Geometry of Q-manifolds and Gauge Theories II

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- The presymplectic AKSZ type approach to local field theory actions was earlier proposed by K. Alkalaev and M. Grigoriev (2013) and later developed by M. Grigoriev (2016)
- The coordinate version of the parent formalism was proposed by G. Barnich and M. Grigoriev (partially motivated by unfolded approach to higher spin gauge theory of M. Vasiliev)

Jet bundles and the geometry of PDEs

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Given any local section σ of π, denote by $\sigma_{(k)}$ the corresponding $k-$jet prolongation, where $k = 0, \ldots, \infty$.

For any vector field v on X, denote by D_v the total derivation along v, which maps functions on $J^k(\pi)$ to functions on $J^{k+1}(\pi)$: given a function f one has

$$L_v \sigma^*_{(k)} f = \sigma^*_{(k)} D_v f$$

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Now the total derivation with respect to x^α reads as

$$D_\alpha = \frac{\partial}{\partial x^\alpha} + u^a_\alpha \frac{\partial}{\partial u^a} + u^a_{\alpha\beta} \frac{\partial}{\partial u^a_\beta} + \dots$$

The total lifting of derivations on X is

1. linear with respect to multiplication on functions on X
2. respects the (super) Lie bracket of vector fields on X, i.e.

$$[D_{v_1}, D_{v_2}] = D_{[v_1, v_2]}$$

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All properties together imply that the space of infinite jets is canonically supplied with a horizontal involutive distribution, called the Cartan distribution and denoted by $\mathcal{C}$.

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In this way, we obtain an infinitely prolonged *PDE*. By construction, the corresponding subspace of $J^\infty(\pi)$ contains the Cartan distribution.

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A PDE is a pair $(E_X, \mathcal{C})$, where E_X is a manifold and $\mathcal{C}(E_X)$ (denoted by just $\mathcal{C}$ in what follows if it doesn't lead to confusions) is an involutive distribution $\mathcal{C}(E_X) \subset TE_X$ called Cartan distribution. It is typically assumed (as it's done later) that

- E_X is a locally trivial bundle $\pi_X : E_X \to X$ over the manifold X of independent variables.
- Canonical projection π_X induces an isomorphism $\mathcal{C}_p(E_X) \to T_{\pi_X(p)}X$ for all $p \in E_X$. In particular $\mathcal{C}$ is of constant rank, which is equal to $\dim(X)$.
- $(E_X, \mathcal{C})$ can be embedded into some jet bundle as an infinitely prolonged equation, at least locally.

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Solutions of the PDE are sections of π_X, which are tangent to $\mathcal{C}$, i.e. integral submanifolds of the Cartan distribution, tangent to the Cartan distribution.

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$$v \mapsto D_v$$

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In local base coordinates it reads as follows

$$\mathrm{d}_h = \mathrm{d}x^\alpha D_\alpha$$

The super geometric interpretation of the Cartan structure and the horizontal differential is the following: the Cartan distribution on the total space, viewed as a vector bundle with the shifted degree by 1, is a $Q-$manifold such that the canonical projection

$$(\mathcal{C}[1], \mathrm{d}_h) \to (T[1]X, \mathrm{d})$$

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In general, it is not locally trivial as a $Q-$bundle, i.e. one can not represent it locally as a product of two $Q-$manifolds.

For example, it is not locally trivial in the case of ordinary jet spaces, regarded as "empty" differential equations.

Reparametrization invariant PDEs

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All PDEs of finite type are reparametrization invariant.

In particular, all ODEs are reparametrization invariant.

Evolutionary vector fields as symmetries of PDEs

Let $(E_X, \mathcal{C})$ be a PDE over X, $\pi_X : E_X \to X$ be the corresponding projection. A vertical vector field on the total space, preserving $\mathcal{C}$, is a called an evolutionary vector field.

