The strength of better-quasi-orderings, via ordinal analysis

Anton Freund

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Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Emmy
Noether-
Programm
DFG Deutsche
Forschungsgemeinschaft

Part I: Better-Quasi-Orderings

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Quasi-orders are reflexive and transitive. **Partial orders** also enjoy anti-symmetry. The latter always holds in a quotient.

We will see the following subsequently stronger notions:

- well-founded quasi orders,
- well-quasi-orderings (**wqo**s),
- better-quasi-orderings (**bqo**s).

When the order is linear, these notions coincide and one speaks of a well-order.

Definition and Theorem: A quasi-order Q is a well-quasi-ordering if any of these equivalent conditions holds:

(1) every sequence $f : \mathbb{N} \to Q$ is good, i.e., there are $i <_{\mathbb{N}} j$ with $f(i) \leq_Q f(j)$,

(2) there are no infinitely descending sequences and no infinite antichains in Q,

(3) any $X \subseteq Q$ has a finite basis $X_0 \subseteq X$, i.e., such that each $x \in X$ admits an $x_0 \in X_0$ with $x_0 \leq_Q x$,

(4) every linear extension of the order on Q is a well-order (assuming the order is anti-symmetric).

Exercise: Prove the equivalence between the conditions.

For historical background, see Kruskal's "The theory of well-quasi-ordering: A frequently discovered concept" (1972).

Concerning reverse mathematics, the equivalences hold over $\mathsf{WKL}_0 + \mathsf{CAC}$ but not all over RCA_0 (Cholak, Marcone and Solomon, "Reverse Mathematics and the Equivalence of Definitions for Well and Better Quasi-Orders", JSL 2004).

Why well-quasi-orderings? – Application 1

Skolem asked in 1956 if the exponential polynomials over $\mathbb{N}$ are well-ordered by eventual domination.

Ehrenfeucht gave a one-page proof in 1973 ("Polynomial functions with exponentiation are well ordered").

This proof exploits Kruskal's theorem from the theory of well-partial-orderings (details on the blackboard).

Note: The order-type of the well-order is still open (cf. Berarducci and Mamino, "Asymptotic analysis of Skolem's exponential functions", JSL 2022).

Why well-quasi-orderings? – Application 2

Robertson and Seymour (2004) famously proved that the finite graphs are well-quasi-ordered by the minor relation.

As a consequence, it must be possible to test many properties of graphs in polynomial time (though we may not know how).

For example, when $k \in \mathbb{N}$ is fixed, one can polynomially decide if there are disjoint paths between k given pairs of vertices.

Note: Concerning reverse mathematics, H. Friedman, Robertson and Seymour (1987) showed that the graph minor theorem cannot be proved in Π^1_1-CA_0 (via ordinal analysis).

Towards better-quasi-orderings

Given a quasi-order $(Q, \leq)$, we order the powerset $\mathcal{P}(Q)$ by

$$X \leq^{\exists}_{\forall} Y \quad \Leftrightarrow \quad \forall x \in X \exists y \in Y : x \leq y.$$

Proposition: If Q is wqo, then any sequence $X_0, X_1, \ldots$ of finite $X_i \subseteq \mathcal{P}(Q)$ is good.

Proof: See blackboard.

Exercise: Show that the proof goes through in ACA_0.

We also have a reversal (Marcone, see later).

Example (Rado):

The following gives a wqo R on

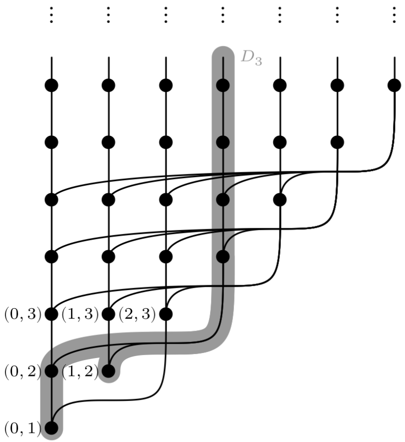
$[\mathbb{N}]^2 = \{(m,n) \mid m < n \text{ from } \mathbb{N}\}$

where the powerset is not wqo:

$$(m,n)\,\mathrm{R}(m',n') \\ \Updownarrow \\ (m = m' \text{ and } n \leq n') \text{ or } n < m'$$

Proposition (Laver):

Whenever Q but not $\mathcal{P}(Q)$ is wqo, $([\mathbb{N}]^2, \mathrm{R})$ embeds into Q.



