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Symmetries, spectra and eigenfunctions of simple $\mathcal{B}$-free lattice systems

M. Baake
Bielefeld

(input from many people, including
U. Grimm, R.V. Moody, P.A.B. Pleasants (≤ 2000),
N. Mañibo, D. Lenz, T. Schindler, ... (recent))

$V = \{ x \in \mathbb{Z} : x \text{ square-free} \} = \mathbb{Z} \setminus \bigcup_{p \in \mathcal{P}} p^2 \mathbb{Z}$ [$\mathcal{B}$-free, $\mathcal{B} = \{p^2 : p \in \mathcal{P}\}$]

$\mathbb{X} = \overline{\mathbb{Z} + V} \quad \circlearrowleft \quad \boxed{(\mathbb{X}, \mathbb{Z}) \text{ TDS}} \quad \mathbb{X}$ hereditary

$V \leftrightarrow 1_V \in \{0,1\}^{\mathbb{Z}}$: view $\mathbb{X}$ as subshift

Properties:

- V not Delone, but $V - V = \mathbb{Z}$
- $\operatorname{dens}(V) = \frac{6}{\pi^2} = 1/\zeta(2)$, $\quad \mathcal{A} = \{ [-n, n] : n \in \mathbb{N} \}$
- $(\mathbb{X}, \mathbb{Z}, \mu_M)$ has pp spectrum
  ↳ Mirsky measure
- $\Sigma = \{ q \in \mathbb{Q} \cap [0,1) : \operatorname{den}(q) \text{ cube-free} \}$
  $= \bigoplus_{p \in \mathcal{P}} p^{-2} \mathbb{Z} / \mathbb{Z} \simeq \bigoplus_{p \in \mathcal{P}} C_{p^2}$
- trivial top. point spectrum (e.f.: later)
- $\operatorname{cent}_{\mathrm{Homeo}(\mathbb{X})}(\mathbb{Z}) = \mathbb{Z} =: \mathcal{C}$
  $\operatorname{norm}_{\mathrm{Homeo}(\mathbb{X})}(\mathbb{Z}) = \mathbb{Z} \rtimes C_2 =: \mathcal{R}$ ("reversible")
  $\operatorname{cent}^{*}_{\mathrm{Homeo}}(\mathbb{X})(\mathbb{Z}) = \mathbb{Z}$
- $h_{top} = \log(2)/\zeta(2) > 0$, $\quad \mu_{max} = \mu_M \otimes \mu_{\text{coin toss}}$

(More) general setup:

$\Gamma \subset \mathbb{R}^d$ lattice, $\mathcal{B} = \{\Gamma_i : i \in \mathbb{N}\}$ with

$\boxed{V = \Gamma \setminus \bigcup_{\Lambda \in \mathcal{B}} \Lambda}$

(1) $\Gamma_i \subset \Gamma$, $1 < [\Gamma : \Gamma_i] \leq [\Gamma : \Gamma_{i+1}]$ $\quad (i \in \mathbb{N})$

(2) $\Gamma_i + \Gamma_j = \Gamma$ for all $i \neq j$

(3) $\sum_{i \in \mathbb{N}} \lambda(\Gamma_i)^{-d} < \infty$, $\lambda(\Gamma_i) = \min_{x \in \Gamma_i \setminus \{0\}} \|x\|$

$d = 1$, $\Gamma = \mathbb{Z}$: Erdős $\mathcal{B}$-free numbers

$\curvearrowright$ analogues for all properties, explicit.

<u>Here:</u> A 2D paradigmatic example

$\Gamma = \mathbb{Z}^2$, $\mathcal{B} = \{ p \cdot \mathbb{Z}^2 : p \in \mathcal{P} \}$

$V = \mathbb{Z}^2 \setminus \bigcup_{p \in \mathcal{P}} p \cdot \mathbb{Z}^2 = \{ (m,n) \in \mathbb{Z}^2 : \gcd(m,n) = 1 \}$

"<u>visible lattice points</u>"