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- An evolutionary vector field commutes with D_v for all vector fields v on X;
- If an evolutionary vector field can be exponentiated, i.e. it is an infinitesimal flow of some diffeomorphism of the total space E_X, then:
  - this diffeomorphism is a bundle isomorphism
  - it preserves the Cartan distribution

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- Taking into account that functions on the infinite jet space $J^\infty(\pi)$ are generated by functions on E and their total derivatives, one concludes that evolutionary vector fields are uniquely fixed by its action on $\mathcal{F}(E)$.

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In local coordinates:

$$v(u^a_I) = D^I_x v(u^a)$$

where $D^I_x = D^{i_1}_{x^1} \dots D^{i_n}_{x^n}$ for $I = (i_1, \dots, i_n)$

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- In particular, gauge symmetries of local field theories are evolutionary vector fields
- Given that differential graded (Koszul) resolution of a PDE, which is embedded into a jet space, must commute with jet prolongations, it is determined by a degree 1 evolutionary super vector field
- Finally, a $Q-$manifold in the context of PDEs is a PDE together with a degree 1 evolutionary self-commuting vector field

K-JET PROLONGATION OF A VECTOR FIELD

- Given a vector field v on the total space of $\pi \colon E \to X$, there exists a canonical $k-$jet prolongation $v^{(k)}$ on $J^k(\pi)$ for all k, which preserves the Cartan distribution on $J^k(\pi)$ (the span of tangent spaces to all $k-$jets of local sections of π)

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- This prolongation is compatible with the projections $J^k(\pi) \to J^l(\pi)$ for all $k > l$. In other words, there is a canonical infinite jet prolongation of v to a vector field on $J^\infty(\pi)$, preserving the Cartan structure, such that $v^{(k)}$ coincides with its restriction to $\mathcal{F}(J^k(\pi)) \subset \mathcal{F}(J^\infty(\pi))$.
- The latter statement is equivalent to the existence of a canonical prolongation to $\mathcal{C}[1]$, commuting with d_h.

K-JET PROLONGATION OF A VECTOR FIELD

In local coordinates (x^a, u^a), let

$$v = \sum_{\alpha} f^{\alpha}(x, u)\,\partial_{x^{\alpha}} + \sum_{a} g^{a}(x, u)\,\partial_{u^{a}}$$

Then $v^{(\infty)}$ is the sum of horizontal and vertical evolutionary parts

$$v^{(\infty)} = v_h^{(\infty)} + v_e^{(\infty)}$$

where

$$v_h^{(\infty)} = \sum_{\alpha} f^{\alpha}(x, u)\,D_{x^{\alpha}}$$

and

$$v_e^{(\infty)}(u^a) = g^a(x, u) - f^{\alpha}(x, u)\,u^a_{\alpha}, \quad u^a_{\alpha} = D_{x^{\alpha}}\,u^a$$

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Therefore we obtain two super commuting differentials on E_X, d_h and s. Their sum $\mathrm{d}_h + s$ is again a differential.

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The differentials d_h and s have the bi-degrees $(1, 0)$ and $(0, 1)$, respectively. The total differential $\mathrm{d}_h + s$ has the total degree 1.

Gauge PDEs

Gauge pre-PDE is a $\mathbb{Z}$-graded Q-bundle $(E_{T[1]X}, Q)$ over $(T[1]X, d_X)$, where $(T[1]X, d_X)$ is considered as a graded Q-manifold with the canonical degree (form degree) and the canonical Q-structure (de Rham differential).

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Gauge pre-PDE $(E_{T[1]X}, Q)$ is called contractible if as a bundle over $T[1]X$ it is locally trivial, admits a global Q-section, and its fiber is a contractible Q-manifold.

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Gauge pre-PDE $(E_{T[1]X}, Q)$ is an equivalent reduction of $(E'_{T[1]X}, Q')$ if $(E'_{T[1]X}, Q')$ is a locally-trivial Q-bundle over $(E_{T[1]X}, Q)$ (in the category of Q-bundles over $T[1]X$) whose fiber is contractible and which admits a global Q-section $i : E_{T[1]X} \to E'_{T[1]X}$.

