Transport maps as direct connections on groupoids

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Plan

- Motivation: transport maps in regularity structures for sections of vector bundles.
- Lie groupoids replacing structure groups.
- Direct connections replacing parallel displacement.
- Torsion and curvature.
- Jet prolongation of groupoids and of direct connections.

Motivation

Consider singular stochastic PDE

$$\boxed{\partial_t u = \Delta u + F(u, \nabla u, \xi)}$$

where $u = u(t, x)$ function (or distribution) on $\mathbb{R}_+ \times \mathbb{R}^d$

ξ white noise

F non-linear in u $\Rightarrow$ product of singular distributions: ill-posed!

Need **regularization** u_ε by smooth $\xi^{(\varepsilon)}$ with $\varepsilon \to 0$

renormalization to ensure convergence of u_ε.

(These steps are not the topic here.)

Trick: solve by symbolic expansion [Hairer 2014]

- **Local expansion of α-Hölder functions (distributions) at x_0:** mimic Taylor expansion

$$f(x) = f(x_0) + \sum_{1\leqslant|\mathbf{k}|\leqslant n\leqslant\alpha} \frac{f^{(\mathbf{k})}(x_0)}{\mathbf{k}!}(x-x_0)^{\mathbf{k}} + r(x_0,x), \quad |r(x_0,x)| \leqslant C\,||x-x_0||^{\alpha}$$

by adding terms for ξ and for the heat kernel

$$\boxed{u_\varepsilon(x) = u_\varepsilon(x_0) + \sum_{\tau\in T} a^\varepsilon_\tau(x_0)\,\big(\Pi^\varepsilon_{x_0}\tau\big)(x) + r_\varepsilon(x_0,x)}$$

- τ are **graded symbols** for coordinate polyomials $X^{\mathbf{k}}$, white noise Ξ and derivatives of convolution with the heat kernel $I_{\mathbf{k}}(\tau)$,
- $\Pi^\varepsilon_{x_0}\tau$ is a $|\tau|$-Hölder function which generalises $\big(\Pi^\varepsilon_{x_0}X^{\mathbf{k}}\big)(x) = (x-x_0)^{\mathbf{k}}$.

- **Relate local expansions at x_0 and y_0:** use **transport maps** $\Gamma_{x_0\,y_0}$ induced by transaltion $y_0 - x_0 \in \mathbb{R}^d$ (which act on symbols τ).

Regularity structures on $\mathbb{R}^d$ [Hairer 2014]

Def. An **abstract regularity structure** is (A, T, G) with
$A \subset \mathbb{R}$ set of homogeneities α (contains 0, bounded from below, discrete)
$T = \bigoplus_{\alpha \in A} T_\alpha$ graded vector space of symbols τ (with norm, $T_0 = \mathbb{R} \cdot \mathbf{1}$)
$\rho \colon G \to \mathrm{Aut}(T)$ Lie group action (s.t. $\rho(g)\mathbf{1} = \mathbf{1}$ and $\rho(g)\tau - \tau \in \bigoplus_{\beta < |\tau|} T_\beta$)

Def. A **model for** (A, T, G) **on** $\mathbb{R}^d$ is (Π, Γ) with

$\boxed{\Pi : \mathbb{R}^d \to \mathcal{L}(T, \mathcal{S}'(\mathbb{R}^d))}$ and $\boxed{\Gamma = \rho \circ \gamma \quad \text{s.t.} \quad \Pi_x \Gamma(x,y) = \Pi_y}$

$\boxed{\gamma : \mathbb{R}^d \times \mathbb{R}^d \to G \quad \text{s.t.} \quad \gamma(x,x) = 1_G \quad \text{and} \quad \gamma(x,y)\,\gamma(y,z) = \gamma(x,z)}$

plus local uniform compatibility with A.

