Combinatorial Dyson-Schwinger Equations in Tensorial Field Theory

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Tensor theories

Generalization of random matrices ($r = 2$) to rank $r > 2$ tensors

- random surfaces $\to$ random geometry from $r > 2$ dim. triangulations
- large-N expansion of Tensor models: Gurau degree ω^{G} generalizes genus
- "branched polymers" (continuous random tree) at LO
- subleading regimes: "planar phase" (Brownian map), multi-criticality...

Field theory with tensor-invariant interactions:

- genus/degree expansion related to renormalization group flow!
- tractable (solvable, integrable?) at LO (in the "UV" regime)
- richer (non-convergent?) structure beyond LO
- interpretation as quantum geometry (quantum gravity) for specific models ("group field theory")

From perturbative to non-perturbative

Ultimate goal: random/quantum geometry at criticality!

$\rightarrow$ non-perturbative techniques necessary!

- hints from functional RG: Wilson-Fisher type fixed point, further new fixed points, relation to vector theories [Pithis, JT 2020, 2021]
- functional methods for Dyson-Schwinger eq. ($\rightarrow$ Grosse-Wulkenhaar model)
- combinatorial (Hopf-algebraic) Dyson-Schwinger equations (cDSE)
- LO diagrammatics have tree structure $\rightarrow$ tractable cDSE

Steps to get there

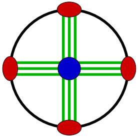
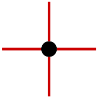
1. Hopf algebra of local QFT generalizes to Tensorial field theory (even more general: any non-local interactions)
2. Map from diagrams to (renormalized) amplitudes for actual computations (here BPHZ momentum scheme)
3. cDSE defined by Hochschild 1-cocycles of the theory
4. use relation to known cDSEs and methods to solve

Perturbative field theory

Fields $\phi : \mathbb{R}^D \to \mathbb{R}$ with covariance/propagator $P(\boldsymbol{x}, \boldsymbol{x}') = \int \mathrm{d}\mu[\phi]\, \phi(\boldsymbol{x})\phi(\boldsymbol{x}')$:

$$G^{(n)}(\boldsymbol{x}_1, ..., \boldsymbol{x}_n) = \int \mathrm{d}\mu[\phi]\, \mathrm{e}^{\mathrm{i}S_{\mathrm{IA}}[\phi]} \prod_{i=1}^{n} \phi(\boldsymbol{x}_i)$$

$$S_{\mathrm{IA}}[\phi] = \int_{\mathbb{R}^D} \mathrm{d}\boldsymbol{x}\, \lambda_k \phi(\boldsymbol{x})^k = \lambda_k \int_{\mathbb{R}^D} \prod_{i=1}^{k} \mathrm{d}\boldsymbol{q}_i\, \delta\Big(\sum_{i=1}^{k} \boldsymbol{q}_i\Big) \prod_{i=1}^{k} \tilde{\phi}(\boldsymbol{q}_i)$$

Point-like interactions, e.g. quartic $k = 4$:  $\cong$ 

Perturbative exp. $\mathrm{e}^{\mathrm{i}S_{\mathrm{IA}}[\phi]} = \sum_l \frac{(\mathrm{i}S_{\mathrm{IA}})^l}{l!} \Rightarrow$ formal power series over graphs γ:

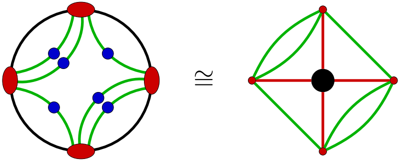
$$G^{(n)}(\boldsymbol{p}_1, ..., \boldsymbol{p}_n) = \sum_{\substack{\gamma \in \mathbf{G}_1, \\ N^e_\gamma = n}} \frac{1}{|\mathrm{Aut}\,\gamma|} \prod_{e \in \mathcal{E}_\gamma} \int \mathrm{d}\boldsymbol{q}_e \tilde{P}(\boldsymbol{q}_e) \prod_{v \in \mathcal{V}_\gamma} \mathrm{i}\lambda_v \delta\Big(\sum_{e @ v} \boldsymbol{q}_e\Big)$$

Combinatorially non-local field theory (cNLFT)

Combinatorially non-local interactions for fields $\phi : (\mathbb{R}^d)^r \to \mathbb{R}$:

$$S_{\text{IA}}[\phi] = \lambda_\gamma \int \prod_{i=1}^{k} \mathrm{d}\boldsymbol{q}_i \prod_{(ia,jb)} \delta(q_i^a - q_j^b) \prod_{i=1}^{k} \tilde{\phi}(\boldsymbol{q}_i)$$

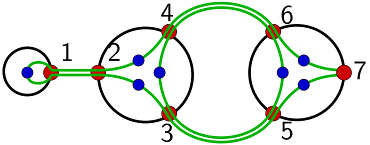
pairwise convolution of individual entries $q^a \in \mathbb{R}^d$, $a = 1, ..., r$



