Geometry of Q-manifolds and Gauge Theories II

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Higher Structures and Field Theory
September 15, 2020
ESI - Vienna
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Consider $J^k(\pi)$ and $J^\infty(\pi)$, the spaces of $k$–jets and infinite jets of local sections of $\pi$, respectively.
Jet bundles and the geometry of PDEs

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Given any local section $\sigma$ of $\pi$, denote by $\sigma^{(k)}$ the corresponding $k$–jet prolongation, where $k = 0, \ldots, \infty$. 
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Given any local section $\sigma$ of $\pi$, denote by $\sigma_{(k)}$ the corresponding $k$–jet prolongation, where $k = 0, \ldots, \infty$.

For any vector field $v$ on $X$, denote by $D_v$ the total derivation along $v$, which maps functions on $J^k(\pi)$ to functions on $J^{k+1}(\pi)$: given a function $f$ one has

$$L_v \sigma_{(k)}^* f = \sigma_{(k)}^* D_v f$$
By construction, the space of functions on $J^\infty(\pi)$ is closed under the action of $D_v$. 
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Let $x^\alpha$ be local coordinates on the base and $u^a$ be local fiber coordinates. Then the associated coordinate system on $J^k(\pi)$ is $(x^\alpha, u^a_I)$, where $I$ is a (super) symmetric multi-index (in one of possible conventions) corresponding to partial derivatives along base coordinates.
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Now the total derivation with respect to $x^\alpha$ reads as

$$D_\alpha = \frac{\partial}{\partial x^\alpha} + u^a_\alpha \frac{\partial}{\partial u^a} + u^a_{\alpha\beta} \frac{\partial}{\partial u^a_\beta} + \ldots$$
The total lifting of derivations on $X$ is

1. linear with respect to multiplication on functions on $X$
2. respects the (super) Lie bracket of vector fields on $X$, i.e.

$$[D_{v_1}, D_{v_2}] = D_{[v_1, v_2]}$$

Provided the first property is holding, it is sufficient to require that, in local coordinates, $[D_\alpha, D_\beta] = 0$. 
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All properties together imply that the space of infinite jets is canonically supplied with a horizontal involutive distribution, called the Cartan distribution and denoted by $\mathcal{C}$. 
In local coordinates, a partial differential equation (PDE) on sections of $\pi$ is determined by a system of equations

$$H_k(x, u_I) = 0$$

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If a local section is a solution to $H_k = 0$, it is also a solution to $D_\alpha H_k = 0$. Therefore, in addition to the original system equations, one should consider all prolongations of the form $D_\alpha D_\beta \ldots H_k = 0$ for all (super) symmetric finite sequences of indices $\alpha, \beta, \ldots$. 
Jet bundles and the geometry of PDEs

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In this way, we obtain an infinitely prolonged PDE. By construction, the corresponding subspace of $J^\infty(\pi)$ contains the Cartan distribution.
Jet bundles and the geometry of PDEs

This leads to a geometrical formulation of PDEs.
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A PDE is a pair \((E_X, C)\), where \(E_X\) is a manifold and \(C(E_X)\) (denoted by just \(C\) in what follows if it doesn’t lead to confusions) is an involutive distribution \(C(E_X) \subset TE_X\) called Cartan distribution. It is typically assumed (as it’s done later) that

- \(E_X\) is a locally trivial bundle \(\pi_X : E_X \rightarrow X\) over the manifold \(X\) of independent variables.

- Canonical projection \(\pi_X\) induces an isomorphism \(C_p(E_X) \rightarrow T_{\pi_X(p)}X\) for all \(p \in E_X\). In particular \(C\) is of constant rank, which is equal to \(\text{dim}(X)\).

- \((E_X, C)\) can be embedded into some jet bundle as an infinitely prolonged equation, at least locally.
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- \((E_X, \mathcal{C})\) can be embedded into some jet bundle as an infinitely prolonged equation, at least locally.