Ex. Model for polynomial regularity structure on $\mathbb{R}^d$: $A = \mathbb{N}$,
$T = \mathbb{R}_n[X_1, \dots, X_d]$ contains pol. $P(X)$ det. by coeff. $\left(a_{\mathbf{k}}^{(|\mathbf{k}|)}\right)_{0 \leqslant |\mathbf{k}| \leqslant n}$
$\big(\Pi_{x_0} P\big)(x) = \sum_{|\mathbf{k}|=0}^n a_{\mathbf{k}}^{(|\mathbf{k}|)} \frac{(x-x_0)^{\mathbf{k}}}{\mathbf{k}!}$ function on $\mathbb{R}^d$ "centered" at x_0
$G = (\mathbb{R}^d, +)$ acts by translation $\rho(g)\,P(X) = P(X+g)$
$\gamma(x_0, y_0) = x_0 - y_0$ so that $\Gamma(x_0, y_0) P(X) = P(X + x_0 - y_0)$

Thm [Hairer 2014, Bruned-Hairer-Zambotti 2017, etc]
There exist models solving several stochastic PDEs.

Regularity structures on a manifold M [Dahlqvist-Diehl-Driver 2017]

M **closed Riemannian manifold** of dimension d with **Levi-Civita connection** ∇ and local geodesics, and with **distributions** $\mathcal{D}'(M)$.

Def. A **regularity structure** on M is (A, T, G) where now

$\boxed{T = \bigoplus_{\alpha \in A} T_\alpha \to M}$ is a graded vector bundle ($T_0 = M \times \mathbb{R} \to M$).

A **model for** (A, T, G) **on** M **with transport precision** $\beta \in \mathbb{R}$ is a collection $\big(U_x, \Pi_x, \Gamma(x,y)\big)_{x \in M}$ with U_x an open neighborhood of x and

$$\boxed{\Pi_x : T_x \to \mathcal{D}'(U_x) \qquad \text{and} \qquad \Gamma(x,y) : T_y \to T_x}$$

with $\Gamma(x,y)$ defined on a **diagonal domain** in $M \times M$ (i.e. for x, y close) and $\underline{\Pi_x \Gamma(x,y) \neq \Pi_y}$ but the difference is bounded by β.

Ex. Model for polynomial regularity structure on M: $A = \mathbb{N}$,

$T_x = \bigoplus_{\ell=0}^{n} S^\ell T_x^* M$ symmetric powers of covectors (representing jets)

$$\big(\Pi_{x_0}\tau\big)(x) = \sum_{\ell=0}^{n} \frac{1}{\ell!} \tau_\ell\big(\exp_{x_0}^{-1}(x)^{\otimes \ell}\big) \quad \text{and} \quad \big(\Gamma(x,y)\tau\big)_\ell = \mathrm{Sym}(\nabla_x^\ell(\Pi_y \tau)).$$

Remarks leading to groupoids

- These models apply to functions (on $\mathbb{R}^d$ or M) with scalar values, or vector values seen in components, or in manifolds embedded in $\mathbb{R}^N$.

$\Longrightarrow$ Wish PDEs for sections $u : M \to E$ of vector or fibre bundles endowed with a connection fixing parallel displacement.

- Hairer's model equations

$$\Pi_x \Gamma(x, y) = \Pi_y \quad \text{and} \quad \Gamma(x, y)\Gamma(y, z) = \Gamma(x, z)$$

  are linked and say that Γ is a groupoid morphism from the *pair groupoid of* M to a groupoid acting on the fibres of $T \to M$.

- Dahlqvist, Diehl and Driver attach groups to pairs of points of M.

$\Longrightarrow$ Add a principal G-bundle $P \to M$ associated to E and consider the gauge groupoid $\mathcal{G}(P) \rightrightarrows M$.

- Dahlqvist, Diehl and Driver relax the model equations by introducing a precision β.

$\Longrightarrow$ Look for suitable connection on groupoids whose curvature measures the default of (local) groupoid maps to be groupoid morphisms.

$\Longrightarrow$ Next: follow the deformation of the connection through renormalization.

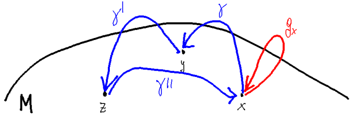
Work in progress with **S. Azzali**, **Y. Boutaïb** and **S. Paycha**.