Combinatorics of interaction: vertex graph $\gamma =$  , not just $k = V_\gamma$

Perturbation theory: 2-graphs

$\rightarrow$ Perturbative series over "ribbon graphs", "stranded diagramms" ...
here general concept: *2-graphs* $\Gamma \in \mathbf{G}_2$ (or maybe better *strand graphs*)



$$G^{\gamma}(\boldsymbol{p}_1, ..., \boldsymbol{p}_{V_\gamma}) = \sum_{\substack{\Gamma \in \mathbf{G}_2, \\ \partial\Gamma = \gamma}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \prod_{v \in \mathcal{V}_\Gamma} \mathrm{i}\lambda_{\gamma_v} \prod_{f \in \mathcal{F}_\Gamma^{\mathrm{int}}} \int_{\mathbb{R}^d} \mathrm{d}q_f \prod_{\{i,j\} \in \mathcal{E}_\Gamma} \tilde{P}(\boldsymbol{q}_i)$$

Feynman rules:

1. coupling $\mathrm{i}\lambda_{\gamma_v}$ for each vertex $v \in \mathcal{V}_\Gamma$ with vertex graph γ_v
2. propagator $\tilde{P}(\boldsymbol{q}_i)$ for each internal edge $e = \{i, j\} \in \mathcal{E}_\Gamma$,
3. Lebesgue integral $\int_{\mathbb{R}^d} \mathrm{d}q_f$ for internal face $f \in \mathcal{F}_\Gamma^{\mathrm{int}}$ ($q_f = q_i^a$ identified)

Renormalization

Integrals might not converge $\rightarrow$ renormalization needed

- various prescriptions how to remove infinite part of the integral
- always necessary: forest formula to subtract subdivergences
- universally described by the Connes-Kreimer Hopf algebra
- principle of locality needed for this

Main result:

Hopf algebra of Feynman graphs generalizes to 2-graphs in cNLFT

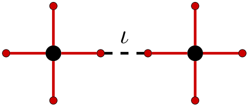
- locality captured by vertex graphs (generalizing "Moyality", "traciality")
- clear and concise algorithm to apply forest formula: ex. BPHZ momentum
- opens up possibility for Hopf-algebra based methods, in particular cDSE

Outline

1. Motivation
   - Tensorial field theory
   - Combinatorial non-locality
2. 2-graphs
   - From 1-graphs to 2-graphs
   - Contraction and boundary
   - Algebra
3. Renormalization
   - Renormalizable field theories
   - The BPHZ momentum scheme
   - Combinatorial DSE

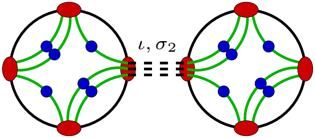
Half-edge graphs + strands

A *1-graph* is a tuple $g = (\mathcal{V}, \mathcal{H}, \nu, \iota)$ with



- a set of *vertices* $\mathcal{V}$
- a set of *half-edges* $\mathcal{H}$
- an adjacency map $\nu : \mathcal{H} \to \mathcal{V}$
- an involution $\iota : \mathcal{H} \to \mathcal{H}$ pairing *edges* (fixed points are external edges)

A *2-graph* $G = (\mathcal{V}, \mathcal{H}, \nu, \iota; \mathcal{S}, \mu, \sigma_1, \sigma_2)$:

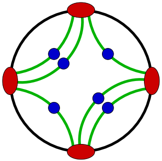


- a set of strand sections $\mathcal{S}$
- an adjacency map $\mu : \mathcal{S} \to \mathcal{H}$
- fixed-point free involution $\sigma_1 : \mathcal{S} \to \mathcal{S}$ with $\forall s \in \mathcal{S}$: $\nu \circ \mu \circ \sigma_1(s) = \nu \circ \mu(s)$
- an involution $\sigma_2 : \mathcal{S} \to \mathcal{S}$ pairing strands at edges: $\forall s \in \mathcal{S}$: $\iota \circ \mu(s) = \mu \circ \sigma_2(s)$ and s is fixed point of σ_2 iff $\mu(s)$ is fixed point of ι.

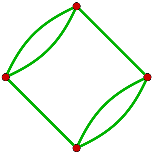
Involutions $\iota, \sigma_1, \sigma_2$ are equivalent to edge sets $\mathcal{E} \subset \mathbf{2}^{\mathcal{H}}$ and $\mathcal{S}^v, \mathcal{S}^e \in \mathbf{2}^{\mathcal{S}}$

Vertex-graph representation

Vertex graph $g_v = (\mathcal{V}_v, \mathcal{H}_v, \nu_v, \iota_v) := \left(\nu^{-1}(v), (\nu \circ \mu)^{-1}(v), \mu|_{\mathcal{H}_v}, \sigma_1|_{\mathcal{H}_v}\right)$



$\longrightarrow \quad g_v =$



Represent 2-graphs via vertex graphs

$$\pi_{\mathrm{vg}} : (\mathcal{V}, \mathcal{H}, \nu, \iota; \mathcal{S}, \mu, \sigma_1, \sigma_2) \mapsto \Big(\bigsqcup_{v \in \mathcal{V}} g_v, \iota, \sigma_2 \Big)$$

Not bijective! In general $g_v = \sqcup_i g_v^{(i)}$, vertex belonging information lost...

