

Symmetries, spectra and eigenfunctions of simple B-free lattice systems

M. Baake
Bielefeld

(input from many people, including
U. Grimm, R.V. Moody, P.A.B. Pleasants, (≤ 2000),
N. Mañibo, D. Lutz, T. Schindler, ... (recent))

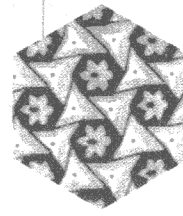
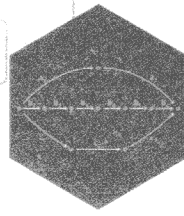
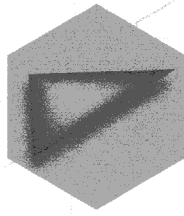
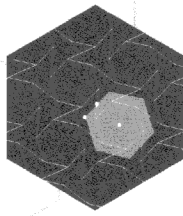
$$V = \{x \in \mathbb{Z} : x \text{ square-free}\} = \mathbb{Z} \setminus \bigcup_{p \in \mathcal{P}} p^2 \mathbb{Z} \quad \left[\begin{array}{l} \text{B-free} \\ \text{B} = \{p^2 : p \in \mathcal{P}\} \end{array} \right]$$

$$X = \overline{\mathbb{Z} + V} \quad \leadsto \quad \boxed{(X, \mathbb{Z}) \text{ TDS}} \quad X \text{ hereditary}$$

$$V \leftrightarrow 1_V \in \{0, 1\}^{\mathbb{Z}} : \text{view } X \text{ as } \underline{\text{subshift}}$$

Properties:

- V not Delone, but $V - V = \mathbb{Z}$
- $\text{dens}(V) = \frac{6}{\pi^2} = 1/\zeta(2)$, $\mathcal{A} = \{[-n, n] : n \in \mathbb{N}\}$
- $(X, \mathbb{Z}, \mu_M)$ has pp spectrum
 $\quad \quad \quad \uparrow$ Mirsky measure
- $\Sigma = \{q \in \mathbb{Q} \cap [0, 1) : \text{den}(q) \text{ cube-free}\}$
 $\quad = \bigoplus_{p \in \mathcal{P}} p^{-2} \mathbb{Z} / \mathbb{Z} \quad \simeq \quad \bigoplus_{p \in \mathcal{P}} C_{p^2}$
- trivial top. point spectrum (c.f.: later)
- $\text{cent}_{\text{Homeo}(X)}(\mathbb{Z}) = \mathbb{Z} =: \mathcal{C}$
 $\text{norm}_{\text{Homeo}(X)}(\mathbb{Z}) = \mathbb{Z} \rtimes C_2 =: \mathcal{R}$ ("reversible")
 $\text{cent}_{\text{Homeo}(X)}^*(\mathbb{Z}) = \mathbb{Z}$
- $h_{\text{top}} = \log(2)/\zeta(2) > 0$, $\mu_{\text{max}} = \mu_M \otimes \mu_{\text{coin toss}}$



(More) general setup:

$\Gamma \subset \mathbb{R}^d$ lattice, $\mathcal{B} = \{\Gamma_i : i \in \mathbb{N}\}$ with

$$(1) \quad \Gamma_i \subset \Gamma, \quad 1 < [\Gamma : \Gamma_i] \leq [\Gamma : \Gamma_{i+1}] \quad (i \in \mathbb{N})$$

$$(2) \quad \Gamma_i + \Gamma_j = \Gamma \quad \text{for all } i \neq j$$

$$(3) \quad \sum_{i \in \mathbb{N}} \lambda(\Gamma_i)^{-d} < \infty, \quad \lambda(\Gamma_i) = \min_{x \in \Gamma_i \setminus \{0\}} \|x\|$$

$d=1, \Gamma = \mathbb{Z}$: Erdős $\mathcal{B}$ -free numbers

$\rightarrow$ analogues for all properties, explicit.

Here: A 2D paradigmatic example

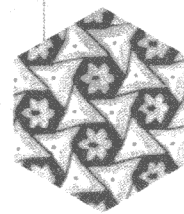
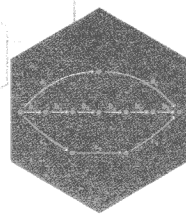
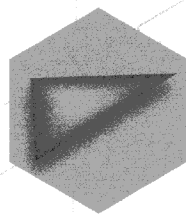
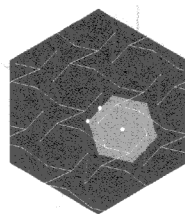
$$\Gamma = \mathbb{Z}^2, \quad \mathcal{B} = \{p \cdot \mathbb{Z}^2 : p \in \mathcal{P}\}$$

$$V = \mathbb{Z}^2 \setminus \bigcup_{p \in \mathcal{P}} p \cdot \mathbb{Z}^2 = \{(m, n) \in \mathbb{Z}^2 : \gcd(m, n) = 1\}$$

"visible lattice points"

Properties:

- V not Delone, but $V - V = \mathbb{Z}^2$
- $\text{dens}(V) = \prod_{p \in \mathcal{P}} (1 - p^{-2}) = 1/\zeta(2), \quad \mathcal{A} = \{[-n, n]^2 : n \in \mathbb{N}\}$
- $X = \overline{\mathbb{Z}^2 + V}, \quad (X, \mathbb{Z}^2) \text{ TDS}; \quad \boxed{\mathcal{C} = \mathbb{Z}^2; \mathcal{R} = \mathbb{Z}^2 \rtimes \text{GL}(2, \mathbb{Z})}$
- μ_M : Mirsky measure (via patch frequencies)
- $(X, \mathbb{Z}^2, \mu_M)$ has pp spectrum
- $\Sigma = \{k \in \mathbb{Q} \cap [0, 1] : \text{den}(k) \text{ square-free}\} = \bigoplus_{p \in \mathcal{P}} \frac{1}{p} \mathbb{Z}^2 / \mathbb{Z}^2$
- triv. topol. point spectrum
- $h_{\text{top}} = \log(2)/\zeta(2) \rightarrow \mu_{\text{max}} = \mu_M \otimes \mu_{\text{coin toss}}$
- $(X, \mathbb{Z}^2, \mu_{\text{max}})$: spectrum pp + ac



V as weak model set:

$$\begin{array}{ccccc}
 \mathbb{R}^2 & \xleftarrow{\pi} & \mathbb{R}^2 \times H & \xrightarrow{\pi_{\text{int}}} & H \\
 \cup & & \cup & & \cup \\
 \mathbb{Z}^2 & \xleftarrow{1:1} & L & \longrightarrow & (\mathbb{Z}^2)^* \\
 \parallel & & & & \cup \\
 L & & \xrightarrow{*} & & L^*
 \end{array}$$

dense

$$H = \bigoplus_{p \in \mathcal{P}} \mathbb{Z}/p\mathbb{Z}^2$$

$$\mathcal{L} = \{(x, x^*) : x \in \mathbb{Z}^2\}$$

$$x^* = (x \bmod p\mathbb{Z}^2)_{p \in \mathcal{P}}$$

$$\leadsto V = \{x \in L : x^* \in W\}$$

$$W = \prod_{p \in \mathcal{P}} (\mathbb{Z}/p\mathbb{Z}^2 \setminus \{0\})$$

$$\text{Vol}(W) = 1/\zeta(2)$$

$\Rightarrow$ spectral properties

(Meyer, Moody, B&Strungaru, Keller, Richard&Strungaru)

$$\hat{H} = \bigoplus_{p \in \mathcal{P}} \frac{1}{p} \mathbb{Z}^2 / \mathbb{Z}^2 = \Sigma$$

Eigenfunctions:

$$k \in \hat{H} \leadsto A_V(k) := \lim_{n \rightarrow \infty} \frac{1}{(2n+1)^2} \sum_{x \in V_n \cap [-n, n]^2} e^{-2\pi i k x}$$

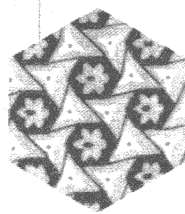
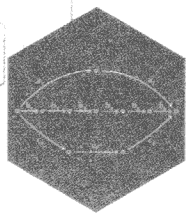
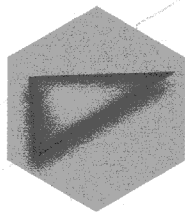
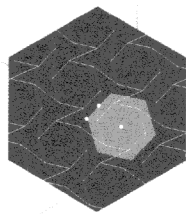
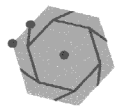
$$\Rightarrow A_{t+V}(k) = e^{-2\pi i k t} A_V(k) \quad \text{etc.}$$

Thm.:

$$A_V(k) = \begin{cases} \frac{1}{\zeta(2)} \prod_{p \mid \text{den}(k)} \frac{1}{1-p^2} & , k \in \hat{H} \\ 0 & , \text{otherwise} \end{cases}$$

Pf.: - uniform distribution ^(of V^*) in W or

- explicit use of limit-periodic structure



$(X, \mathbb{Z}^2, \mu_{\max})$: typ. element via Bernoulli thinning of V

$$\rightarrow A_{V_{\xi}}(k) = \lim_{n \rightarrow \infty} \frac{1}{(2n+1)^2} \sum_{x \in V_n[-n,n]^2} \xi_x e^{-2\pi i k x}$$

$(\xi_x)_{x \in V}$: i.i.d., $\{0,1\}$ -valued, $P(\xi_x=1) = \frac{1}{2}$

$$\stackrel{\text{a.s.}}{=} \frac{1}{2} \cdot A_V(k) \quad (\text{Kolmogorov's SLLN})$$

Thm.: The FB coeff. provide the eigenfunctions in both cases, but:

$(X, \mathbb{Z}^2, \mu_M)$: pure point

$(X, \mathbb{Z}^2, \mu_{\max})$: pp (same Σ) & ac (countably many Lebesgue measures)

Open questions:

- what happens outside Erdős class?
- (extended) symmetries in general?
- top. conjugacy and factor properties?
- k -free numbers in rings of integers?
- spectra for general inv. measure?

$\rightarrow$ more on arXiv