Solutions of the PDE are sections of \(\pi_X\), which are tangent to \(\mathcal{C}\), i.e. integral submanifolds of the Cartan distribution, tangent to the Cartan distribution.
Every vector field on $X$ admits a canonical lift to the total space $\nu \mapsto D_\nu$

which is linear under multiplication on functions on $X$ and which respects the Lie (super) bracket of vector fields.
Jet bundles and the geometry of PDEs

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We obtain a differential $d_h$ on horizontal forms - differential forms on the total space annihilated by vertical vector fields, called the horizontal differential.
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In local base coordinates it reads as follows

$$d_h = dx^\alpha D_\alpha$$
The super geometric interpretation of the Cartan structure and the horizontal differential is the following: the Cartan distribution on the total space, viewed as a vector bundle with the shifted degree by 1, is a $Q$–manifold such that the canonical projection

$$(C[1], dh) \rightarrow (T[1]X, d)$$

is a $Q$–bundle.
Jet bundles and the geometry of PDEs

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In general, it is not locally trivial as a $Q$–bundle, i.e. one can not represent it locally as a product of two $Q$–manifolds.
Jet bundles and the geometry of PDEs

The super geometric interpretation of the Cartan structure and the horizontal differential is the following: the Cartan distribution on the total space, viewed as a vector bundle with the shifted degree by 1, is a $\mathbb{Q}$–manifold such that the canonical projection

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is a $\mathbb{Q}$–bundle.

In general, it is not locally trivial as a $\mathbb{Q}$–bundle, i.e. one can not represent it locally as a product of two $\mathbb{Q}$–manifolds.

For example, it is not locally trivial in the case of ordinary jet spaces, regarded as ”empty” differential equations.
Reparametrization invariant PDEs

If the $Q$–bundle $(C[1], d_h) \to (T[1]X, d)$ is locally trivial, the corresponding PDE is called reparametrization invariant.
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All PDEs of finite type are reparametrization invariant.

In particular, all ODEs are reparametrization invariant.
Evolutionary vector fields as symmetries of PDEs

Let \((E_X, C)\) be a PDE over \(X\), \(\pi_X : E_X \to X\) be the corresponding projection. A vertical vector field on the total space, preserving \(C\), is called an evolutionary vector field.
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- An evolutionary vector field commutes with \(D_\nu\) for all vector fields \(\nu\) on \(X\);
- If an evolutionary vector field can be exponentiated, i.e. it is an infinitesimal flow of some diffeomorphism of the total space \(E_X\), then:
  - this diffeomorphism is a bundle isomorphism
  - it preserves the Cartan distribution
Evolutionary vector fields as symmetries of PDEs

- Taking into account that functions on the infinite jet space $J^\infty(\pi)$ are generated by functions on $E$ and their total derivatives, one concludes that evolutionary vector fields are uniquely fixed by its action on $\mathcal{F}(E)$. 
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In local coordinates:

$$v(u^a_i) = D^l_x v(u^a)$$

where $D^l_x = D^{i_1}_{x_1} \ldots D^{i_n}_{x^n}$ for $l = (i_1, \ldots, i_n)$
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- In particular, gauge symmetries of local field theories are evolutionary vector fields.
- Given that differential graded (Koszul) resolution of a PDE, which is embedded into a jet space, must commute with jet prolongations, it is determined by a degree 1 evolutionary super vector field.
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- Given that differential graded (Koszul) resolution of a PDE, which is embedded into a jet space, must commute with jet prolongations, it is determined by a degree 1 evolutionary super vector field

- Finally, a $Q$–manifold in the context of PDEs is a PDE together with a degree 1 evolutionary self-commuting vector field
• Given a vector field $\nu$ on the total space of $\pi: E \to X$, there exists a canonical $k$–jet prolongation $\nu^{(k)}$ on $J^k(\pi)$ for all $k$, which preserves the Cartan distribution on $J^k(\pi)$ (the span of tangent spaces to all $k$–jets of local sections of $\pi$).
Given a vector field \( \nu \) on the total space of \( \pi : E \rightarrow X \), there exists a canonical \( k \)-jet prolongation \( \nu^{(k)} \) on \( J^k(\pi) \) for all \( k \), which preserves the Cartan distribution on \( J^k(\pi) \) (the span of tangent spaces to all \( k \)-jets of local sections of \( \pi \)).

This prolongation is compatible with the projections \( J^k(\pi) \rightarrow J^l(\pi) \) for all \( k > l \). In other words, there is a canonical infinite jet prolongation of \( \nu \) to a vector field on \( J^\infty(\pi) \), preserving the Cartan structure, such that \( \nu^{(k)} \) coincides with its restriction to \( F(J^k(\pi)) \subset F(J^\infty(\pi)) \).