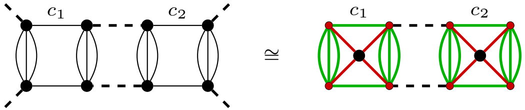
$$\beta_{\mathrm{vg}} : (\mathcal{V}, \mathcal{H}, \nu, \iota; \mathcal{S}, \mu, \sigma_1, \sigma_2) \mapsto \left(\{g_v\}_{v \in \mathcal{V}}, \iota, \sigma_2 \right) \text{ is bijection}$$

Example: edge-coloured graphs

Feynman diagrams of rank-r tensor theories: regular edge-coloured graphs

$(r+1)$-coloured graphs are 2-graphs with r strands per edge

- colour $c = 0$ edges $\to$ 2-graph edges
- colour $c \neq 0$ subgraph components $\to$ vertex graphs
- stranding of edges σ_2 fixed by colour preservation



Bijective only for connected vertex graphs

Topology of edge-coloured graphs

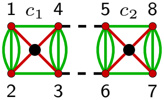
$(r+1)$-coloured graphs $\Longleftrightarrow$ r-dimensional pseudo manifolds [Gurau '11]
(abstract simplicial complexes)

Outline

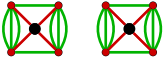
Subgraphs

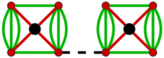
For a 2-graph G, a *subgraph* H is a 2-graph differing from G only in $\mathcal{E}_H \subset \mathcal{E}_G$ and $\mathcal{S}^e_H \subset \mathcal{S}^e_G$. Then one writes $H \subset G$.

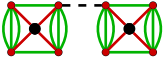
2^{E_G} subgraphs per 2-graph G, for example for

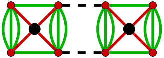
$G =$ 

:

$H_0 =$ 

$H_1 =$ 

$H_2 =$ 

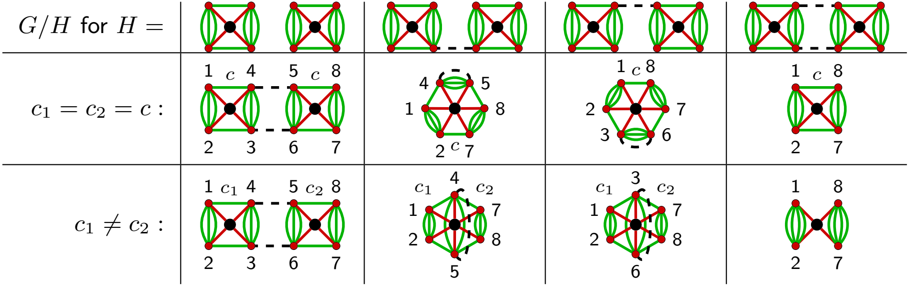
$H_3 =$ 

Contraction

Contraction of $H \subset G$: shrinking all stranded edges of H:

- $\mathcal{V}_{G/H} = \mathcal{K}_H$ the set of connected components of H
- $\mathcal{H}_{G/H} = \mathcal{H}_H^{\text{ext}}$, $\mathcal{S}_{G/H} = \mathcal{S}_H^{\text{ext}}$, only external half-edges of H remain
- $\mathcal{E}_{G/H} = \mathcal{E}_G \setminus \mathcal{E}_H$, $\mathcal{S}^e_{G/H} = \mathcal{S}^e_G \setminus \mathcal{S}^e_H$ (deleting stranded edges of H)
- $\mathcal{S}^v_{G/H} = \{\{s_1, s_{2n}\} | (s_1 ... s_{2n}) \in \mathcal{F}_H^{\text{ext}}\}$, external faces are shrunken to the strands at the new contracted vertices

Example:



Labelled vs. Unlabelled

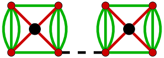
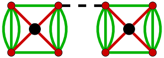
Unlabelled 2-graphs

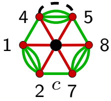
Isomorphism $j : G_1 \to G_2$ is a triple of bijections $j = (j_{\mathcal{V}}, j_{\mathcal{H}}, j_{\mathcal{S}})$ s.t.:

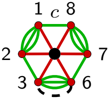
- $\nu_{G_2} = j_{\mathcal{V}} \circ \nu_{G_1} \circ {j_{\mathcal{H}}}^{-1}$ and $\mu_{G_2} = j_{\mathcal{H}} \circ \mu_{G_1} \circ {j_{\mathcal{S}}}^{-1}$
- $\iota_{G_2} = j_{\mathcal{H}} \circ \iota_{G_1} \circ {j_{\mathcal{H}}}^{-1}$
- $\sigma_{1G_2} = j_{\mathcal{S}} \circ \sigma_{1G_1} \circ {j_{\mathcal{S}}}^{-1}$ and $\sigma_{2G_2} = j_{\mathcal{S}} \circ \sigma_{2G_1} \circ {j_{\mathcal{S}}}^{-1}$

Then equivalence $G_1 \cong G_2$, *unlabelled 2-graph*, $\Gamma = [G_1]_{\cong} = [G_2]_{\cong}$.
Compatible with contractions.

