Calabi-Yau complete intersections in fake weighted projective spaces

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Rmk. In dim 5 too many! Skarke-Schöller classified all IP weights systems ($> 10^{11}$).

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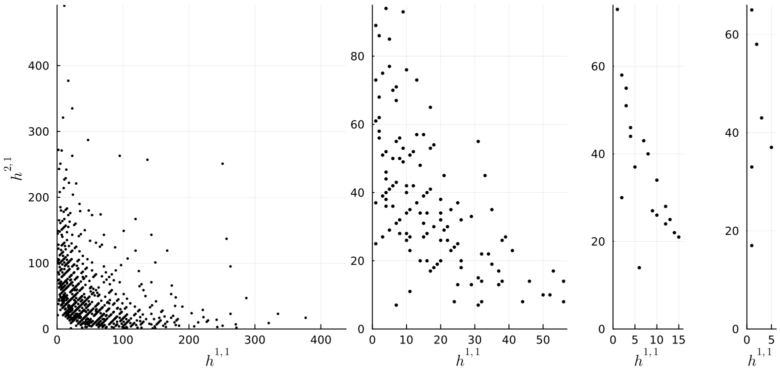
Rmk. These give higher dim CY hypersurfaces, but also CYCI from nef partitions. The classification algorithm can be adapted to generate simplices allowing nef partitions in dim > 6, and we obtain...

CYCI from nef partitions of simplices

$s \backslash \dim$	1	2	3	4	5
1	5	48	1.561	220.794	309.019.970
2	2	10	164	6.045	1.042.424
3	-	3	21	425	20.647
4	-	-	6	43	1.134
5	-	-	-	9	95
6	-	-	-	-	18

Hodge pairs in dimension three



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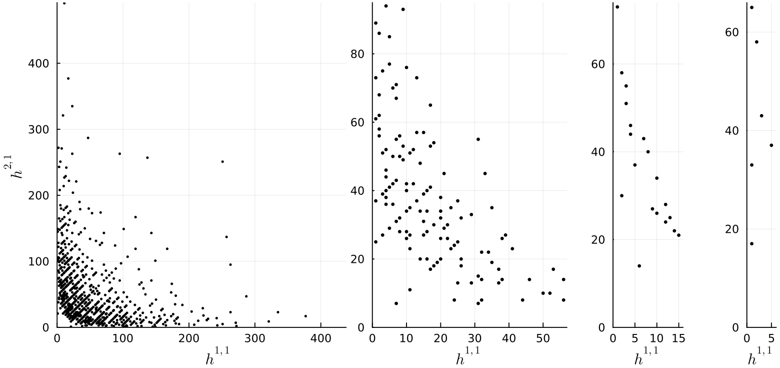

Rmk. 20 Hodge pairs not featured in the Kreuzer-Skarke list:
$(1,25)_2$, $(1,33)_4$, $(1,37)_2$, $(1,61)_2$, $(1,65)_4$, $(1,73)_{2,3}$, $(1,77)_4$, $(1,89)_2$, $(2,30)_3$, $(2,56)_2$, $(2,58)_{2,3,4}$, $(2,68)_2$, $(3,27)_2$, $(3,39)_2$, $(3,55)_3$, $(4,38)_2$, $(6,14)_3$, $(7,7)_2$, $(11,11)_2$.

The Fine interior

Def. For $\Delta \subset \mathbb{Q}^n$ polytope, $u \in \mathbb{Q}^n$, let $\operatorname{ord}_\Delta(u) = \min_{m \in \Delta} \langle m, u \rangle$.
The *Fine interior* of Δ is

$$F(\Delta) = \{m \in \mathbb{Q}^n : \langle m, u \rangle \geq \operatorname{ord}_\Delta(u) + 1 \ \forall u \in \mathbb{Q}^n \setminus \{0\}\}.$$

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Rmk. $F(\Delta) \supseteq \operatorname{int}(\Delta)$. Hence $F(\Delta) = \{0\} \implies \operatorname{int}(\Delta) = \{0\}$.