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A pre-PDE, where $E_{T[1]X}$ is $\mathcal{C}[1]$ for a (super) jet space and $Q = d_h + s$ for an evolutionary degree 1 vector field s, is called a standard gauge pre-PDE. Equivalence of standard gauge pre-PDEs are those which respect the natural bi-grading.

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Gauge pre-PDE $(E_{T[1[X]}, Q)$ is a gauge PDE if:

1. it is equivalent to a nonnegatively graded gauge pre-PDE
2. it is equivalent to a standard gauge pre-PDE

MAXWELL EQUATIONS AS A GAUGE THEORY: "CLASSICAL" PART

- Independent variables: x^i, $i = 1, \ldots, n$

Maxwell equations as a gauge theory: "classical" part

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- Independent variables: x^i, $i = 1, \ldots, n$
- Dependent variables: A_i, $i = 1, \ldots, n$, $\mathrm{gh} A_i = 0$
- The action functional:

$$S[A] = \int L[A]$$

where

$$L[A] = -\frac{1}{2} \sum_{i,j} F_{ij} F_{ij}, \quad F_{ij} = D_i A_j - D_j A_i$$

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and $D_i = D_{x^i}$. We will denote $A_{i,j} = D_j A_i$.

Maxwell equations as a gauge theory: "classical" part

- The equations of motion (Euler-Lagrange equations):

$$\frac{\delta L}{\delta A_i} = \frac{\partial L}{\partial A_i} - \sum_j D_j \left(\frac{\partial L}{\partial A_{i,j}} \right) = 0$$

will give us

$$\frac{\delta L}{\delta A_i} = -\sum_j D_j F_{ij} = 0$$

for all $i = 1, \ldots, n$.

Maxwell equations as a gauge theory: the Koszul-Tate resolution

- New dependent variables: A_*^i, $i = 1, \dots, n$, $\mathrm{gh} A_*^i = -1$, C^*, $\mathrm{gh} C^* = -2$

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- The Koszul-Tate differential δ:

$$
\begin{aligned}
\delta A_i &= 0 \\
\delta A_*^i &= \sum_j D_j F_{ij} \\
\delta C^* &= \sum_i D_i A_*^i
\end{aligned}
$$

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- δ is an evolutionary vector field, that is

$$\delta\left(D_x^I A_*^i\right) = D_x^I\left(\sum_j D_j F_{ij}\right), \quad \delta\left(D_x^I C^*\right) = D_x^I\left(\sum_i D_i A_*^i\right)$$

Maxwell equations as a gauge theory: the Koszul resolution

- δ is acyclic except the degree 0; the 0-degree cohomology gives functions on the equation manifold:

$$H_\delta^k = \begin{cases} 0, & k < 0 \\ \mathcal{F}\left(\Sigma_{Maxwell}\right), & k = 0 \end{cases}$$

Here

$$\Sigma_{Maxwell} = \left\{ D_I \left(\frac{\delta L}{\delta A_i} \right) = 0, \, \forall I, i \right\}$$

Maxwell equations as a gauge theory: the gauge symmetries generator

- New dependent variable: C, $\mathrm{gh}\, C = 1$

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- The differential γ:

$$\begin{aligned} \gamma A_i &= D_i C \\ \gamma C &= 0 \end{aligned}$$

generates gauge symmetries

$$\delta_\epsilon A_i = D_i \epsilon$$

Maxwell equations as a gauge theory: the BRST differential

- Independent variables: x^i, $i = 1, \dots, n$
- Dependent variables: A_i, $\mathrm{gh} A_i = 0$ and A_*^i, $i = 1, \dots, n$, $\mathrm{gh} A_*^i = -1$; C^*, $\mathrm{gh} C^* = -2$, and C, $\mathrm{gh} C = 1$
- The differential s:

$$\begin{aligned} sA_i &= D_i C = C_i, \quad sC = 0 \\ sA_*^i &= \sum_j D_j F_{ij} = \sum_j F_{ij,j} \\ sC^* &= \sum_i D_i A_*^i \end{aligned}$$

- $Q = \mathrm{d}_h + s$, such that $Qx^i = \theta^i$, $QA_i = \sum_k \theta^k A_{i,k} + C_i$, etc.