Example:

$H_1 =$  $\cong H_2 =$ 

$\Rightarrow [G/H_1]_{\cong} = \Big[$ 

$\Big]_{\cong} = [G/H_2]_{\cong} = \Big[$ 

$\Big]_{\cong}$

Boundary and external structure

Residue and skeleton

2-graph has two characteristic 2-graphs without edges $\mathbf{R}^* \subset \mathbf{G}_2$:

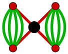
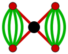
- $\mathrm{res} : \mathbf{G}_2 \to \mathbf{R}^*, \Gamma \mapsto \Gamma/\Gamma$, the "external structure"
- $\mathrm{skl} : \mathbf{G}_2 \to \mathbf{R}^*, \Gamma \mapsto \Theta_0$, the subgraph without edges

Boundary and vertex graphs

Can be used to define the boundary 1-graph of a 2-graph:

- $\partial : \mathbf{G}_2 \to \mathbf{G}_1, \quad \Gamma \mapsto \partial\Gamma := \pi_{\mathrm{vg}}(\mathrm{res}(\Gamma))$

For r-coloured 2-graphs: indeed $(r-1)$-dimensional boundary ps. manifolds

External structure must be sensitive to con. comp. (e.g.  $\sqcup$ ):

- $\widetilde{\partial} : \mathbf{G}_2 \to \mathcal{P}(\mathbf{G}_1),\ \Gamma = \bigsqcup_i \Gamma_i \mapsto \widetilde{\partial}\Gamma := \{\partial\Gamma_i\}_i \quad = \beta_{\mathrm{vg}}(\mathrm{res}(\Gamma))$
- $\widetilde{\varsigma} : \mathbf{G}_2 \to \mathcal{P}(\mathbf{G}_1)\,, \qquad \Gamma \mapsto \widetilde{\varsigma}\Gamma := \{\gamma_v\}_{v\in\mathcal{V}_\Gamma} = \beta_{\mathrm{vg}}(\mathrm{skl}(\Gamma))$

Outline

Coalgebra

Algebra

Let $\mathcal{G} := \langle \mathbf{G}_2 \rangle$ be the $\mathbb{Q}$-algebra generated by all 2-graphs $\Gamma \in \mathbf{G}_2$ with

$$m : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G} \quad , \quad \Gamma_1 \otimes \Gamma_2 \mapsto \Gamma_1 \sqcup \Gamma_2$$

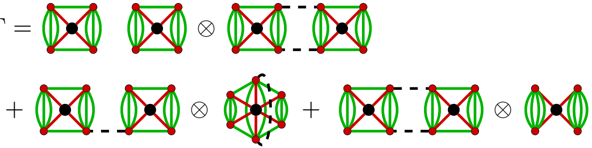
Unital commutative algebra with $u : \mathbb{Q} \to \mathcal{G}, q \mapsto q\mathbb{1}$ ($\mathbb{1}$ empty 2-graph)

Coalgebra

$$\Delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}, \quad \Gamma \mapsto \sum_{\Theta \subset \Gamma} \Theta \otimes \Gamma/\Theta$$

Associative counital coalgebra with counit $\epsilon = \chi_{\mathbf{R}^*} : \mathcal{G} \to \mathbb{Q}$
In fact, also bialgebra (all proofs completely parallel to 1-graphs)

Example: $\Delta\Gamma =$



Subalgebras

Contraction closure

Let $\mathbf{P}, \mathbf{K} \subset \mathbf{G}_2$.

- $\mathbf{P}$-*contraction closure* ${}^{\mathbf{P}}\overline{\mathbf{K}} := \{\Gamma = \Gamma'/\Theta | \Theta \subset \Gamma' \in \mathbf{K}, \Theta \in \mathbf{P}\}$
- *contraction closure* $\overline{\mathbf{K}} := {}^{\mathbf{G}_2}\overline{\mathbf{K}}$

2-graph subbialgebra

- 2-graphs of restricted vertex types $\mathbf{V}$: $\mathbf{G}_2(\mathbf{V}) := \{\Gamma \in \mathbf{G}_2 \,|\, \tilde{\varsigma}\Gamma \in \mathcal{P}(\mathbf{V})\}$
- Prop: $\langle\overline{\mathbf{G}_2(\mathbf{V})}\rangle$ is a subbialgebra of $\mathcal{G}$.
- for field theory with interactions $\mathbf{V} \in \mathbf{G}_1$: "theory space" $\langle\overline{\mathbf{G}_2(\mathbf{V})}\rangle$