Lie groupoids

Def. A **Lie groupoid** $\mathcal{G} \rightrightarrows M$ is a smooth manifold of **arrows** $\gamma_{yx} \in \mathcal{G}_x^y$ above $(y, x) \in M \times M$ determined by surjective submersions called the **source** and the **target** map $s, t : \mathcal{G} \to M$ $\begin{cases} s(\gamma_{yx}) = x \\ t(\gamma_{yx}) = y \end{cases}$, such that

- arrows can be **composed** $\gamma'_{zy}\gamma_{yx} \in \mathcal{G}_x^z$ if $s(\gamma'_{zy}) = t(\gamma_{yx})$ (associative),
- above points there are **units** $u(x) = 1_x \in \mathcal{G}_x^x$ and $M \equiv u(M) \subset \mathcal{G}$,
- each arrow $\gamma_{yx} \in \mathcal{G}_x^y$ has an **inverse** $\gamma_{yx}^{-1} \in \mathcal{G}_y^x$.

The induced map $(t, s) : \mathcal{G} \to M \times M$ is called the **anchor**.



Features:

- Each $\mathcal{G}_x^x$ is a (non empty) Lie group, the **vertex group** or **isotropy**.
- $\mathcal{G}$ has a rich infinitesimal structure given by a **Lie algeborid** $A \to TM$.
- $\mathcal{G}$ can act on fibre or vector bundles $E \to M$.

$\Longrightarrow$ Lie groupoids are (bi-)fibred generalizations of Lie groups whose action on fibre bundles keeps track of fibre transformations (internal symmetry) and bundle automorphisms (global symmetries).

Examples of Lie groupoids

- **Pair groupoid** $\boxed{\mathrm{Pair}(M) = M \times M \rightrightarrows M}$

- **Trivial Lie groupoid** with fibre G $\boxed{M \times G \times M \rightrightarrows M}$

- **Gauge groupoid** of principal G-bdl $P \to M$ $\boxed{\mathcal{G}(P) = P \times_G P \rightrightarrows M}$
  made of equivalence classes $[p, q]$ under $(p, q) \sim (p\,g, q\,g)$ for $g \in G$.

- **Frame groupoid** of vector bdl $E \to M$ $\boxed{\mathrm{Iso}(E) = \bigcup_{x,y} \mathrm{Iso}(E_y, E_x)}$

  If E has rank r and $\boxed{F(E) = \bigcup_x \mathrm{Iso}(\mathbb{R}^r, E_x)}$ is the **frame bundle** of E

  (principal $GL_r(\mathbb{R})$-bundle), then $\boxed{\mathrm{Iso}(E) \cong \mathcal{G}(F(E))}$

  If the structure group of E reduces to $G \subset GL_r(\mathbb{R})$, then
  $\boxed{\mathcal{G}(P) \hookrightarrow \mathrm{Iso}(E)}$

Direct connections on Lie groupoids

Def. A **local map** between two groupoids $\mathcal{G}$ and $\mathcal{G}'$ over M is a map $\phi : \mathcal{U} \subset \mathcal{G} \to \mathcal{G}'$ defined on an open neighborhood $\mathcal{U}$ of the units $u(M) \subset \mathcal{G}$, which commutes with s, t and u. Denote it $\boxed{\phi : \mathcal{G} \ast\!\!\rightarrow \mathcal{G}'}$.

A **local morphism** is local map which also preserves composition (hence inversion).

Def. [Teleman 2004 in the linear case, Kock 2007 similar, ABFP general] A **direct connection** on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a local right inverse of the anchor which preserves the units, i.e. $\boxed{\Gamma : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G}}$ defined on an open neighborhood $\mathcal{U}_\Delta$ of the diagonal $\Delta \subset \mathrm{Pair}(M)$ (**diagonal domain**), such that

$$\boxed{\Gamma(y,x) \in \mathcal{G}_x^y} \text{ for all } (y,x) \in \mathcal{U}_\Delta \quad \text{and} \quad \boxed{\Gamma(x,x) = 1_x \in \mathcal{G}_x^x} \text{ for all } x \in M.$$

Prop $\boxed{\text{A Lie groupoid with a direct connection is a gauge groupoid.}}$

Expected examples

Assume M is a manifold with affine connection ∇^M and local geodesics.