The latter statement is equivalent to the existence of a canonical prolongation to \( C[1] \), commuting with \( d_h \).
In local coordinates \((x^a, u^a)\), let

\[
\nu = \sum_\alpha f^\alpha(x, u) \partial_{x^\alpha} + \sum_a g^a(x, u) \partial_{u^a}
\]

Then \(\nu(\infty)\) is the sum of horizontal and vertical evolutionary parts

\[
\nu(\infty) = \nu_h(\infty) + \nu_e(\infty)
\]

where

\[
\nu_h(\infty) = \sum_\alpha f^\alpha(x, u) D_{x^\alpha}
\]

and

\[
\nu_e(\infty)(u^a) = g^a(x, u) - f^\alpha(x, u) u^a_{\alpha}, \quad u^a_{\alpha} = D_{x^\alpha} u^a
\]
Gauge PDEs

Consider a graded PDE together with a homological evolutionary vector field $s$. 
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As it was previously mentioned, we canonically extend $s$ to $C[1]$ in such a way that it will (super-)commute with all total derivations, and thus with the horizontal differential $d_h$. 
Consider a graded PDE together with a homological evolutionary vector field $s$.

As it was previously mentioned, we canonically extend $s$ to $\mathcal{C}[1]$ in such a way that it will (super-)commute with all total derivations, and thus with the horizontal differential $d_h$.

Therefore we obtain two super commuting differentials on $E_X$, $d_h$ and $s$. Their sum $d_h + s$ is again a differential.
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Notice that $C[1]$ is naturally bi-graded: the first grading comes from the degree of horizontal differential forms, while the second one, called the ghost number, corresponds to the degree of fiber coordinates.
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The differentials $d_h$ and $s$ have the bi-degrees $(1, 0)$ and $(0, 1)$, respectively. The total differential $d_h + s$ has the total degree 1.
**Gauge PDEs**

Gauge pre-PDE is a $\mathbb{Z}$-graded $Q$-bundle $\left( E_{T[1]X}, Q \right)$ over $\left( T[1]X, d_X \right)$, where $\left( T[1]X, d_X \right)$ is considered as a graded $Q$-manifold with the canonical degree (form degree) and the canonical $Q$-structure (de Rham differential).
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Gauge pre-PDE \((E_{T[1]X}, Q)\) is called contractible if as a bundle over \(T[1]X\) it is locally trivial, admits a global \(Q\)-section, and its fiber is a contractible \(Q\)-manifold.
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The equivalence relation generated by the equivalence reduction is called the **equivalence of gauge pre-PDEs**.
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A pre-PDE, where $E_{T[1]X}$ is $C[1]$ for a (super) jet space and $Q = dh + s$ for an evolutionary degree 1 vector field $s$, is called a standard gauge pre-PDE. Equivalence of standard gauge pre-PDEs are those which respect the natural bi-grading.
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Gauge pre-PDE $(E_{T[1]X}, Q)$ is a gauge PDE if:

1. it is equivalent to a nonnegatively graded gauge pre-PDE
2. it is equivalent to a standard gauge pre-PDE
MAXWELL EQUATIONS AS A GAUGE THEORY:
"CLASSICAL" PART

- Independent variables: $x^i, i = 1, \ldots, n$
Maxwell equations as a gauge theory:
"classical" part

- Independent variables: $x^i, i = 1, \ldots, n$
- Dependent variables: $A_i, i = 1, \ldots, n$, $ghA_i = 0$

The action functional:

$S[A] = \int L[A]$

where

$L[A] = -\frac{1}{2} \sum_{i,j} F_{ij} F_{ij} ,
F_{ij} = D_i A_j - D_j A_i$

and $D_i = D x^i$. We will denote $A_{i,j} = D_j A_i$. 
Maxwell equations as a gauge theory:
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Maxwell equations as a gauge theory: "classical" part

- The equations of motion (Euler-Lagrange equations):

\[ \frac{\delta L}{\delta A_i} = \frac{\partial L}{\partial A_i} - \sum_j D_j \left( \frac{\partial L}{\partial A_{i,j}} \right) = 0 \]

will give us

\[ \frac{\delta L}{\delta A_i} = - \sum_j D_j F_{ij} = 0 \]

for all \( i = 1, \ldots, n \).
MAXWELL EQUATIONS AS A GAUGE THEORY: THE KOSZUL-TATE RESOLUTION

- New dependent variables: $A^i_*, i = 1, \ldots, n$, $\text{gh} A^i_* = -1$, $C^*$, $\text{gh} C^* = -2$
Maxwell equations as a gauge theory: the Koszul-Tate resolution