(Illustration from Y. Pequignot, Better-quasi-order: ideals and spaces, PhD Thesis, Paris/Lausanne 2015)

In $[\mathbb{N}]^2$, write $(m, n) \lhd (k, l)$ if $m < n = k < l$.

Proposition: Assume that any $f : [\mathbb{N}]^2 \to Q$ is good, i.e., that there are $s \lhd t$ with $f(s) \leq f(t)$. Then $\mathcal{P}(Q)$ is wqo. The converse holds as well.

Proof: See blackboard.

Let $[\mathbb{N}]^{<\omega}$ and $[\mathbb{N}]^{\omega}$ consist of the finite and infinite increasing sequences in $\mathbb{N}$. Write $s \sqsubset t$ if s is a strict initial segment of t. Write $s \subset t$ if every entry of s occurs in t but not conversely.

We call $B \subseteq [\mathbb{N}]^{<\omega}$ a **block** if

1. every $X \in [\mathbb{N}]^{\omega}$ admits an $s \sqsubset X$ with $s \in B$,
2. there are no $s, t \in B$ with $s \sqsubset t$.

If there are not even $s, t \in B$ with $s \subset t$, we call B a **barrier**. For our purposes, blocks and barriers are interchangeable over WKL_0 (Marcone).

Example: $[\mathbb{N}]^2$ and $\{s \in [\mathbb{N}]^{<\omega} : |s| = \min(s) + 1\}$ are barriers.

For $X = x_0, x_1, \ldots$, let $X^- = x_1, x_2, \ldots$. For $s, t \in [\mathbb{N}]^{<\omega}$, write $s \lhd t$ if there is $X \in [\mathbb{N}]^\omega$ with $s \sqsubset X$ and $t \sqsubset X^-$.

Definition (Nash-Williams 1965):
A quasi-order Q is a better-quasi-ordering if every 'array' $f : B \to Q$ on a block / barrier B is good, i.e., if there are $s, t \in B$ with $s \lhd t$ and $f(s) \leq_Q f(t)$.

- Every bqo is wqo.
- Every (linear!) well-order is bqo.

Let $H(Q)$ be the set-theoretic universe over urelements Q (hereditarily countable sets suffice). We order it by

$$x \leq_H y \Leftrightarrow \begin{cases} x \leq_Q y & \text{if } x, y \in Q, \\ \forall x' \in x \exists y' \in y : x' \leq_H y' & \text{if } x, y \notin Q, \\ \exists y' \in y : x \leq_H y' & \text{if } x \in Q \text{ and } y \notin Q, \\ \forall x' \in x : x' \leq_H y & \text{if } x \notin Q \text{ and } y \in Q. \end{cases}$$

Theorem: Q is bqo iff $H(Q)$ is wqo iff $H(Q)$ is bqo.

Proof: See Section 3.3 of Pequignot, "Towards better: A motivated introduction to better-quasi-orders".

In reverse mathematics, this is equivalent to ATR_0 (Manca).

A second perspective

A function $F : [\mathbb{N}]^\omega \to Q$ is continuous (Σ^0_1-definable) iff there is a block B and an $f : B \to Q$ with

$$F(X) = f(s) \quad \text{for the unique } s \in B \text{ with } s \sqsubset X.$$

An order Q is bqo iff every continuous $F : [\mathbb{N}]^\omega \to Q$ admits an $X \in [\mathbb{N}]^\omega$ with $F(X) \leq_Q F(X^-)$.

