 + \cdots + \partial_p^2 - \partial_{p+1}^2 - \cdots - \partial_{p+q}^2$.

Observation

The joint kernel $I_{\min} = \{u \in I_{\varepsilon,\lambda} : P_j(\partial)u = 0 \, \forall j\}$ is a subrepresentation $\pi_{\min}$ of $\pi_{\varepsilon,\lambda}$ with invariant Hermitian form given by (a regularization of) the convolution expression

$$\langle u, v \rangle = \int_{\overline{\mathfrak{n}} \times \overline{\mathfrak{n}}} |\Delta(x-y)|^{\lambda-\rho,\varepsilon} u(x) \overline{v(y)} \, dx \, dy \qquad (\Delta \text{ some } M\text{-inv. polynomial on } \overline{\mathfrak{n}})$$

$\rightsquigarrow$ apply the Euclidean Fourier transform on $\overline{\mathfrak{n}}$!

The Euclidean Fourier transform

The Euclidean Fourier transform $\mathcal{I}_{\min} \subseteq \mathcal{S}'(\bar{\mathfrak{n}}) \to \mathcal{S}'(\mathfrak{n})$, $u \mapsto \widehat{u}$ provides a new realization $(\widehat{\pi}_{\min}, \widehat{\mathcal{I}}_{\min})$ of the minimal representation $(\pi_{\min}, \mathcal{I}_{\min})$.Note that

- $P_j(\partial)u = 0 \quad \Leftrightarrow \quad P_j(\xi)\widehat{u} = 0 \quad \Leftrightarrow \quad \operatorname{supp} \widehat{u} \subseteq \{P_j = 0\}$
- $\displaystyle\int_{\bar{\mathfrak{n}}\times\bar{\mathfrak{n}}} |\Delta(x-y)|^{\lambda-\rho,\varepsilon} u(x)\overline{u(y)}\,dx\,dy = \int_{\bar{\mathfrak{n}}} (\widehat{|\Delta|^{\lambda-\rho,\varepsilon}})(\xi) \cdot |\widehat{u}(\xi)|^2\,d\xi$

Theorem (Rossi–Vergne, Sahi, Sahi-Dvorsky, Kobayashi–Ørsted, M.–Schwarz)

- (Excluding some cases) The representation $\widehat{\pi}_{\min}$ is unitary and irreducible on $L^2(\mathcal{O}, d\mu)$, where $\mathcal{O} = \{\xi : P_j(\xi) = 0\,\forall\, j\}$ and $d\mu$ is a certain $\operatorname{Ad}(MA)$-equivariant measure on $\mathcal{O}$.
- The Lie algebra action $d\pi_{\min}$ is given by polynomial differential operators up to order 2.

Idea: Generalize the above steps (conformally invariant differential operators, Fourier transform) to degenerate principal series induced from Heisenberg parabolic subgroups.

Outline

Degenerate principal series for Heisenberg parabolics

- G – simple real Lie group
- $P = MAN$ – (maximal) parabolic subgroup with N a Heisenberg group
- $\pi_{\varepsilon,\lambda} = \mathrm{Ind}_P^G(\varepsilon \otimes e^\lambda \otimes \mathbf{1})$ – degenerate principal series representation ($\varepsilon \in \widehat{M}$, $\lambda \in \mathfrak{a}_\mathbb{C}^*$)

For many groups G it is known that $\pi_{\min} \subseteq \pi_{\varepsilon,\lambda}$ for some specific $\varepsilon \in \widehat{M}$, $\lambda \in \mathfrak{a}^*$.

Goal

- Identify $\pi_{\min}$ as a subrepresentation of $\pi_{\varepsilon,\lambda}$
- Obtain an L^2-realization by taking a Fourier transform.

- Restriction to $\overline{N} \subseteq G/P$ realizes $\pi_{\varepsilon,\lambda}$ on a subspace $\mathcal{S}(\overline{N}) \subseteq I_{\varepsilon,\lambda} \subseteq C^\infty(\overline{N})$

$\rightsquigarrow$ use the structure of $\overline{N} \simeq \overline{\mathfrak{n}} = V \ltimes \mathbb{R}$ to describe:

- conformally invariant differential operators on $\overline{N}$ $\rightsquigarrow$ subrepresentation
- standard intertwining operators $\rightsquigarrow$ invariant inner product
- Fourier transform $\rightsquigarrow$ L^2-realization

5-graded Lie algebras and symplectic invariants

The fact that N is a Heisenberg group is equivalent to the Lie algebra $\mathfrak{g}$ admitting a 5-grading:

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\bar{\mathfrak{n}}} \oplus \underbrace{\mathfrak{g}_0}_{\mathfrak{m}\oplus\mathfrak{a}} \oplus \underbrace{\mathfrak{g}_1 \oplus \mathfrak{g}_2}_{\mathfrak{n}} \qquad \text{with } \dim \mathfrak{g}_{\pm 2} = 1.$$