- Parallel displacement τ on $P \to M$ along small geodesics (equivalent to a principal connection ω on P) defines a direct connection Γ^τ on $\mathcal{G}(P) \rightrightarrows M$.

  Same for $E \to M$ and $\mathrm{Iso}(E)$ [Teleman 2004].

- Viceversa, a direct connection Γ on $\mathcal{G}(P) \rightrightarrows M$ induces an infinitesimal connection on the Lie algebroid $A(P) \to TM$, hence a principal connection ω^Γ on P.

- Apply maps $\omega \mapsto \tau \mapsto \Gamma^\tau \mapsto \omega^{\Gamma^\tau}$, then $\boxed{\omega^{\Gamma^\tau} = \omega}$ on P.
- Viceversa, if apply maps $\Gamma \mapsto \omega^\Gamma \mapsto \tau^\Gamma \mapsto \Gamma^{\tau^\Gamma}$, then $\boxed{\Gamma^{\tau^\Gamma} \neq \Gamma}$ on $\mathcal{G}(P)$ in general.
- There are direct connections on $\mathcal{G}(P)$ which are not parallel displacements (simplest example in two slides).

Curvature and flat direct connections

Let $\Gamma : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G}$ be a direct connection defined on $\mathcal{U}_\Delta$.

Def. For $x \in M$, set $\quad \mathcal{U}^1_\Delta(x) = \{y \in M \mid (x,y),(y,x) \in \mathcal{U}_\Delta\} \subset M$.

Torsion of Γ at x is the map $\boxed{T^\Gamma(\_,x) : \mathcal{U}^1_\Delta(x) \longrightarrow \mathcal{G}^x_x}$ given by

$$\boxed{T^\Gamma(y,x) := \Gamma(x,y)\,\Gamma(y,x) \in \mathcal{G}^x_x} \qquad y \in \mathcal{U}^1_\Delta(x).$$

Γ is **torsion-free** if $T^\Gamma(y,x) = 1_x$ for any y, i.e. $\Gamma(x,y) = \Gamma(y,x)^{-1}$.

Def. For $x \in M$, set

$$\mathcal{U}^2_\Delta(x) = \{(z,y) \in M \times M \mid (y,x),(z,y),(z,x) \in \mathcal{U}_\Delta\} \subset M \times M.$$

Curvature of Γ at x is the map $\boxed{R^\Gamma(\_,\_,x) : \mathcal{U}^2_\Delta(x) \longrightarrow \mathcal{G}^x_x}$ given by

$$\boxed{R^\Gamma(z,y,x) := \Gamma(z,x)^{-1}\Gamma(z,y)\,\Gamma(y,x) \ \in \mathcal{G}^x_x}, \qquad (z,y) \in \mathcal{U}^2_\Delta(x).$$

Γ is **flat** if $R^\Gamma(\_,\_,x) = 1_x$ for any x, i.e. Γ **is a groupoid morphism**.

- If Γ is flat then it is torsion-free. But not the other way round.
- A parallel transport is always torsion-free (torsion can not be seen on P!) and **it is a flat direct connection iff the principal connection is flat**.

Examples

- $M = \mathbb{R}$ with flat connection $\nabla^M_{\partial_x}(h(x)\,\partial_x) = h'(x)\,\partial_x$.
- $E = M \times \mathbb{R} \to M$ with global section $e_1(x) = (x,1) \in E_x$ and linear connection $\nabla^E_{\partial_x} : \Gamma(E) \to \Gamma(E)$ given by $f \in C^\infty(M)$ s.t. $\nabla^E_{\partial_x} e_1 = f\, e_1$.
- The induced **parallel transport** along a geodesic from x to y is the isomorphism $\tau(y,x) : E_x \to E_y$ defined by $\tau(y,x)\,\xi_0\, e_1(x) = \xi(y)\, e_1(y)$ solution of the ODE

$$\nabla^E_{\partial_x}\big(\xi(x)\, e_1(x)\big) = \big(\xi'(x) + \xi(x) f(x)\big) e_1(x) = 0$$

with initial value $\xi(x)\, e_1(x) = \xi_0\, e_1(x)$. Set $F(x) = \int -f(x)dx$.
Then the direct connection on $\mathrm{Iso}(E)$ is

$$\boxed{\tau(y,x) : E_x \to E_y, \quad e_1(x) \mapsto \tau(y,x)\, e_1(x) = e^{F(y)-F(x)}\, e_1(y)}$$

