Peierls Phenomenon and Integrability

P. Wiegmann

University of Chicago

Based on recent paper by Valdemar Melin, Yuta Sekiguchi, P. W., and Konstantin Zarembo

July 25, 2024

OUTLINE

- Peierls Phenomenon

Outline

- Peierls Phenomenon
- Peierls Phenomenon and periodic solutions of solitons equations (say, KdV)

OUTLINE

- Peierls Phenomenon
- Peierls Phenomenon and periodic solutions of solitons equations (say, KdV)
- Peierls Phenomenon and Gross-Neveu model with large rank Lie group

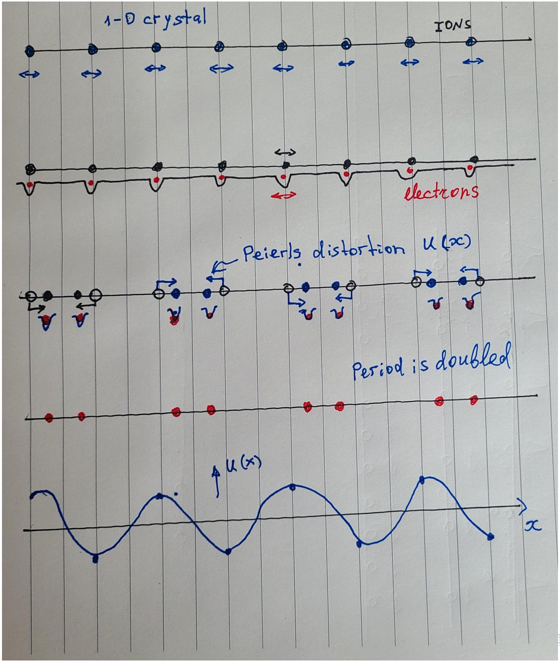
Outline

- Peierls Phenomenon
- Peierls Phenomenon and periodic solutions of solitons equations (say, KdV)
- Peierls Phenomenon and Gross-Neveu model with large rank Lie group
- Bethe Ansatz for integrable theories with Lie group symmetries

OUTLINE

- Peierls Phenomenon
- Peierls Phenomenon and periodic solutions of solitons equations (say, KdV)
- Peierls Phenomenon and Gross-Neveu model with large rank Lie group
- Bethe Ansatz for integrable theories with Lie group symmetries
- Singular limit of the large rank

Peierls Phenomenon



Peierls Phenomenon

Peierls' theorem (first espoused in 1930, written in 1954):

One-dimensional equally spaced chain with one electron per ion is unstable!

Peierls Phenomenon

Peierls' theorem (first espoused in 1930, written in 1954):

One-dimensional equally spaced chain with one electron per ion is unstable!

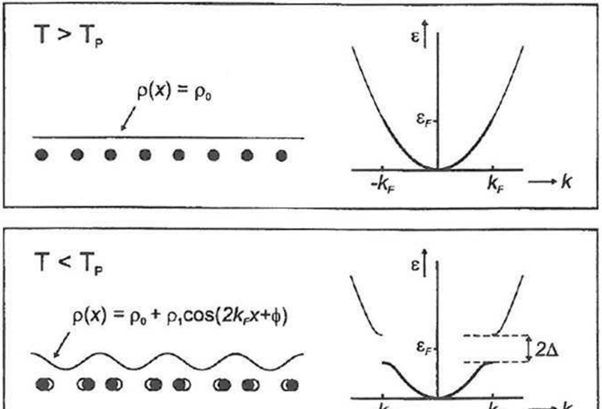
Electronic crystals

PEIERLS PHENOMENON

Peierls' theorem (first espoused in 1930, written in 1954):

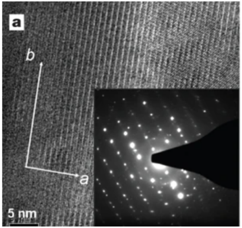
One-dimensional equally spaced chain with one electron per ion is unstable!