- New dependent variables: $A^i_*, i = 1, \ldots, n$, $\text{gh} A^i_* = -1$, $C^*$, $\text{gh} C^* = -2$

- The Koszul-Tate differential $\delta$:

\[
\begin{align*}
\delta A_i &= 0 \\
\delta A^i_* &= \sum_j D_j F_{ij} \\
\delta C^* &= \sum_i D_i A^i_*
\end{align*}
\]
Maxwell equations as a gauge theory: the Koszul-Tate resolution

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- The Koszul-Tate differential $\delta$:

$$\delta A_i = 0$$
$$\delta A^i_* = \sum_j D_j F_{ij}$$
$$\delta C^* = \sum_i D_i A^i_*$$

- $\delta$ is an evolutionary vector field, that is

$$\delta \left( D^l_x A^i_* \right) = D^l_x \left( \sum_j D_j F_{ij} \right), \quad \delta \left( D^l_x C^* \right) = D^l_x \left( \sum_i D_i A^i_* \right)$$
Maxwell equations as a gauge theory: the Koszul resolution

- $\delta$ is acyclic except the degree 0; the 0-degree cohomology gives functions on the equation manifold:

$$H^k_\delta = \begin{cases} 0, & k < 0 \\ \mathcal{F}(\Sigma_{\text{Maxwell}}), & k = 0 \end{cases}$$

Here

$$\Sigma_{\text{Maxwell}} = \left\{ D_I \left( \frac{\delta L}{\delta A_i} \right) = 0, \forall I, i \right\}$$
Maxwell equations as a gauge theory: the gauge symmetries generator

- New dependent variable: $C$, $g h C = 1$
Maxwell equations as a gauge theory: the gauge symmetries generator

- New dependent variable: $C$, $g h C = 1$
- The differential $\gamma$:

\[
\begin{align*}
\gamma A_i &= D_i C \\
\gamma C &= 0
\end{align*}
\]

generates gauge symmetries

\[
\delta_\epsilon A_i = D_i \epsilon
\]
Maxwell equations as a gauge theory: the BRST differential

- Independent variables: $x^i$, $i = 1, \ldots, n$
- Dependent variables: $A_i$, $ghA_i = 0$ and $A^*_i$, $i = 1, \ldots, n$, $ghA^*_i = -1$; $C^*$, $ghC^* = -2$, and $C$, $ghC = 1$
- The differential $s$:

$$sA_i = D_i C = C_i, \quad sC = 0$$

$$sA_i^* = \sum_j D_j F_{ij} = \sum_j F_{ij,j}$$

$$sC^* = \sum_i D_i A_i^*$$

- $Q = dh + s$, such that $Qx^i = \theta^i$, $QA_i = \sum_k \theta^k A_{i,k} + C_i$, etc.
MAXWELL EQUATIONS AS A GAUGE THEORY: THE MINIMAL MODEL

Now we introduce another set of jet coordinates, which is decomposed into the following two subsets:
Maxwell equations as a gauge theory: the minimal model

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Subset 1

\[ \{ A_i^{*j_1...j_m}, QA_i^{*j_1...j_m}, C_{j_1...j_m}^*, QC_{j_1...j_m}^*, A_{(j_1j_2...j_m)}, QA_{(j_1j_2...j_m)} \} \]

where

\[ A_{(j_1j_2...j_m)} = A_{j_1j_2...j_m} + A_{j_2j_3...j_mj_1} + \ldots + A_{j_{m-1}j_1...j_m} \]
Maxwell equations as a gauge theory: the minimal model

Subset 2

\{x^i, \theta^i, C_{\text{min}}, P_{i,j_1...j_m}\}

where \(C_{\text{min}} = C + \sum_k \theta^k A_k\) and \(P_{i,j_1...j_m} = F'_{i(j_1,j_2...j_m)}\) for

\[F'_{i j_1 j_2 ... j_m} = F_{i j_1 j_2 ... j_m} - \frac{1}{n} \left( \sum_j F_{i j j_3 ... j_m} \right) \delta_{j_1 j_2}\]

\(F'_{i j_1 j_2 ... j_m}\) is the traceless part of \(F_{i(j_1,j_2...j_m)}\).
Maxwell equations as a gauge theory: the minimal model