This direct connection is flat. For instance:
$\nabla^E_{\partial_x} e_1(x) = -2x\, e_1(x)$ gives $\tau(y,x) e_1(x) = e^{y-x+y^2-x^2} e_1(y)$,
$\nabla^E_{\partial_x} e_1(x) = -3x^2\, e_1(x)$ gives $\tau(y,x) e_1(x) = e^{y-x+y^3-x^3} e_1(y)$.

- Instead, the following direct connections **are not parallel transports**:
$\Gamma(y,x) e_1(x) = e^{y-x+(y-x)^2} e_1(y)$, with torsion $T^\Gamma(y,x) = e^{2(y-x)^2} \neq 1_x$,
$\Gamma(y,x) e_1(x) = e^{y-x+(y-x)^3} e_1(y)$, torsion-free but non-flat.

Jet prolongation of groupoids

- $E \to M$ vector bundle of rank r with structure group $G \subset GL_r(\mathbb{R})$, principal G-bundle $P \to M$ s.t. $E \cong P \times_G \mathbb{R}^r$ and $\mathcal{G}(P) \subset \mathrm{Iso}(E)$.

Def. The ***n*-jet bundle** $\boxed{J^n E \to M}$ is the vector bundle of n-jets $j^n_x u$ of local sections $u : M \ast\!\!\to E$ around x (i.e. equivalence classes of local sections with the same **contact of order** n **at** x).

Thm [Kolář-Michor-Slovak 1993]
The structure group of $J^n E \to M$ is the semidirect group

$$\boxed{W^n_d G = GL^n_d \ltimes T^n_d G} \qquad \boxed{\begin{aligned} GL^n_d &= \mathrm{inv} J^n_0(\mathbb{R}^d, \mathbb{R}^d) \\ T^n_d G &= J^n_0(\mathbb{R}^d, G) \end{aligned}}$$

and the associated principal $W^n_d G$-bundle is the **jet prolongation**

$$\boxed{W^n P = F^n M \times_M J^n P} \qquad \boxed{F^n M = \mathrm{inv} J^n_0(\mathbb{R}^d, M)}.$$

Def. The ***n*-jet prolongation** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is the groupoid $\boxed{J^n \mathcal{G} \rightrightarrows M}$ of n-jets of **local bisections** $\sigma : M \to \mathcal{G}$ s.t. $s \circ \sigma = \mathrm{id}$ and $t \circ \sigma = \varphi_\sigma$ is a local diffeomorphism.

Thm [Kolář 2008] $\boxed{W^n F(E) \cong F J^n E}$ $\boxed{J^n \mathcal{G}(P)) \cong \mathcal{G}(W^n P)}$

Geometric regularity structures

- M manifold with affine connection ∇^M, $E \to M$ be a vector bundle of rank r with structure group $G \subset GL_r(\mathbb{R})$, associated principal G-bundle $P \to M$ and gauge groupoids $\mathcal{G}(P) \subset \mathrm{Iso}(E)$.

Def. [ABFP] A **geometric polynomial structure** on $E \to M$ is a regularity structure

$$\boxed{\big(A, J^n E, J^n \mathcal{G}(P)\big)}$$

with $A = [[0, n]]$ and $J^n \mathcal{G}(P) = \mathcal{G}(W^n P) \subset \mathrm{Iso}(J^n E)$ acting on $J^n E$.