Maxwell equations as a gauge theory: the minimal model

Now we introduce another set of jet coordinates, which is decomposed into the following two subsets:

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Subset 1

$$\left\{ A^i_{*,j_1\ldots j_m}, QA^i_{*,j_1\ldots j_m}, C^*_{j_1\ldots j_m}, QC^*_{j_1\ldots j_m}, A_{(j_1,j_2\ldots j_m)}, QA_{(j_1,j_2\ldots j_m)} \right\}$$

where

$$A_{(j_1,j_2\ldots j_m)} = A_{j_1,j_2\ldots j_m} + A_{j_2,j_3\ldots j_m j_1} + \ldots + A_{j_m,j_1\ldots j_{m-1}}$$

Maxwell equations as a gauge theory: the minimal model

Subset 2

$\{x^i, \theta^i, C_{min}, P_{i,j_1\ldots j_m}\}$

where $C_{min} = C + \sum_k \theta^k A_k$ and $P_{i,j_1\ldots j_m} = F'_{i(j_1,j_2\ldots j_m)}$ for

$$F'_{ij_1,j_2\ldots j_m} = F_{ij_1,j_2\ldots j_m} - \frac{1}{n}\left(\sum_j F_{ij,j,j_3\ldots j_m)}\right)\delta_{j_1 j_2}$$

$F'_{ij_1,j_2\ldots j_m}$ is the traceless part of $F_{i(j_1,j_2\ldots j_m)}$.

Maxwell equations as a gauge theory: the minimal model

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$F'_{ij_1,j_2\ldots j_m}$ is the traceless part of $F_{i(j_1,j_2\ldots j_m)}$. One has

$$Qx^i = \theta^i, \quad QC_{min} = \frac{1}{2}\sum_{i,j}\theta^i\theta^j P_{i,j}$$

$$QP_{i,j_1,j_2\ldots,j_m} = \sum_k \theta^k\left(\frac{m+1}{m+2}P_{i,kj_1\ldots j_m} + \frac{1}{m+2}P_{k,ij_1,\ldots j_m}\right)$$

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- Independent variables: $\{x^i, \theta^i\}$, $i = 1, \ldots, n$
- Dependent variables: $P_{i,j_1 j_2,\ldots,j_m}$, $\mathrm{gh} P_{\ldots} = 0$; C_{min}, $\mathrm{gh} C_{min} = 1$, where $P_{i,j_1 j_2,\ldots,j_m}$ is symmetric w.r.t. $j_1, j_2, \ldots, j_m$ and satisfies

$$P_{(i,j_1 \ldots j_m)} = P_{i,j_1 j_2 \ldots j_m} + P_{j_1,j_2,\ldots,j_m i} + \ldots + P_{j_m,i j_1 j_2 \ldots j_{m-1}} = 0$$

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- The first one consists of standard gauge PDEs together with morphisms and equivalences preserving $\mathbb{Z} \times \mathbb{Z}$ bi-grading, while the second one consists of gauge PDEs, which are only $\mathbb{Z}-$graded.
- There is a forgetful functor from the first category to the second one which replaces the bi-grading with the total grading.
- There is also a canonical functor in the opposite direction which associates to a gauge PDE $(E_{T[1[X]}, Q)$ the differential equation whose solutions are $Q-$sections of $E_{T[1[X]} \to T[1]X$.

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More precisely, let $E_X \to X$ be the jet space of local sections for a (super) bundle over X together with a degree 1 homological evolutionary vector field s, whose negative cohomology are vanishing.

This describes a PDE with gauge symmetries in the usual sense: normally s is obtained by the homological perturbation of a couple of homological degree 1 evolutionary vector fields, the first of which is the Koszul resolution for a PDE embedded into the jet space (in other words the Koszul resolution determines such an embedding).