- $\mathrm{ad}(\mathfrak{m})|_{\mathfrak{g}_{\pm 2}} = 0$ and $\mathfrak{a} = \mathbb{R}H$ with $\mathrm{ad}(H)|_{\mathfrak{g}_j} = j \cdot \mathrm{id}_{\mathfrak{g}_j}$
- H can be completed to an $\mathfrak{sl}(2)$-triple by $E \in \mathfrak{g}_2$ and $F \in \mathfrak{g}_{-2}$.
- $V = \mathfrak{g}_{-1}$ is a symplectic vector space with symplectic form ω given by $[X, Y] = \omega(X, Y)F$.
- $\mathfrak{m}$ acts sympletically on V by the adjoint representation, i.e. $\mathfrak{m} \subseteq \mathfrak{sp}(V, \omega)$.

There are three symplectic invariants associated with (V, ω):

$$\begin{aligned}
&\mu : V \to \mathfrak{m} \subseteq \mathfrak{g}_0, && \mu(X) = \mathrm{ad}(X)^2 E && \text{(moment map)}\\
&\Psi : V \to V, && \Psi(X) = \mathrm{ad}(X)^3 E && \text{(cubic map)}\\
&Q : V \to \mathbb{R}, && Q(X)F = \mathrm{ad}(X)^4 E && \text{(quartic)}
\end{aligned}$$

Remark (Faulkner, Slupinski–Stanton)

$\mathfrak{g}$ can be reconstructed from the tuple $(V, \omega, \mathfrak{m}, \mu)$.

Conformally invariant differential operators

Barchini–Kable–Zierau use the invariants μ, Ψ and Q to construct systems of differential operators on the Heisenberg group $\overline{N}$ by quantization:

$$\Omega_\mu(T) \quad (T \in \mathfrak{m}), \qquad \Omega_\Psi(v) \quad (v \in V), \qquad \Omega_Q.$$

Theorem (Barchini–Kable–Zierau '08)

The system $\Omega_\mu(T)$ $(T \in \mathfrak{m})$ of differential operators is conformally invariant for $\pi_{\varepsilon,\lambda}$ if and only if $\lambda = \lambda_{\min}$. (+ similar statements for Ω_Ψ and Ω_Q)

$\rightsquigarrow$ $\ker \Omega_\mu(\mathfrak{m}) \subseteq I_{\varepsilon,\lambda}$ is a subrepresentation (possibly $\{0\}$)

In some special cases it was shown that the joint kernel $\ker \Omega_\mu(\mathfrak{m}) \subseteq I_{\varepsilon,\lambda}$ is the minimal representation of G, but mostly algebraically and without providing the explicit Hilbert space (Gross–Wallach '96, Kable '12, Kubo–Ørsted '18). $\rightsquigarrow$ Fourier transform!

Remark

To be precise, one has to modify the above theorem slightly: For every simple/abelian ideal $\mathfrak{m}' \subseteq \mathfrak{m}$ the system $\Omega_\mu(\mathfrak{m}')$ is conformally invariant for some $\lambda = \lambda(\mathfrak{m}')$.

Which Fourier transform?

Observations

1. The differential operators $\Omega_\mu(T)$ on $\overline{\mathfrak{n}} \simeq V \times \mathbb{R}$ do not have constant coefficients (invariant under translation), but are left-invariant (invariant under Heisenberg translation).
2. The G-invariant Hermitian form for $\lambda \in \mathbb{R}$ is given by the convolution expression

$$(u, v) \mapsto \int_{\overline{N}\times\overline{N}} |\Delta(x^{-1} \cdot y)|^{\lambda-\rho,\varepsilon} u(x)\overline{v(y)}\, dx\, dy,$$

where $\Delta(z, t) = t^2 - Q(z)$ for $(z, t) \in V \ltimes \mathbb{R} = \overline{\mathfrak{n}} \simeq \overline{N}$.

$\rightsquigarrow$ use the Heisenberg group Fourier transform!

The Heisenberg group Fourier transform

The infinite-dimensional irreducible unitary representations σ_λ of the Heisenberg group $\overline{N}$ can be realized on $L^2(\Lambda)$, $\Lambda \subseteq V$ Lagrangian, and are parameterized by $\lambda \in \mathbb{R}^\times = \mathfrak{z}(\overline{\mathfrak{n}})^* \setminus \{0\}$.

Heisenberg group Fourier transform

The Heisenberg group Fourier transform $\widehat{u}$ of $u \in L^1(\overline{N})$ is the operator-valued map

$$\widehat{u} : \mathbb{R}^\times \to \mathrm{End}(L^2(\Lambda)), \quad \widehat{u}(\lambda) = \int_N u(n)\sigma_\lambda(n)\, dn.$$

Note: The non-commutativity of $\overline{N}$ is reflected by the non-commutativity of $\mathrm{End}(L^2(\Lambda))$.