Electronic crystals

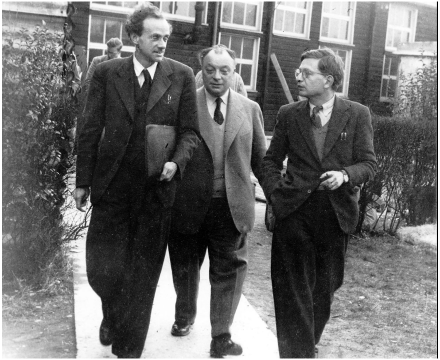


Electronic crystal

W. Little 1964



Rudolf Peierls: 1907-1995



Peierls problem: Discrete version

1) Schrödinger equation: $c_n \psi_{n+1} + c_{n-1}\psi_{n-1} = \varepsilon \psi_n$

2) Find the spectrum as a functional of $C = \{c_1, \dots\}$: $\varepsilon[C]$

3) Compute the energy: sum over all eigenvalues below μ: $E[C] = \sum_{\varepsilon<\mu} \varepsilon[C] + \sum_n c_n^2$

Peierls Problem: Find C which gives the minimum to the energy $\min_C E[C]$

Peierls problem: Discrete version

1) Schrödinger equation: $$c_n\psi_{n+1} + c_{n-1}\psi_{n-1} = \varepsilon\psi_n$$

2) Find the spectrum as a functional of $C = \{c_1, \dots\}$: $$\varepsilon[C]$$

3) Compute the energy: sum over all eigenvalues below μ: $$E[C] = \sum_{\varepsilon<\mu} \varepsilon[C] + \sum_n c_n^2$$

Peierls Problem: Find C which gives the minimum to the energy $\min_C E[C]$

Krichever Solution: The extrema are given by the *finite-gap solutions* of the Toda chain.

The minimum is given by the *one-gap* solution.

Peierls problem and the Lax operator

1) The Schrödinger equation with a variable hopping:

$L\psi = c_n \psi_{n+1} + c_{n-1}\psi_{n-1} = \varepsilon \psi_n$ was identified with the Lax operator

2) Extrema of energy were identified with finite-gap periodic solutions

Peierls problem: Continuos version

(1+1)-Dirac Hamiltonian: $H = \psi^\dagger \tau_1 \big(i\partial_x + \tau_3 \sigma \big) \psi + \frac{1}{2\lambda} \sigma^2$

Dirac equation: $\begin{cases} -i\big(\partial_x - \sigma(x)\big)\psi_+ = \varepsilon \psi_- \\ -i\big(\partial_x + \sigma(x)\big)\psi_- = \varepsilon \psi_+ \end{cases}$

Peierls problem: Continuos version

(1+1)-Dirac Hamiltonian: $H = \psi^\dagger \tau_1 \big(i\partial_x + \tau_3 \sigma\big)\psi + \frac{1}{2\lambda}\sigma^2$

Dirac equation: $\begin{cases} -i\big(\partial_x - \sigma(x)\big)\psi_+ = \varepsilon\psi_- \\ -i\big(\partial_x + \sigma(x)\big)\psi_- = \varepsilon\psi_+ \end{cases}$

1. Compute the energy as a functional of σ: $E[\sigma] = \underset{H<\mu}{\mathrm{Tr}}\, H = \sum_{\varepsilon<\mu} \epsilon + \frac{1}{2\lambda}\sigma^2$
2. Minimize with respect to σ: $\min_\sigma E[\sigma]$
3. Compute the spectrum of H in the most favorable σ

Peierls problem: Continuos version

(1+1)-Dirac Hamiltonian: $H = \psi^\dagger \tau_1 \big(i\partial_x + \tau_3 \sigma\big)\psi + \frac{1}{2\lambda}\sigma^2$

Dirac equation: $\begin{cases} -i\big(\partial_x - \sigma(x)\big)\psi_+ = \varepsilon\psi_- \\ -i\big(\partial_x + \sigma(x)\big)\psi_- = \varepsilon\psi_+ \end{cases}$

1. Compute the energy as a functional of σ: $E[\sigma] = \operatorname*{Tr}_{H<\mu} H = \sum_{\varepsilon<\mu} \epsilon + \frac{1}{2\lambda}\sigma^2$
2. Minimize with respect to σ: $\min_\sigma E[\sigma]$
3. Compute the spectrum of H in the most favorable σ

The minimum of energy is achieved if σ is a periodic solution of mKdV

$$\sigma_t - 6\sigma^2\sigma_x + \sigma_{xxx} = 0, \quad \sigma = \text{function}\,(x - ct).$$

CNOIDAL WAVE

$$\begin{aligned} \text{mKdV:} \quad & \sigma_t - 6\sigma^2\sigma_x + \sigma_{xxx} = 0 \\ \text{Miura:} \quad & q = \sigma^2 + \sigma_x \\ \text{KdV:} \quad & q_t - 6qq_x + q_{xxx} = 0\,. \end{aligned}$$

