(DISCUSSION OF)

PARACONTROLLED CALCULUS AND REGULARITY STRUCTURES

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What's happening

Two approaches to singular SPDEs

- Regularity structures (Hairer '14)
  - increment description of regularity ("$|f(x) - f(y)| \lesssim |x-y|^{\alpha}$")
  - very systematic and general, $\exists$ black box local well-posedness results for wide class of parabolic eqn.
  - recently also "global" estimates (Chandra - Moinat - Weber '19, Moinat - Weber '20)

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  - very systematic and general, $\exists$ black box local well-posedness results for wide class of parabolic eqn.
  - recently also "global" estimates (Chandra-Moinat-Weber '19, Moinat-Weber '20)
- Paracontrolled distributions (Gubinelli-Imkeller-P. '15)
  - Fourier description of regularity ("$|\mathcal{F}^{-1}(\mathbb{1}_{|\cdot|\sim 2^j}\mathcal{F}f)| \lesssim 2^{-j\alpha}$")
  - Bailleul-Bernicot '19 extend GIP '15, but still no black box results
  - useful for global estimates (plugs right into PDE theory) $\rightsquigarrow$ many
  - Applies to some dispersive eqn. (Gubinelli-Koch-Oh '18, Ai-Ifrim-Tataru '21)

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Two approaches to singular SPDEs

- Regularity structures
- Paracontrolled distributions

Are they the same?

Background story

- Gubinelli-Imkeller-P. '15: E.g. 2d PAM $\quad \partial_t u = \Delta u + u\xi$ $\quad \in C^{-1-}$

$$\rightsquigarrow \boxed{u = u \circledcirc Z + [u]}$$

C^{1-} $\quad C^{2-}$ for $(\partial_t - \Delta) Z = \xi$

Problem: $u\xi$ not well def.

Solution of GIP'15: $C(u, Z, \xi) = (u \circledcirc Z) \odot \xi - u(Z \odot \xi)$

is bounded operator ($\Rightarrow$ well posed equ. for $[u]$)

- Gubinelli-Imkeller-P. '15: First link between PD $\longleftrightarrow$ RS:

$F = \mathcal{R} f$ (reconstruction) $\Rightarrow$ $F - \sum_\tau f_\tau \Pi \odot \tau \in C^\gamma$; + L-P construction of $\mathcal{R} f$

$\mathcal{D}^\gamma$

- Bailleul-Bernicot '16: Paracontrolled calculus on manifolds

Dahlquist-Diehl-Driver '1x (Littlewood-Paley blocks $\Rightarrow$ semigroup of Δ)

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  $F = \mathcal{R}f$ (reconstruction) $\Rightarrow$ $F - \sum_\tau f_\tau \Pi \circledcirc \tau \in C^\gamma$; + L-P construction of $\mathcal{R}f$

  $f \in \mathcal{D}^\gamma$

- Bailleul - Bernicot '16: Paracontrolled calculus on manifolds
  (Littlewood - Paley blocks $\rightarrow$ semigroup of Δ)

- Bailleul - Bernicot '19: Higher order paracontrolled calculus (on manifolds):
  iterate commutator estimates, e.g. $C^{(2)}(u,v,z,\xi) = C(u \circledcirc v, z, \xi) - u\, C(v, z, \xi)$

  Issue: algebraic complexity for very singular eqn., renormalization open

  $\hookrightarrow$ but see Bailleul - Bruned '21

- Martin - P. '20: For fixed regularity structure: modelled distributions $\Leftrightarrow$ paracontrolled

  Issue: Regularity structure must be already given, not so nice paraproduct, need "structure condition"

Contributions of the presented work

- Description of regularity structures via paraproducts
- Nicer paraproduct description of modelled distributions than in Martin-P. '20
- Strong understanding of the algebraic underpinning

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(Potential) Applications

- Insights in renormalization of higher order paracontrolled calculus → see also Bailleul-Bruned '21
- Systems of paracontrolled equations in RS → global estimates? (Hoshino '19, '20)
- Simple proofs of higher order commutator estimates in paracontrolled calculus
- Geometric theory?
- Dispersive eqn. in RS?

Thank You!

Questions:

- Structure condition in Martin–P. '20: Q: What's the replacement?
  needed for $f^1 \in \mathcal{C}^{1+\delta}$
  $D^k\left(f_\sigma(\cdot) - \langle \sigma, \Gamma_{\cdot x} f(x) \rangle\right)_x = 0 \quad \forall |k| < \gamma - |\sigma| \rightarrow f^1 - \tilde{f} \odot X = f^1 \in \mathcal{C}^{1+\delta}$
  $\forall \tilde{f} \in \mathcal{C}^\delta$
- Tried to prove Schauder estimates for non-scaling-invariant operators?
- $[\tau]^g, [\tau]^\mu : \mathbb{R}^d \to \mathbb{R}$ $\to$ vector-valued noise componentwise?
  Could any of this work in ∞ dim?
- Is higher order paracontrolled system here equivalent to Bailleul–Bernicot '19?