<u>Properties:</u>

- V not Delone, but $V - V = \mathbb{Z}^2$
- $\mathrm{dens}(V) = \prod_{p \in \mathcal{P}} (1 - p^{-2}) = 1/\zeta(2)$, $\quad \mathcal{A} = \{ [-n,n]^2 : n \in \mathbb{N} \}$
- $\mathbb{X} = \overline{\mathbb{Z}^2 + V}$, $(\mathbb{X}, \mathbb{Z}^2)$ TDS; $\quad \boxed{\mathcal{C} = \mathbb{Z}^2; \ \mathcal{R} = \mathbb{Z}^2 \rtimes GL(2,\mathbb{Z})}$
- μ_M: Mirsky measure (via patch frequencies)
- $(\mathbb{X}, \mathbb{Z}^2, \mu_M)$ has pp spectrum
- $\Sigma = \{ k \in \mathbb{Q}^2 \cap [0,1)^2 : \mathrm{den}(k) \text{ square-free} \} = \bigoplus_{p \in \mathcal{P}} \frac{1}{p} \mathbb{Z}^2 / \mathbb{Z}^2$
- triv. topol. point spectrum
- $h_{top} = \log(2)/\zeta(2)$ $\curvearrowright$ $\mu_{max} = \mu_M \otimes \mu_{\text{coin toss}}$
- $(\mathbb{X}, \mathbb{Z}^2, \mu_{max})$: spectrum pp + ac

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V as weak model set:

$$\begin{tikzcd}
\mathbb{R}^2 & \mathbb{R}^2 \times H \arrow[l, "\pi"'] \arrow[r, "\pi_{int}"] & H \\
\mathbb{Z}^2 \arrow[u, phantom, sloped, "\subset"] & \mathcal{L} \arrow[l, "1:1"'] \arrow[r] \arrow[u, phantom, sloped, "\subset"] & (\mathbb{Z}^2)^* \arrow[u, phantom, sloped, "\subset", "\text{dense}"'] \\
L \arrow[u, phantom, sloped, "="] \arrow[rr, "*"] & & L^* \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$

$H = \bigotimes_{p \in \mathcal{P}} \mathbb{Z}^2/p\mathbb{Z}^2$

$\mathcal{L} = \{(x, x^*) : x \in \mathbb{Z}^2\}$

$x^* = (x \bmod p\mathbb{Z}^2)_{p \in \mathcal{P}}$

$\curvearrowright$ $\boxed{V = \{x \in L : x^* \in W\}}$ $\qquad W = \prod_{p \in \mathcal{P}} \left(\mathbb{Z}^2/p\mathbb{Z}^2 \setminus \{0\}\right)$

$\mathrm{vol}(W) = 1/\zeta(2)$

$\Rightarrow$ spectral properties (Meyer, Moody, B & Strungaru, Keller, Richard & Strungaru)

$$\hat{H} = \bigoplus_{p \in \mathcal{P}} \tfrac{1}{p}\mathbb{Z}^2/\mathbb{Z}^2 = \Sigma$$

Eigenfunctions:

$k \in \hat{H}$ $\curvearrowright$ $A_V(k) := \lim_{n \to \infty} \frac{1}{(2n+1)^2} \sum_{x \in V \cap [-n,n]^2} e^{-2\pi i k x}$

$\Rightarrow$ $\boxed{A_{t+V}(k) = e^{-2\pi i k t} A_V(k)}$ etc.

Thm.:

$$A_V(k) = \begin{cases} \frac{1}{\zeta(2)} \prod_{p \mid \mathrm{den}(k)} \frac{1}{1-p^2}, & k \in \hat{H} \\ 0, & \text{otherwise} \end{cases}$$

Pf.: - uniform distribution $\overset{\text{of } V^*}{}$ in W or
- explicit use of limit-periodic structure

$(\mathbb{X}, \mathbb{Z}^2, \mu_{max})$: typ. element via Bernoulli thinning of V

$$\rightsquigarrow \quad A_{V_\xi}(k) = \lim_{n\to\infty} \frac{1}{(2n+1)^2} \sum_{x\in V\cap[-n,n]^2} \xi_x \, e^{-2\pi i k x}$$

$(\xi_x)_{x\in V}$: i.i.d., $\{0,1\}$-valued, $\mathbb{P}(\xi_x = 1) = \frac{1}{2}$

$$\overset{a.s.}{=} \frac{1}{2} \cdot A_V(k) \qquad \text{(Kolmogorov's SLLN)}$$

<u>Thm.:</u> The FB coeff. provide the eigenfunctions in both cases, but:

$(\mathbb{X}, \mathbb{Z}^2, \mu_M)$: pure point

$(\mathbb{X}, \mathbb{Z}^2, \mu_{max})$: pp (same Σ) & ac (countably many Lebesgue measures)

Open questions:

- what happens outside Erdős class?
- (extended) symmetries in general?
- top. conjugacy and factor properties?
- k-free numbers in rings of integers?
- spectra for general inv. measure?

$\rightsquigarrow$ more on arXiv