Properties of the Heisenberg group Fourier transform

- $\widehat{Xu}(\lambda) = d\sigma_\lambda(X) \circ \widehat{u}(\lambda)$ for every left-invariant vector field $X \in \mathfrak{n}$
- $\widehat{u * v}(\lambda) = \widehat{u}(\lambda) \circ \widehat{v}(\lambda)$
- $u \mapsto \widehat{u}$ extends to an isometric isomorphism $L^2(N) \simeq L^2(\mathbb{R}^\times, \mathrm{HS}(L^2(\Lambda)); |\lambda|^{\dim \Lambda}\, d\lambda)$.
- (F. '20) $u \mapsto \widehat{u}$ extends to an embedding $I_{\varepsilon,\lambda} \hookrightarrow \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathrm{Hom}(\mathcal{S}(\Lambda), \mathcal{S}'(\Lambda))$ for $\lambda > -\rho$.

Fourier transform of the subrepresentation

Question 1: What is the Fourier transform of the equation $\Omega_\mu(T)u = 0$?

Theorem (F. '20)

1. For every $T \in \mathfrak{m}$ we have
$$\widehat{\Omega_\mu(T)u}(\lambda) = d\omega_{\text{met}}(T) \circ \widehat{u}(\lambda),$$
where $d\omega_{\text{met}}$ is the metaplectic representation of $\mathfrak{m} \subseteq \mathfrak{sp}(V, \omega)$ on $L^2(\Lambda)$.
2. If G is non-Hermitian, then the joint kernel of all $d\omega_{\text{met}}(T)$, $T \in \mathfrak{m}$, is essentially one-dimensional, spanned by ξ. (M-distinguished vector in ω_{met})

(For $G = \mathrm{SL}(n, \mathbb{R})$ and $\mathrm{O}(p, q)$: need vector-valued principal series and generalization of $\Omega_\mu(T)$)

If now $\Omega_\mu(T)u = 0$ for all $T \in \mathfrak{m}$, then $\widehat{u}(\lambda) : \mathcal{S}(\Lambda) \to \mathbb{C}\xi$, so
$$\widehat{u}(\lambda)\varphi = \langle u_0(\lambda), \varphi \rangle \cdot \xi \qquad \text{for some } u_0 \in \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda).$$
$\rightsquigarrow$ We obtain a map $u \mapsto u_0$ from $\ker \Omega_\mu(\mathfrak{m}) \subseteq I_{\varepsilon,\lambda}$ into $\mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$.
$\rightsquigarrow$ We obtain a representation $\pi_{\min}$ of G on $I_{\min} \subseteq \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$ (possibly $= \{0\}$).

The L^2-realization

Question 2: When is $I_{\min} \neq \{0\}$? What is the G-invariant inner product on $I_{\min}$?

Theorem (F. '20)

1. Assume G is non-Hermitian. Then the map $u \mapsto u_0$ defines a representation $\pi_{\min}$ of G on $\mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$ whose Lie algebra action is given by polynomial differential operators up to order 3 (w/ explicit formulas in terms of μ, Ψ, Q; *not* case-by-case).
2. Assume in addition that $G \not\simeq F_{4(4)}$ and $G \not\simeq \mathrm{O}(p,q)$, $p,q \geq 4$, $p+q$ odd. Then the representation $\pi_{\min}$ is unitary and irreducible on $L^2(\mathbb{R}^\times \times \Lambda; |\lambda|^s \, d\lambda \, dx)$ and is the minimal representation of G (if it exists).
3. + explicit description of the lowest K-type (key ingredient of the proof; case-by-case)

Relation to previous work

- The obtained model of the minimal representation on $L^2(\mathbb{R}^\times \times \Lambda; |\lambda|^s \, d\lambda \, dx) \simeq L^2(\mathbb{R}^\times \times \Lambda)$ agrees with the ones by Gelfand, Torasso and Kazhdan–Savin
  $\rightsquigarrow$ uniform construction/formulas, recover the formula for $\pi(w_0)$
- New for quaternionic groups $E_{6(2)}$, $E_{7(-5)}$, $E_{8(-24)}$ and for $\mathrm{SO}(p,q)$

Outlook

Questions/Problems

- Some groups possess both a Siegel parabolic subgroup and a Heisenberg parabolic subgroup (e.g. $G = \mathrm{SO}(p,q), E_{6(6)}, E_{7(7)}$). In those cases where both constructions yield an L^2-realization of the minimal representation, how are they related?
- Brylinski–Kostant construct the minimal representation on holomorphic functions/sections on the minimal nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$. How is this realization related to the L^2-realization?
- Use the explicit formulas for the Lie algebra action in the L^2-realization to obtain branching laws for the restriction of the minimal representation to non-compact reductive subgroups, for instance:
  - $H = M \times \mathrm{SL}(2,\mathbb{R})$
  - $H = M' \times \mathrm{SL}(3,\mathbb{R})$
  - $H = M'' \times G_{2(2)}$
  - $\mathrm{SU}(2) \subseteq H \subseteq G$ for G of quaternionic type

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