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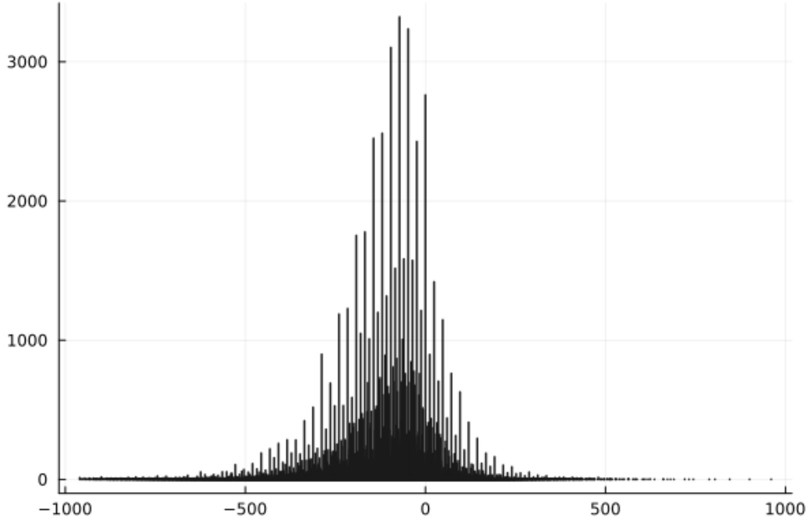
Thm. (Kasprzyk, 2008): 674.688 canonical polytopes in dim 3.

Thm. (—, 2026): 710.450 canonical simplices in dim 4.
Furthermore:

$\dim F(\Delta)$	0	1	2	3	4
# simplices	387.310	112.672	95.713	70.130	44.625

Stringy Euler numbers



Weight vectors

Def. $\Delta \subset \mathbb{Q}^n$ lattice simplex with $0 \in \mathrm{int}(\Delta)$ and primitive vertices $v_0, \ldots, v_n$. Its *weight vector* is the primitive $w \in \mathbb{Z}_{>0}^{n+1}$ s.t. $\sum_i v_i w_i = 0$.

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Same weight vector, different simplices!

Degree matrices

Constr. Write the vertices of Δ as columns of $P = [v_0 \ \dots \ v_n]$, and consider the projection

$$\mathbb{Z}^{n+1} \xrightarrow{Q} K := \mathbb{Z}^{n+1}/\mathrm{im}(P^t).$$

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Then $K \cong \mathbb{Z} \times \mathbb{Z}_{\mu_1} \times \cdots \times \mathbb{Z}_{\mu_r}$. Write $\omega_i = Q(e_i) \in K$, and view Q as a *degree matrix*:

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Rmk. Let Z_Δ be the TV arising form the face fan of Δ. Then $\mathrm{Cl}(Z) = K$ and the ω_i's are the torus-invariant divisor classes.

Fake weighted projective spaces

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Rmk. Fwps are precisely the TV arising from the face fans of simplices. They include all wps.

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All Gorenstein wps can be computed algorithmically using unit fraction partitions!

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Not. Write $\omega_i = (w_i, \eta_{i1}, \ldots, \eta_{ir})$ with $w_i \in \mathbb{Z}$, $\eta_{ij} \in \mathbb{Z}_{\mu_j}$, and set:

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Cor. Z is Gorenstein if and only if:

$$\begin{cases} L \mid \sum_i w_i, \\ M_j \mid \dfrac{\sum_i w_i}{L}, \quad j = 1, \dots, r, \\ \eta_{nj} = -(\eta_{0j} + \dots + \eta_{n-1j}) \in \mathbb{Z}/\mu_j\mathbb{Z}, \quad j = 1, \dots, r. \end{cases}$$

Nef-partitions of simplices

Def. A *nef-partition* of Δ is a partition $p_1 \sqcup \cdots \sqcup p_s = \{0, \ldots, n\}$ such that

$$\omega_{p_j} := \sum_{i \in p_j} \omega_i \in \mathrm{Pic}(Z), \;\; j = 1, \ldots, s.$$

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Rmk. If $\omega_{p_j} = \omega_i$, a general section of ω_{p_j} contains a monomial supported on the variable corresponding to ω_i. Eliminating this variable and equation, we obtain an isomorphic CYCI in a fwps of dim $d - 1$ and codim $s - 1$. Hence, we require $\omega_{p_j} \neq \omega_i$.

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- $b = (b_1/\mu_1, \ldots, b_s/\mu_s)$ with $0 \leq b_\ell \leq \mu_\ell - 1$

set: $R_{ij}(c, b) \coloneqq \mu_1 \big(w_j c + (w_i \eta_j - w_j \eta_i) \cdot b \big) \mod \mu_1 w_i$.

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set: $R_{ij}(c, b) := \mu_1\big(w_j c + (w_i\eta_j - w_j\eta_i)\cdot b\big) \mod \mu_1 w_i$.

Thm. Δ canonical $\iff$ $\Sigma_{j\neq i} R_{ij}(c, b) \geq \mu_1 w_i$ for all i and (c, b).

Thank you for your attention!