We obtain an equivalent definition of bqos if we allow all Borel functions $F : [\mathbb{N}]^\omega \to Q$ (Simpson).

Excursion: Fraïssé's conjecture (1/3)

Fraïssé's conjecture (proved by Laver in 1971) says that the countable linear orders are wqo under embeddability (stronger: the σ-scattered linear orders are bqo).

The original proof uses the 'minimal bad array lemma', which is equivalent to Π^1_2-CA_0 over ATR_0 (F.-Pakhomov-Soldá 2024).

Montalbán (2017) gave a proof in Π^1_1-CA_0. The known lower bound is ATR_0 (Shore 1993).

Excursion: Fraïssé's conjecture (2/3)

Montalbán showed that indecomposable linear orders correspond to well-founded signed trees, via

$$\mathrm{lin}(+/-\langle T_0, T_1, \ldots\rangle) = \textstyle\sum_{n\in\omega/\omega^*} \mathrm{lin}(T_{\pi_0(n)}).$$

Embeddings of orders correspond to label preserving $f : T \to T'$ such that $s \sqsubset t$ implies $f(s) \sqsubset f(t)$.
The latter can be weakened to $f(s) \sqsubseteq f(t)$ if one admits a third incomparable label ('weak embeddings').

Excursion: Fraïssé's conjecture (3/3)

Montalbán shows that if every Δ^0_2-function $F : [\mathbb{N}]^\omega \to Q$ is good, then Q-labelled trees are bqo under weak embeddings, over ATR_0 (partial reversal by Manca).

This is analogous to the result about $H(Q)$, which corresponds to trees with leaf labels. To accommodate internal labels, use that Δ^0_2-functions correspond to eventually stable $f : [\mathbb{N}]^{<\omega} \to Q$.

Over Π^1_1-CA_0, one can replace Δ^0_2 by continuous. This reduces Fraïssé's conjecture to the statement that the antichain with three elements is bqo.

Finite better-quasi-orderings

The open Ramsey theorem says that any continuous colouring $F : [\mathbb{N}]^\omega \to \{0, \ldots, n-1\}$ is constant on the subsets of some infinite $Y \subseteq \mathbb{N}$. For fixed $n \in \mathbb{N}$, this is equivalent to ATR_0.

Corollary (ATR_0): Every finite order is bqo.

Proposition (Marcone): In RCA_0 one can prove that orders with two elements are bqo.

Proof: Use that odd circles are not 2-colourable (blackboard).

Goal of this course: If the antichain with three elements is bqo, one has at least ACA_0^+ (i.e., ω-th Turing jumps).

Some Reading for Part I

- Y. Pequignot, "Towards better: A motivated introduction to better-quasi-orders", EMS Surv. Math. Sci. 4 (2017).
- A. Marcone, "The reverse mathematics of wqos and bqos", in Schuster, Seisenberger and Weiermann (eds)., "Well-quasi orders in computation, logic, language and reasoning", Trends in Logic 53, Springer, 2020.
- A. Montalbán, "Fraïssé's conjecture in Π^1_1-comprehension", Journal of Mathematical Logic 17:2 (2017).

Part I: Better-Quasi-Orderings

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Prequel: Proving $\mathsf{I}\Sigma_1$ consistent in PA

An obvious idea is to formalize soundness via induction over proofs.

For this we must say that formulas are true, but we have no global truth predicate (Tarski).

There are partial truth predicates for limited quantifier complexity, which covers the axioms of $\mathsf{I}\Sigma_1$.

However, proofs may still contain detours of arbitrary complexity. This can be resolved via **cut elimination**.

Tait-style sequents are finite sets of formulas in negation normal form. One writes them without set brackets and reads them as disjunctions.