A **model for** $(A, J^n E, \mathcal{G}(W^n P))$ is $\boxed{\text{direct connection } \Gamma^{(n)} \text{ on } J^n \mathcal{G}(P)}$ and a collection $\big(U_x, \Pi_x\big)_{x \in M}$ with U_x a uniformly normal open n. of x and

$$\boxed{\Pi_x : J^n_x E \to \mathcal{D}'(U_x)}$$

(Next: model equations, flatness and precision β (need analysis!).)

Thm [ABFP] $\boxed{\begin{array}{l}\text{Any direct connection } \Gamma : \mathrm{Pair}(M) \twoheadrightarrow \mathcal{G} \text{ can be} \\ \text{prolonged to the jet groupoid } \Gamma^{(n)} : \mathrm{Pair}(M) \twoheadrightarrow J^n \mathcal{G}.\end{array}}$

Jet prolongation of direct connections

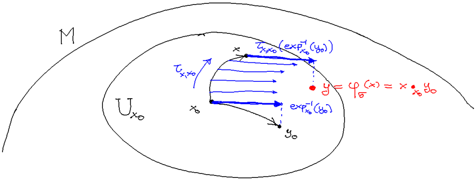
Lemma

There exists a canonical **geodesic direct connection** $\delta^{(n)}$ on $J^n\mathrm{Pair}(M)$.

Proof. For $x_0 \in M$ let U_{x_0} be a uniformly normal neighb. of x_0 in M.
Then $\mathcal{U} = \{(y,x) \mid x \in M, y \in U_x\}$ is a neighb. of Δ in $\mathrm{Pair}(M)$.
For any $(y_0, x_0) \in \mathcal{U} \subset \mathrm{Pair}(M)$ take the geodesic $\gamma : [0,1] \to M$ s.t. $\gamma(0) = x_0$ and $\gamma(1) = y_0$ and define a local map $\sigma : U_{x_0} \to \mathrm{Pair}(M)$ by

$$\boxed{\sigma(x) = \Big(\varphi_\sigma(x), x\Big)} \quad \text{with} \quad \boxed{\varphi_\sigma(x) = \exp_x\left(\tau(x,x_0)(\exp_{x_0}^{-1}(y_0))\right)}$$

where $\exp_{x_0}^{-1}(y_0) \in T_{x_0}M$ is along γ and $\tau(x, x_0)$ is the parallel transport on TM along a geodesic from x_0 to $x \in U_{y_0}$.



Finally, define $\delta^{(n)} : \mathrm{Pair}(M) \ast\!\!\longrightarrow J^n\mathrm{Pair}(M)$ by $\boxed{\delta^{(n)}(x_0, y_0) = j^n_{x_0}\sigma}$.

Thm Any direct connection $\Gamma : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G}$ can be prolonged to the jet groupoid $\Gamma^{(n)} : \mathrm{Pair}(M) \ast\!\!\longrightarrow J^n\mathcal{G}$.

Proof. If $\Gamma : \mathcal{V} \subset \mathrm{Pair}(M) \to \mathcal{G}$, the intersection $\mathcal{U} \cap \mathcal{V}$ is a diagonal domain and for any $(y_0, x_0) \in \mathcal{U} \cap \mathcal{V}$ there exists a geodesic bisection $\sigma : U_{x_0} \to \mathrm{Pair}(M)$ defined as above.
Then $\Gamma \circ \sigma_{|\mathcal{V}}$ is a local bisection of $\mathcal{G}$ and can define

$$\boxed{\Gamma^{(n)}(y_0, x_0) = j^n_{x_0}\big(\Gamma \circ \sigma_{|\mathcal{V}}\big) = j^n_{(y_0,x_0)}\Gamma \circ \delta^{(n)}(x_0, y_0)}\,.$$

Next:

- Look for higher dimensional examples of direct connections which are not parallel displacements.
- Look for examples of direct connections on jet groupoids $J^n\mathcal{G}$ which are not induced by some on $\mathcal{G}$.
- Adapt to α-Hölder sections of bundles and include precision β.
- Start from a parallel displacement and follow the renormalization process.
- Study the whole geometry of **groupoids with direct connections** and compare with **usual gauge theory**!
  (work in progress with **S. Azzali**, **A. Garmendia** and **S. Paycha**)

Thank you for the attention!