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\( F'_{i j_1 j_2\ldots j_m} \) is the traceless part of \( F_{i(j_1 j_2\ldots j_m)} \). One has

\[ Q x^i = \theta^i, \quad QC_{\text{min}} = \frac{1}{2} \sum_{i,j} \theta^i \theta^j P_{i,j} \]

\[ Q P_{i,j_1 j_2\ldots j_m} = \sum_k \theta^k \left( \frac{m+1}{m+2} P_{i,k j_1\ldots j_m} + \frac{1}{m+2} P_{k,i j_1\ldots j_m} \right) \]
Maxwell equations as a gauge theory: the minimal model

- Independent variables: \( \{x^i, \theta^i\}, \ i = 1, \ldots, n \)
- Dependent variables: \( P_{i,j_1j_2,\ldots,j_m}, \ g\ h P_{\ldots} = 0; \ C_{min}, \ g\ h C_{min} = 1 \)

where \( P_{i,j_1j_2,\ldots,j_m} \) is symmetric w.r.t. \( j_1, j_2, \ldots, j_m \) and satisfies

\[
P_{(i,j_1\ldots j_m)} = P_{i,j_1j_2\ldots j_m} + P_{j_1j_2,\ldots,j_mi} + \ldots + P_{j_m,j_1j_2\ldots j_{m-1}} = 0
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- The differential \( Q: \)

\[
Qx^i = \theta^i, \quad QC_{min} = \frac{1}{2} \sum_{ij} \theta^i \theta^j P_{i,j}
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\[
QP_{i,j_1j_2\ldots j_m} = \sum_k \theta^k \left( \frac{m + 1}{m + 2} P_{i,kj_1\ldots j_m} + \frac{1}{m + 2} P_{k,ij_1\ldots j_m} \right)
\]
We have two categories:

- The first one consists of standard gauge PDEs together with morphisms and equivalences preserving $\mathbb{Z} \times \mathbb{Z}$ bi-grading,
- while the second one consists of gauge PDEs, which are only $\mathbb{Z}$-graded.
- There is a forgetful functor from the first category to the second one which replaces the bi-grading with the total grading.
- There is also a canonical functor in the opposite direction which associates to a gauge PDE $(E^T_{[1]}X, Q)$ the differential equation whose solutions are $Q$-sections of $E^T_{[1]}X \to T_{[1]}X$. 
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More precisely, let $E_X \rightarrow X$ be the jet space of local sections for a (super) bundle over $X$ together with a degree 1 homological evolutionary vector field $s$, whose negative cohomology are vanishing.
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More precisely, let $E_X \to X$ be the jet space of local sections for a (super) bundle over $X$ together with a degree 1 homological evolutionary vector field $s$, whose negative cohomology are vanishing.

This describes a PDE with gauge symmetries in the usual sense: normally $s$ is obtained by the homological perturbation of a couple of homological degree 1 evolutionary vector fields, the first of which is the Koszul resolution for a PDE embedded into the jet space (in other words the Koszul resolution determines such an embedding).
Parent formalism

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Now we take the corresponding Cartan distribution $C$ on the (super) jet space, the $Q$–bundle $C[1] \to T[1]X$ with the total $Q = dh + s$ and ”forget” about the bi-grading of the total space. We obtain a gauge PDE, which ”remembers” only the total $\mathbb{Z}$–grading.
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From now we are allowed to work with this new object as if it was a gauge PDE from the very beginning, replacing it with an equivalent gauge PDE (which inherits the same important properties, eg. all associated natural cohomologies are the same).
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However, the parent formalism gives us the way how to restore the original standard gauge PDE up an equivalence of standard gauge PDEs. As mentioned before, we take differential equation for $Q$–sections of the gauge PDE regarded as a $Q$–bundle over $T[1]X$. This gives us a standard gauge PDE which is equivalent to the original one.
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If the gauge PDE is reparametrization invariant (by the definition, the underlined $Q$–bundle is locally trivial), we obtain (at least locally) an AKSZ-type differential equation.
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\(Q^{(\infty)}\) splits into the horizontal and vertical (evolutionary) parts,

\[
Q^{(\infty)} = Q_h^{(\infty)} + Q_e^{(\infty)}
\]

as well as the Euler vector field \(\epsilon^{(\infty)}\) (which determines the corresponding \(\mathbb{Z}\)–grading on the total space of \(E_{T[1]X}\))

\[
\epsilon^{(\infty)} = \epsilon_h^{(\infty)} + \epsilon_e^{(\infty)}
\]
The only non-trivial commutation relations between the four obtained components are

\[ [\epsilon_h(\infty), Q_h(\infty)] = Q_h(\infty) \]

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Therefore we have a bi-complex with the two commuting \( \mathbb{Z} \)-gradings, given by \( \epsilon_h^{(\infty)} \) and \( \epsilon_e^{(\infty)} \), respectively.
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Therefore we have a bi-complex with the two commuting \( \mathbb{Z} \)-gradings, given by \( \epsilon_h^{(\infty)} \) and \( \epsilon_e^{(\infty)} \), respectively.