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Now we take the corresponding Cartan distribution $\mathcal{C}$ on the (super) jet space, the $Q-$bundle $\mathcal{C}[1] \to T[1]X$ with the total $Q = \mathrm{d}_h + s$ and "forget" about the bi-grading of the total space. We obtain a gauge PDE, which "remembers" only the total $\mathbb{Z}-$grading.

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From now we are allowed to work with this new object as if it was a gauge PDE from the very beginning, replacing it with an equivalent gauge PDE (which inherits the same important properties, eg. all associated natural cohomologies are the same).

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However, the parent formalism gives us the way how to restore the original standard gauge PDE up an equivalence of standard gauge PDEs. As mentioned before, we take differential equation for $Q-$sections of the gauge PDE regarded as a $Q-$bundle over $T[1]X$. This gives us a standard gauge PDE which is equivalent to the original one.

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If the gauge PDE is reparametrization invariant (by the definition, the underlined $Q-$bundle is locally trivial), we obtain (at least locally) an AKSZ-type differential equation.

Parent formalism: an explicit global construction

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$Q^{(\infty)}$ splits into the horizontal and vertical (evolutionary) parts,

$$Q^{(\infty)} = Q_h^{(\infty)} + Q_e^{(\infty)}$$

as well as the Euler vector field $\epsilon^{(\infty)}$ (which determines the corresponding $\mathbb{Z}-$grading on the total space of $E_{T[1]X}$)

$$\epsilon^{(\infty)} = \epsilon_h^{(\infty)} + \epsilon_e^{(\infty)}$$

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The only non-trivial commutation relations between the four obtained components are

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One can show that $Q_h^{(\infty)}$ and $Q_e^{(\infty)}$ can be canonically identified with d_h and s for the standard gauge PDE for $Q-$sections of π, respectively.

Presymplectic gauge PDEs

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AKSZ model as a presymplectic gauge PDE

In this case $E = E_{T[1]}X = M \times T[1]X$, where (M, Q_M, ω) is a symplectic Q-manifold. The Q-structure on the total space is $Q = \mathrm{d}_X + Q_M$, while the vertical presymplectic form is given by ω.

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The transgression formula for AKSZ

One can reformulate the transgression formula (A.K., T. Strobl) from the previous lecture as follows: for a symplectic $Q-$manifold (M, Q_M, ω) with the symplectic form ω of degree $p > 0$ one has

- $\mathrm{d}\omega = 0$
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This implies that

$$\omega = (\mathrm{d} + L_{Q_M})(\chi + I)$$

where $\chi = \frac{1}{p}\iota_\epsilon \omega$ and $I = \frac{1}{p+1}\iota_{Q_M}\chi$.

THE TRANSGRESSION FORMULA FOR AKSZ

In particular, one has

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Let X an n-dimensional manifold ($n > 1$), (M, Q_M, ω) be a symplectic Q-manifold of degree $p = n - 1$, and ϕ be a (degree-preserving) map from $T[1]X$ to M. Then the (classical part of the) AKSZ sigma model action for the source space $T[1]X$ and the target M is

$$S_{AKSZ}[\phi] = \int_{T[1]X} \tilde{\phi}^*(\chi + I)$$

Now let us remark that, whenever $(E_{T[1]X}, Q, \omega)$ is a presymplectic gauge (pre-)PDE, such that

$$\omega = (\mathrm{d}_v + L_Q)(\chi + l)$$

where χ and l are vertical $1-$form and $0-$form, respectively, and σ is a (degree preserving) section of $E_{T[1]}X$, one can construct an action in a similar way to the AKSZ case:

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Here we use the generalized Cartan map induced by σ by taking into account that the corresponding field strength is a vertical vector field, which allows us to apply the generalized Cartan map to vertical forms on the total space.