Example: The induction axiom corresponds to the sequent

$$\neg\varphi(0), \exists x(\varphi(x) \land \neg\varphi(Sx)), \forall y\, \varphi(y).$$

Six simple rules yield a complete proof system:

$$\frac{}{\Gamma, \theta, \neg\theta}\ (\theta \text{ atomic}) \qquad \frac{\Gamma, \varphi_0 \qquad \Gamma, \varphi_1}{\Gamma, \varphi_0 \land \varphi_1} \qquad \frac{\Gamma, \varphi_i}{\Gamma, \varphi_0 \lor \varphi_1}$$

$$\frac{\Gamma, \varphi(y)}{\Gamma, \forall x\, \varphi(x)}\ (y \text{ free}) \qquad \frac{\Gamma, \varphi(t)}{\Gamma, \exists x\, \varphi(x)} \qquad \frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}\ (\text{`Cut'})$$

Theorem (Gentzen): If a sequent can be proved with the cut rule, it can be proved without it.

Proof: See blackboard.

Note that in all rules other than cut, the premises are subformulas of the conclusion.

Despite the theorem, cuts are a practical necessity, e.g., to recover modus ponens (blackboard).

Theorem (Orevkov / Statman):
Cuts bring superexponential speedup.

The Consistency of PA (à la Gentzen-Schütte)

In the foregoing, one cannot replace $\mathrm{I}\Sigma_1$ by PA, as the induction axioms have unbounded complexity.

To resolve this, use infinite proof trees with Hilbert's ω-rule:

$$\frac{\Gamma, \varphi(\overline{0}) \qquad \Gamma, \varphi(\overline{1}) \qquad \cdots}{\Gamma, \forall x\, \varphi(x)}$$

One gets a system that can prove induction (see blackboard) and admits cut elimination.

The Cantor normal form inspires term notations for ordinals up to $\varepsilon_0 = \min\{\gamma \,|\, \omega^\gamma = \gamma\}$:

If $\alpha_1 \succeq \ldots \succeq \alpha_n$ are terms, so is $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$, with (hereditarily) lexicographic comparisons.

For later use, we also consider notations up to ε_α (the α-th element of $\{\gamma \,|\, \omega^\gamma = \gamma\}$), where α is a fixed linear order:

Enrich the term system with constants ε_β for $\beta < \alpha$, but exclude terms of the form $\omega^{\varepsilon_\beta}$.

Theorem: (a) If Γ has an infinite proof of height $\leq \alpha$ with cut formulas of height $< n+1$ (write $\vdash^{\alpha}_{n+1} \Gamma$), then $\vdash^{\omega^{\alpha}}_{n} \Gamma$.

(b) If Γ has an infinite proof of height $\leq \alpha$ (with no bound on cuts), it has a cut-free proof of height $\leq \varepsilon_{\alpha}$, i.e., we have

$$\vdash^{\alpha}_{\omega} \Gamma \quad \Rightarrow \quad \vdash^{\varepsilon_{\alpha}}_{0} \Gamma.$$

Proof: See blackboard.

The theorem can be formalized in $\mathsf{I}\Sigma_1$ (e.g., à la Buchholz).

Corollary: The following are equivalent over $\mathsf{I}\Sigma_1$:

(i) PA is Σ^0_1-sound,

(ii) ε_0 is primitive recursively well-founded.

Theorem (Marcone-Montalbán / Afshari-Rathjen):

The following are equivalent over RCA_0:

(i) Every set has an ω-jump (ACA_0^+),

(ii) every set lies in an ω-model of ACA_0,

(iii) whenever α is a well order, so is ε_α.

Proof of (iii)⇒(ii): By ω-completeness, we get (ii) if there is no infinite proof of contradiction from ACA_0. Given (iii), this follows via ordinal analysis (see blackboard).

For later use, we also consider the notation system

$$\omega^\alpha = \{\omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \,|\, \alpha_1 \geq \ldots \geq \alpha_n \text{ in } \alpha\}$$

with lexicographic comparisons (for fixed α).