Example: Matrix/Tensor field theory

- 2-graphs characterized by fixed # of strands at edges = tensor rank r
- for rank-r interactions $\mathbf{V}_r$: $\overline{\mathbf{G}_2(\mathbf{V}_r)} = \mathbf{G}_2(\mathbf{V}_r)$ contraction closed
- r-coloured diagrams generate subbialgebra $\langle\mathbf{G}_2(\mathbf{V}_r)\rangle$

Hopf algebra of 2-graphs

interest: group structure on algebra homomorphisms $\phi, \psi : \mathcal{G} \to \mathcal{A}$ wrt

convolution product: $\phi * \psi := m_{\mathcal{A}} \circ (\phi \otimes \psi) \circ \Delta_{\mathcal{G}}$

Hopf algebra of 2-graphs

- The bialgebra of 2-graphs $\mathcal{G}$ is a Hopf algebra, i.e. there is a *coinverse* S:

$$S * \mathrm{id} = \mathrm{id} * S = u \circ \epsilon\,.$$

- The set $\Phi_{\mathcal{A}}^{\mathcal{G}}$ of algebra homomorpisms from $\mathcal{G}$ to a unital commutative algebra $\mathcal{A}$ is a group with inverse $S^{\phi} = \phi \circ S$ for every $\phi \in \Phi_{\mathcal{A}}^{\mathcal{G}}$,

$$S^{\phi} * \phi = \phi * S^{\phi} = u_{\mathcal{A}} \circ \epsilon_{\mathcal{G}}.$$

- The subbialgebra $\langle\overline{\mathbf{G}_2(\mathbf{V})}\rangle$ for specific vertex graphs $\mathbf{V} \subset \mathbf{G}_1$ is a Hopf subalgebra of $\mathcal{G}$.

Outline

Renormalizability

cNLFT $T = (\mathbf{E}, \mathbf{V}, \omega, d)$ given by dimension $d \in \mathbb{N}$, $\mathbf{E}, \mathbf{V} \subset \mathbf{G}_1$, weights

$$\omega : \mathbf{E} \cup \mathbf{V} \to \mathbb{Z}$$

Feynman diagrams $\mathbf{G}_2^T := \mathbf{G}_2(\mathbf{V})$ generate a Hopf algebra $\mathcal{G}_T := \langle \overline{\mathbf{G}_2^T} \rangle$

Hopf algebra of divergent Feynman 2-graphs

- *Superficial degree of divergence* $\omega^{\mathrm{sd}}(\Gamma) = \sum_{v \in \mathcal{V}_\Gamma} \omega(\gamma_v) - \sum_{e \in \mathcal{E}_\Gamma} \omega(\gamma_e) + d \cdot F_\Gamma$
- T is *renormalizable* iff $\boxed{\omega^{\mathrm{sd}}(\Gamma) = \omega(\partial\Gamma) - \delta_\Gamma}$ for all Γ with $\omega^{\mathrm{sd}}(\Gamma) > 0$;

$$\mathbf{P}_T^{\mathrm{s.d.}} := \left\{ \Gamma = \bigsqcup\nolimits_{i \in I} \Gamma_i \in \mathbf{G}_2^T \ 1\mathrm{PI} \ | \forall i \in I : \Gamma_i \notin \mathbf{R} \Rightarrow \omega^{\mathrm{sd}}(\Gamma_i) \geq 0 \right\}$$

- $\mathcal{H}_T^{\mathrm{f2g}} = \langle \mathbf{P}_T^{\mathrm{s.d.}} \rangle$ is the Hopf algebra of divergent 2-graphs of T
- Hopf subalgebra of $\mathcal{G}_T$ when contraction closed due to renormalizablity.

Tensorial field theory

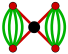
$\phi^n_{d,r}$ *tensorial field theory* [BenGeloun'14]:

- similar to $d_r = d(r-1)$ dimensional local field theory
- interactions $\mathbf{V}$ are r-coloured graphs, $\omega(\gamma_v) = d_r - \frac{d_r - 2\zeta}{2} V_{\gamma_v}$
- Feynman diagrams Γ are 2-graphs bijective to $(r+1)$-coloured graphs

Divergence degree (for general propagator $\omega(\gamma_e) = 2\zeta$):

$$\omega^{\mathrm{sd}}(\Gamma) = d_r - \frac{d_r - 2\zeta}{2} V_{\partial\Gamma} - d\left(\delta^{\mathrm{G}}_{\Gamma} + K_{\partial\Gamma} - 1\right) .$$

$\delta^{\mathrm{G}}_{\Gamma} = \frac{2\omega^{\mathrm{G}}_{\Gamma} - 2\omega^{\mathrm{G}}_{\partial\Gamma}}{(r-1)!}$, Gurau degree $\omega^{\mathrm{G}} = \sum_J g_J$ (J generalized Heegaard surfaces)