One can show that \( Q_h^{(\infty)} \) and \( Q_e^{(\infty)} \) can be canonically identified with \( d_h \) and \( s \) for the standard gauge PDE for \( Q \)-sections of \( \pi \), respectively.
Presymplectic gauge PDEs

Let \((E_{T[1]X}, Q)\) be a gauge (pre-)PDE regarded as a Q-bundle over \(T[1]X\), where \(\dim X = n\).
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A compatible presymplectic structure is a degree \(n - 1\) vertical 2-form \(\omega\) on the total space satisfying

\[
dv \omega = 0, \quad L_Q \omega = 0. \tag{1}
\]

AKSZ model as a presymplectic gauge PDE

In this case \(E = E_{T[1]X} = M \times T[1]X\), where \((M, Q_M, \omega)\) is a symplectic Q-manifold. The Q-structure on the total space is \(Q = dX + Q_M\), while the vertical presymplectic form is given by \(\omega\).

Notice that here \(dv\) is just \(dM\).
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One can reformulate the transgression formula (A.K., T. Strobl) from the previous lecture as follows: for a symplectic $Q$–manifold $(M, Q_M, \omega)$ with the symplectic form $\omega$ of degree $p > 0$ one has

- $d\omega = 0$
- $L_{Q_M} \omega = 0$ and
- $L_\epsilon \omega = p\omega$

where $\epsilon$ is the Euler vector field on $M$, which determines the grading.
The transgression formula for AKSZ

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- $L_{\mathcal{Q}_\mathcal{M}}\omega = 0$ and
- $L_\epsilon\omega = p\omega$

where $\epsilon$ is the Euler vector field on $\mathcal{M}$, which determines the grading.

This implies that

$$\omega = (d + L_{\mathcal{Q}_\mathcal{M}})(\chi + l)$$

where $\chi = \frac{1}{p} L_\epsilon \omega$ and $l = \frac{1}{p+1} L_{\mathcal{Q}_\mathcal{M}} \chi$. 
In particular, one has

- $\omega = d\chi$ and
- $\iota_{Q_M} \omega = dh$, where $h = \frac{p}{p+1} \iota_{Q_M} \chi$. 

The transgression formula for AKSZ
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Let $X$ an n-dimensional manifold $(n > 1)$, $(M, Q_M, \omega)$ be a symplectic Q-manifold of degree $p = n - 1$, and $\phi$ be a (degree-preserving) map from $T[1]X$ to $M$. Then the (classical part of the) AKSZ sigma model action for the source space $T[1]X$ and the target $M$ is

$$S_{AKSZ}[\phi] = \int_{T[1]X} \tilde{\phi}^* (\chi + l)$$
Now let us remark that, whenever \((E_{T[1]X}, Q, \omega)\) is a presymplectic gauge (pre-)PDE, such that

\[
\omega = (d_v + L_Q)(\chi + l)
\]

where \(\chi\) and \(l\) are vertical 1–form and 0–form, respectively, and \(\sigma\) is a (degree preserving) section of \(E_{T[1]X}\), one can construct an action in a similar way to the AKSZ case:

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Here we use the generalized Cartan map induced by \(\sigma\) by taking into account that the corresponding field strength is a vertical vector field, which allows us to apply the generalized Cartan map to vertical forms on the total space.
If $\omega$ is vertically non-degenerate, the solutions to the EOM are in one-to-one correspondence with $Q$-sections of $E \to T[1]X$.
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- In general it is not true (all interesting non-topological examples, such as Maxwell, Yang-Mills and Einstein gravity models, correspond to degenerate horizontal 2-forms)
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- In general it is not true (all interesting non-topological examples, such as Maxwell, Yang-Mills and Einstein gravity models, correspond to degenerate horizontal 2-forms)

- Then one should be able to quotient out the kernel of $\omega$, viewed in a reasonable way
In this case $E_{\mathcal{T}[1]}X$ corresponds to the jet bundle for a bundle over $X$ and $Q = dh + s$, where $s$ is an extension of a vertical evolutionary homological vector field.
Standard gauge presymplectic PDEs

In this case $E_{T[1]X}$ corresponds to the jet bundle for a bundle over $X$ and $Q = d_h + s$, where $s$ is an extension of a vertical evolutionary homological vector field.