Theorem (Girard / Hirst):

The following are equivalent over RCA_0:

(i) Arithmetical comprehension (ACA_0),

(ii) whenever α is a well order, so is ω^α.

Some Reading for Part II

- M. Rathjen and W. Sieg, "Proof Theory", Stanford Encyclopedia of Philosophy, `https://plato.stanford.edu/archives/win2024/entries/proof-theory/`.
- M. Rathjen, "The realm of ordinal analysis", in Cooper and Truss (eds.), "Sets and Proofs", LMS Lecture Note Series, Cambridge University Press, 1999.
- A. Freund, "Unprovability in Mathematics: A First Course on Ordinal Analysis", `https://arxiv.org/abs/2109.06258`.

Some More Reading for Part II

- W. Buchholz, "Notation systems for infinitary derivations", Archive for Mathematical Logic 30 (1991).
- A. Marcone and A. Montalbán, "The Veblen functions for computability theorists", JSL 76 (2011).
- B. Afshari and M. Rathjen, "Reverse mathematics and well-ordering principles: A pilot study", APAL 160 (2009).
- M. Rathjen, "Well-Ordering Principles in Proof Theory and Reverse Mathematics", in Ferreira, Kahle and Sommaruga (eds.), "Axiomatic Thinking II", 2022.

Part I: Better-Quasi-Orderings

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Part III: Synthesis

Lemma (Marcone): There are order reflecting maps

(a) from ω^α into $\mathcal{P}(\omega \otimes \alpha)$,

(b) from $\omega \otimes \alpha$ into $\mathcal{P}(\omega \oplus \alpha)$.

Proof: See blackboard.

Corollary: Over RCA_0, if disjoint sums of well orders are bqo, then we have ACA_0.

Lemma: If α is a well-order, then $\alpha \oplus \alpha$ embeds into $H(\overline{3})$, for the antichain $\overline{3} = \{0, 1, 2\}$.

Proof: Form two copies of the von Neumann ordinals by

$$\dot{\alpha} = \{0,1\} \cup \{\dot{\gamma} \,|\, \gamma < \alpha\} \quad \text{and} \quad \ddot{\alpha} = \{1,2\} \cup \{\ddot{\gamma} \,|\, \gamma < \alpha\}.$$

Proposition (F.): Over RCA_0, if $\overline{3}$ is bqo, then so are disjoint sums of well orders.

Proof: In ATR_0, we could infer that $H(\overline{3})$ is bqo. The proof effectivizes on the transitive closure of $\alpha \oplus \alpha \subseteq H(\overline{3})$ (blackboard).

Corollary: Over RCA_0, if $\overline{3}$ is bqo, we have ACA_0.

Proposition: There is an order reflecting map from ε_α into $H_f((\omega^2 \cdot \alpha) \oplus 1)$. Here ε_α is essentially as large as possible.

Theorem (F.): Over RCA_0, if $\overline{3}$ is bqo, we have ACA_0^+; and if $\overline{3}$ is Δ^0_2-bqo, we have ATR_0.

Theorem (Manca): We have $\mathsf{ACA}_0^+ \nvdash$ "$\overline{3}$ is bqo".

Fedor Pakhomov has a sketch of a proof showing that "$\overline{3}$ is bqo" is equivalent to ATR_0 (personal communication).

Some Reading for Part III

- A. Freund, "On the logical strength of the better quasi order with three elements", Trans. AMS 376 (2023).
- A. Freund, F. Pakhomov and G. Soldà, "The logical strength of minimal bad arrays", Proc. AMS 152 (2024).
- A. Freund, A. Marcone, F. Pakhomov and G. Soldà, "Provable better quasi orders", NDJFL 66 (2025).
- F. Pakhomov and G. Soldà, "On Nash-Williams' Theorem regarding sequences with finite range", `arXiv:2405.13842` (2024).

Thank you very much!

Do you have questions or comments?