- theories renormalizable for interactions up to $n = \lfloor \frac{2d_r}{d_r - 2\zeta} \rfloor$
- just-renormalizable $\phi^4_{d,r}$ theories: $d_r = 4\zeta$ (e.g. $\zeta = \frac{1}{2}$: $\phi^4_{2,2}$, $\phi^4_{1,3}$)
- coproduct preserves δ^{G} [Raasakka/Tanasa'13] $\Rightarrow$ renormalizability for $\delta^{\mathrm{G}}_{\Gamma} > 0$
- $K_{\partial\Gamma} > 1$ possible: e.g. $\phi^6_{1,4}$ theory [BenGeloun/Rivasseau'13] needs  $\in \mathbf{V}$

Outline

Momemtum scheme in cNLFT

algebra homo. $A:\mathcal{G}\to\mathcal{A}$ to the alg. $\mathcal{A}$ of integrals with rational integrands

$$A_\Gamma = A(\Gamma) : \{p_f\}_{f\in\widetilde{\mathcal{F}}^{\mathrm{ext}}_\Gamma,} \mapsto A_\Gamma(\{p_f\}) := \prod_{v\in\mathcal{V}_\Gamma}\lambda_{\gamma_v}\prod_{f\in\mathcal{F}^{\mathrm{int}}_\Gamma}\int_{\mathbb{R}^d}\mathrm{d}q_f\prod_{\{i,j\}\in\mathcal{E}_\Gamma}\tilde{P}(\boldsymbol{q}_i)$$

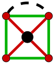
Momentum subtraction operator: Taylor expansion

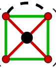
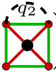
$$R[A_\Gamma](\{p_f\}) := \left(T^{\omega}_{\{p_f\}}A_\Gamma\right)(\{p_f\}) = \sum_{|\vec{k}|\le\omega^{\mathrm{sd}}(\Gamma)}\frac{1}{\vec{k}!}\frac{\partial^{|\vec{k}|}A^{\Lambda}_\Gamma}{\prod_f\partial p_f^{k_f}}(0)\prod_{f\in\widetilde{\mathcal{F}}^{\mathrm{ext}}_\Gamma}p_f^{k_f}$$

Renormalized amplitude for primitive divergent 2-graphs (no subdivergences):

$$A_{\mathrm{R}}(\Gamma) := (A - R\circ A)(\Gamma)$$

Example: Tadpole diagrams in tensorial theories

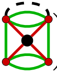
$\phi^4_{d=2,r=2}$ theory with $\tilde{P}(\boldsymbol{p}) = \frac{1}{|p_1|+|p_2|+1}$ (i.e. $\omega(\gamma_e) = 1$):  $\cong$ 

$A_{\mathrm{R}}\Big($  $\Big)(p_1) \equiv A_{\mathrm{R}}\Big(\, p_1$ 

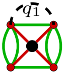
$\Big) = \lambda \left(1 - T^1_{p_1}\right) \int_{\mathbb{R}^2} \mathrm{d}q_2 \, \frac{1}{|p_1| + |q_2| + 1}$

$$= 2\pi\lambda \Big((|p_1| + 1)\log(|p_1| + 1) - |p_1| \Big)$$

$\phi^4_{d=1,r=3}$ theory with $\tilde{P}(\boldsymbol{p}) = \frac{1}{|p_1|+|p_2|+|p_3|+1}$: two tadpoles for each colour

$A_{\mathrm{R}}\Big(\, p_1$  $\Big) = \lambda \left(1 - T^1_{p_1}\right) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\mathrm{d}q_2 \mathrm{d}q_3}{|p_1| + |q_2| + |q_3| + 1}$

$$= 4\lambda \Big((|p_1| + 1)\log(|p_1| + 1) - |p_1| \Big)$$

$A_{\mathrm{R}}\Big(\, p_2, p_3$ 

$\Big) = \lambda \left(1 - T^0_{p_2,p_3}\right) \int_{\mathbb{R}} \frac{\mathrm{d}q_1}{|q_1| + |p_2| + |p_3| + 1}$

$$= -2\lambda \log(|p_2| + |p_3| + 1)$$

Subdivergences

In a renormalizable local field theory T:

- BPHZ: $\forall\Gamma$ with $\omega^{\mathrm{sd}}(\Gamma)\geq 0$ there is a counter term s.t. $A_{\mathrm{R}}(\Gamma)$ converges
- Zimmermann: forest formula for counter term of nested subdivergences
- Kreimer: counter term $S^{\mathrm{A}}_{\mathrm{R}}:\mathcal{H}^{\mathrm{fg}}\to\mathcal{A}$ from antipode S in Hopf alg. $\mathcal{H}^{\mathrm{fg}}$:

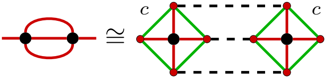
$$A_{\mathrm{R}}=S^{\mathrm{A}}_{\mathrm{R}}*A$$