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- In general it is not true (all interesting non-topological examples, such as Maxwell, Yang-Mills and Einstein gravity models, correspond to degenerate horizontal 2-forms)
- Then one should be able to quotient out the kernel of ω, viewed in a reasonable way

Standard gauge presymplectic PDEs

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The transgression formula $\omega = (\mathrm{d}_v + L_Q)(\chi + l)$ is true if and only if the following properties are holding:

1. $\omega = \mathrm{d}_v \chi$;
2. $\iota_s \omega = \mathrm{d}_v h - \mathrm{d}_h \chi$

where $h = \iota_s \chi - l$.

Indeed, the transgression formula is equivalent to

1. $\omega = \mathrm{d}_v \chi$;
2. $\mathrm{d}_h \chi + L_s \chi + \mathrm{d}_v l = 0$
3. $(L_s + \mathrm{d}_h) l = 0$

While the first properties are the same, the equation

$$\mathrm{d}_h \chi + L_s \chi + \mathrm{d}_v l = 0$$

can be represented as

$$\mathrm{d}_h \chi + \iota_s \mathrm{d}_v \chi - \mathrm{d}_v \iota_s \chi + \mathrm{d}_v l = \mathrm{d}_h \chi + \iota_s \omega - \mathrm{d}_v h = 0$$

therefore the first two properties are equivalent.

Now the last condition $(L_s + \mathrm{d}_h)l = 0$ is fulfilled automatically thanks to the degree reason: one can verify that

$$\mathrm{d}_v(L_s + \mathrm{d}_h)l = -(L_s + \mathrm{d}_h)\mathrm{d}_v(l) = (L_s + \mathrm{d}_h)^2\chi = 0$$

therefore $(L_s + \mathrm{d}_h)(l)$ is coming from the base $T[1]X$, so it is actually a differential form on X. But the degree of this expression is $n + 1$, thus it must be zero.

Example: descend process

(A. Sharapov) One has

- $\omega_n = \mathrm{d}_v \chi_n$ and
- $\iota_s \omega_n = \mathrm{d}_v h_n - \mathrm{d}_h \chi_{n-1}$,

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Now we notice that $L_s \omega_n = -\mathrm{d}_h \omega_{n-1}$, where $\omega_{n-1} = \mathrm{d}_v \chi_{n-1}$ is a vertical 2-form of the horizontal degree $n-1$.

Moreover,

$$\mathrm{d}_h \mathrm{d}_v (\iota_s \omega_{n-1}) = -\mathrm{d}_h L_s \omega_{n-1} = L_s \mathrm{d}_h \omega_{n-1} = -L_s^2 \omega_n = 0$$

Example: descend process

Therefore, under certain topological conditions, $\mathrm{d}_v(\iota_s\omega_{n-1})$ vanishes up to the divergence, which implies that there exist h_{n-1} and χ_{n-2}, such that

$$\iota_s\omega_{n-1} = \mathrm{d}_v h_{n-1} - \mathrm{d}_h\chi_{n-2}$$

and so on: we iterate this process (by the degree reason it will eventually stop) and get $\omega = \omega_n + \omega_{n-1} + \ldots$, $\chi = \chi_n + \chi_{n-1} + \ldots$, and $h = h_n + h_{n-1} + \ldots$, and finally we obtain that

1. $\omega = \mathrm{d}_v\chi$;
2. $\iota_s\omega = \mathrm{d}_v h - \mathrm{d}_h\chi$

is holding and hence the transgression formula $\omega = (\mathrm{d}_v + L_Q)(\chi + I)$ for $I = \iota_s\chi - h$ is true.

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Let us assume that the transgression formula holds:

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The latter implies that

$$\tilde{\sigma}^*(\chi + l) = \sigma^*(-\iota_s \chi + l) = -\sigma^* h$$

So the AKSZ type action is the "original" classical Lagrangian

$$\int_{T[1]X} \sigma^*(-h)$$

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- We start with a standard symplectic gauge PDE, corresponding to a BV-extension of a classical theory;
- By mixing the degrees, we find an equivalent presymplectic gauge PDE (maybe a minimal model);
- Using the AKSZ-type action function determined by the transgression of the compatible presymplectic 2-form we can come back to the standard case