The transgression formula $\omega = (d_v + L_Q)(\chi + l)$ is true if and only if the following properties are holding:

1. $\omega = d_v \chi$;
2. $\iota_s \omega = d_v h - d_h \chi$

where $h = \iota_s \chi - l$. 
Indeed, the transgression formula is equivalent to

1. \( \omega = d_v \chi; \)
2. \( d_h \chi + L_s \chi + d_v l = 0 \)
3. \( (L_s + d_h)l = 0 \)

While the first properties are the same, the equation

\[
dx_h \chi + L_s \chi + d_v l = 0
\]

can be represented as

\[
d_h \chi + \iota s d_v \chi - d_v \iota s \chi + d_v l = d_h \chi + \iota s \omega - d_v h = 0
\]

therefore the first two properties are equivalent.
Now the last condition \((L_s + d_h)l = 0\) is fulfilled automatically thanks to the degree reason: one can verify that

\[
d_v(L_s + d_h)l = -(L_s + d_h)d_v(l) = (L_s + d_h)^2 \chi = 0
\]

therefore \((L_s + d_h)(l)\) is coming from the base \(T[1]X\), so it is actually a differential form on \(X\). But the degree of this expression is \(n + 1\), thus it must be zero.
(A. Sharapov) One has

- $\omega_n = d_v \chi_n$ and
- $\iota_s \omega_n = d_v h_n - d_h \chi_{n-1},$

where $\omega_n$ and $\chi_n$ are vertical 2-form and 1-form of the top horizontal degree, respectively.
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The second condition means that $s$ is a Hamiltonian vector field up to the divergence.
Example: Descend Process

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where \( \omega_n \) and \( \chi_n \) are vertical 2-form and 1-form of the top horizontal degree, respectively.

The second condition means that \( s \) is a Hamiltonian vector field up to the divergence.

Now we notice that \( L_s \omega_n = -d_h \omega_{n-1} \), where \( \omega_{n-1} = d_v \chi_{n-1} \) is a vertical 2-form of the horizontal degree \( n-1 \).

Moreover,

\[
\begin{align*}
\text{d}_h \text{d}_v (\iota_s \omega_{n-1}) &= -\text{d}_h L_s \omega_{n-1} = L_s \text{d}_h \omega_{n-1} = -L_s^2 \omega_n = 0
\end{align*}
\]
Therefore, under certain topological conditions, \( d_v(\iota_s \omega_{n-1}) \) vanishes up to the divergence, which implies that there exist \( h_{n-1} \) and \( \chi_{n-2} \), such that

\[
\iota_s \omega_{n-1} = d_v h_{n-1} - d_h \chi_{n-2}
\]

and so on: we iterate this process (by the degree reason it will eventually stop) and get \( \omega = \omega_n + \omega_{n-1} + \ldots \), \( \chi = \chi_n + \chi_{n-1} + \ldots \), and \( h = h_n + h_{n-1} + \ldots \), and finally we obtain that

1. \( \omega = d_v \chi \);
2. \( \iota_s \omega = d_v h - d_h \chi \)

is holding and hence the transgression formula \( \omega = (d_v + L_Q)(\chi + l) \) for \( l = \iota_s \chi - h \) is true.
Let us assume that the transgression formula holds:

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\[ d_X \sigma^* = \sigma^* d_h \]

The latter implies that

\[ \tilde{\sigma}^*(\chi + l) = \sigma^*(-\iota_s \chi + l) = -\sigma^* h \]

So the AKSZ type action is the "original" classical Lagrangian

\[ \int_{T[1]X} \sigma^*(-h) \]
The general idea

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• We start with a standard symplectic gauge PDE, corresponding to a BV-extension of a classical theory;

• By mixing the degrees, we find an equivalent presymplectic gauge PDE (maybe a minimal model);

• Using the AKSZ-type action function determined by the transgression of the compatible presymplectic 2-form we can come back to the standard case.