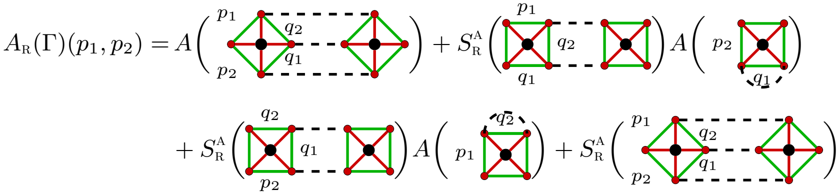
$$S^{\mathrm{A}}_{\mathrm{R}}(\Gamma)=-R\left[(S^{\mathrm{A}}_{\mathrm{R}}*A\circ P)(\Gamma)\right]=-\sum_{\substack{\Theta\in\mathcal{H}^{\mathrm{fg}}\\ \Theta\subsetneq\Gamma}}R\Big[S^{\mathrm{A}}_{\mathrm{R}}(\Theta)A(\Gamma/\Theta)\Big]$$

Renormalization in cNLFT

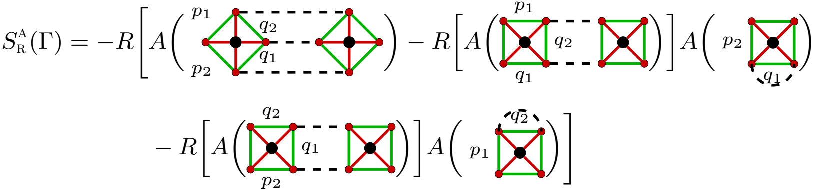
- counter term $S^{\mathrm{A}}_{\mathrm{R}}$ in the same way on the Hopf algebra of 2-graphs
- if cNLFT T is renormalizable, $A_{\mathrm{R}}=S^{\mathrm{A}}_{\mathrm{R}}*A$ on $\mathcal{H}^{\mathrm{f2g}}_{T}$ gives ren. amplitudes
- BPHZ momentum scheme: $S^{\mathrm{A}}_{\mathrm{R}}$ is algebra homomorphism since R is a Rota-Baxter operator $\big(R[AB]+R[A]R[B]=R[R[A]B+A\,R[B]]\big)$ as in local QFT

Example: sunrise diagram in $\phi^4_{2,2}$ theory

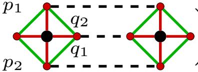
Sunrise 2-graph $\Gamma =$ 



Last counter term calculated recursively:



$$A_{\mathrm{R}}\Bigg($$



$$\Bigg)$$

$$= \lambda^2$$

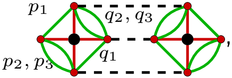
$$\left(1 - T^1_{p_1,p_2}\right) \int_{\mathbb{R}^2} \mathrm{d}q_1 \int_{\mathbb{R}^2} \mathrm{d}q_2 \Bigg(\frac{1}{|p_1|+|q_2|+1} \frac{1}{|q_1|+|q_2|+1} \frac{1}{|q_1|+|p_2|+1} + \frac{1}{|q_1|+|p_2|+1} \left(-T^0_{p_1,q_1}\right) \frac{1}{|p_1|+|q_2|+1} \frac{1}{|q_1|+|q_2|+1} + \frac{1}{|p_1|+|q_1|+1} \left(-T^0_{q_2,p_2}\right) \frac{1}{|q_1|+|q_2|+1} \frac{1}{|q_2|+|p_2|+1} \Bigg)$$

$$= \lambda^2$$

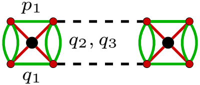
$$\frac{4\pi^2}{|p_1|+|p_2|+1} \Bigg[|p_1||p_2|\zeta_2 + (|p_1|+|p_2|+1) \sum_{i=1,2} \Big((|p_i|+1)\log(|p_i|+1) - |p_i| \Big) - \prod_{i=1,2} (|p_i|+1)\log(|p_i|+1) + \sum_{i=1,2} |p_i|(|p_i|+1)\mathrm{Li}_2(-|p_i|) \Bigg]$$

- in agreement with [Hock2020]
- multiple polylogarithms as in local QFT, but $\zeta_2 = \pi^2/6$ is peculiar

Example: sunrise diagram in $\phi^4_{1,3}$ theory

Sunrise diagram 

, ex. of non-melonic divergent diagram:

- only logarithmic divergence $\omega^{\mathrm{sd}}(\Gamma) = 3 - 3 = 0$
- only one proper divergent 1PI subgraph 

- $\Rightarrow$ no overlapping divergence $\Rightarrow$ factorizing A_{R}

$$A_{\mathrm{R}}(p_1, p_2, p_3) = \lambda^2 \; (1 - T^0_{p_1,p_2,p_3}) \int_{\mathbb{R}} \mathrm{d}q_1 \, \frac{1}{|q_1| + |p_2| + |p_3| + 1}$$
$$\times \, (1 - T^0_{p_1,q_1}) \int_{\mathbb{R}} \mathrm{d}q_2 \int_{\mathbb{R}} \mathrm{d}q_3 \, \frac{1}{|q_1| + |q_2| + |q_3| + 1} \, \frac{1}{|p_1| + |q_2| + |q_3| + 1}$$

- more restricted set of LO diagrams ("melonic") in tensorial theories
- what's the number theory (class of amplitude functions) of tensorial fields?

Outline

Combinatorial Dyson-Schwinger equations

Comb. Green's fct. expanded in # faces F (=# loops for planar maps in MFT):

$$X^\gamma = r_\gamma \pm \sum_{\substack{\Gamma \in \mathcal{H}_T^{\mathrm{f2g}} \\ \partial\Gamma = \gamma}} \alpha^{F_\Gamma} \frac{\Gamma}{|\mathrm{Aut}\,\Gamma|} = r_\gamma \pm \sum_{j=1}^{\infty} \alpha^j c_j^\gamma$$

Comb. Dyson-Schwinger eq. hold with usual comb. factors in B_+^Γ [Kreimer '08]:

$$X^\gamma = r_\gamma \pm \sum_{k \geq 1} \alpha^k \left[\sum_{\substack{\Gamma \text{ prim.} \\ F_\Gamma = k \\ \partial\Gamma = \gamma}} B_+^\Gamma \right] (X^\gamma Q_\gamma)$$

(so far shown for relevant examples, general argument for cNLFT w.i.p.)

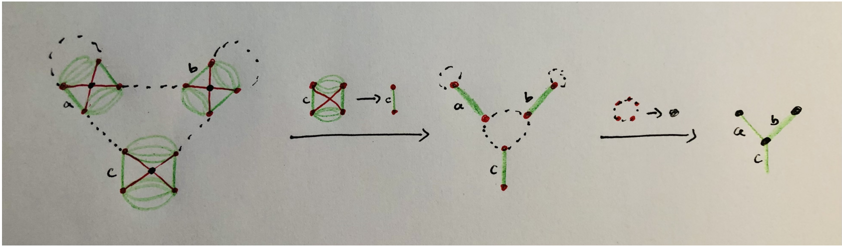
Example: cDSE in $\phi^4_{1,5}$ tensorial field theory

$\phi^4_{d=1,r=5}$ is combinatorially the simplest just renormalizable theory

$$\omega^{\mathrm{sd}}(\Gamma) = 4 - V_{\partial\Gamma} - (\delta^{\mathrm{G}}_{\Gamma} + K_{\partial\Gamma} - 1)$$

Only melonic diagrams ($\delta^{\mathrm{G}} = 0, K_{\partial\Gamma} = 1$) need renormaliz. (as $\delta^{\mathrm{G}} \geq r - 2$ else)

Quartic melonic diagrams can be mapped to planar trees
(intermediate field rep./loop-vertex expansion [Delepouve, Gurau, Rivasseau '14]):

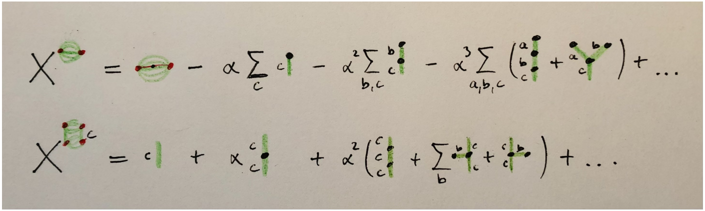


$\phi^4_{1,5}$ renormalization Hopf algebra is one of coloured planar trees

- but edges are coloured (not vertices like in Hopf algebra of decorated trees)
- 2pt graphs are rooted trees, 4pt graphs are trees with 2 markings!

Features of tensorial theory

- only tadpole and fish diagram are primitive $\rightarrow$ nice cDSE
- only con. boundary ("unbroken") 4pt function is in $\mathcal{H}^{\mathrm{f2g}}$
- sum over colours for 2pt function, but not for 4pt function



Consequences:

- coproduct at order k does not factor in $c_{k-j} \otimes c_j$ (overall $\sum_c$)
- different comb. factors (maxf(Γ)) for adjacent interactions $b \neq c$ vs. $b = c$
- can be smoothened including broken 4pt functions ([Tanasa et. al. '13,'15])
- but this gives a different Hopf algebra (not renormalization of $\phi^4_{1,5}$)

w.i.p.: improve and understand using Ward identities (Hopf ideals)

Outlook

- Result: algebraic structure of renormalization generalizes to cNLFT
- gives concise algorithm to calculate amplitudes explicitly (classify!)
- Random geometry/quantum gravity occurs at criticality
  $\rightarrow$ understand non-perturbative cNLFT, in particular tensorial!
- use cDSE to identify algebraic structure underlying solvability of matrix field theory (w.i.p. with A. Hock)
- generalize to tensors of rank $r > 2$

Thanks for your attention!