Cnoidal wave

mKdV: $\sigma_t - 6\sigma^2\sigma_x + \sigma_{xxx} = 0$

Miura: $q = \sigma^2 + \sigma_x$

KdV: $q_t - 6qq_x + q_{xxx} = 0\,.$

Cnoidal wave: $\sigma(x) = \sigma_0 k^{1/2}\,\mathrm{sn}(x|k)$

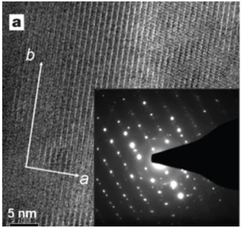
Particles=Period: $N/N_0 = 2k^{1/2}K(k)$

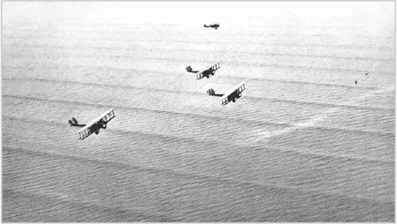
Gap: $\sigma_0 = \Lambda e^{-\pi/\lambda}$

Cnoidal wave

mKdV:	$\sigma_t - 6\sigma^2\sigma_x + \sigma_{xxx} = 0$
Miura:	$q = \sigma^2 + \sigma_x$
KdV:	$q_t - 6qq_x + q_{xxx} = 0\,.$

Cnoidal wave:	$\sigma(x) = \sigma_0 k^{1/2}\,\mathrm{sn}(x\|k)$
# Particles=Period:	$N/N_0 = 2k^{1/2}K(k)$
Gap:	$\sigma_0 = \Lambda e^{-\pi/\lambda}$

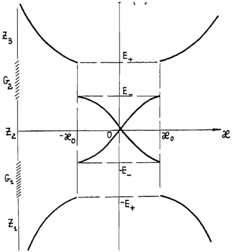




US Army bombers flying over near-periodic swell in shallow water, close to the Panama

Spectral Curve

$$-i\big(\partial_x \mp \sigma(x)\big)\psi_\pm = \varepsilon(p)\psi_\mp$$
$$dN(\varepsilon)/N_0 = dp$$



$$dp = \frac{|\varepsilon^2 - S|}{\sqrt{R}} d\varepsilon$$
$$R(\varepsilon) = (\varepsilon^2 - E_+^2)(\varepsilon^2 - E_-^2)$$
$$2S = -\overline{\sigma^2} + E_+^2 + E_-^2$$

Edges of the spectrum : $E_\pm = \frac{\sigma_0}{2}(k^{-1/2} \pm k^{1/2})$

Quantization of the spectral curve

Valdemar Melin, Yuta Sekiguchi, P. W., and Konstantin Zarembo

How to obtain periodic solutions of classical integrable equations from quantum integrable models?

Valdemar Melin, Yuta Sekiguchi, P. W., and Konstantin Zarembo

How to obtain periodic solutions of classical integrable equations from quantum integrable models?

Quantum version of Peierls problem and the spectral curves

Quantum version: Gross-Neveu model

$$H = \psi^\dagger \tau_2 \big(i\partial_x + \tau_3 \sigma\big)\psi + \tfrac{1}{2\lambda}\sigma^2$$

Adiabatic approximation: σ is determined by the extremum of $\mathrm{Tr}\, H$

Quantum version: Gross-Neveu model

$$H = \psi^\dagger \tau_2 \left(i\partial_x + \tau_3 \sigma \right) \psi + \tfrac{1}{2\lambda} \sigma^2$$

Adiabatic approximation: σ is determined by the extremum of $\mathrm{Tr}\, H$

Quantum version: σ itself is a quantum field.
In this case we have Gross-Neveu model

$$H = \psi^\dagger \tau_2 i\partial_x \psi + \tfrac{\lambda}{2} (\psi^\dagger \psi)^2$$

Quantum version: Gross-Neveu model

$$H = \psi^\dagger \tau_2 \big(i\partial_x + \tau_3 \sigma\big)\psi + \tfrac{1}{2\lambda}\sigma^2$$

Adiabatic approximation: σ is determined by the extremum of $\operatorname{Tr} H$

Quantum version: σ itself is a quantum field.
In this case we have Gross-Neveu model

$$H = \psi^\dagger \tau_2 i\partial_x \psi + \tfrac{\lambda}{2}(\psi^\dagger\psi)^2$$

Large N as a semiclassical parameter: $\psi \to (\psi_1, \dots \psi_N)$

$$H = \sum_{1\le k\le N} \psi_k^\dagger \tau_2 i\partial_x \psi_k + \tfrac{\lambda}{2}\Big(\sum_{1\le k\le N} \psi_k^\dagger \psi_k\Big)^2$$

We recover the Peierls model in the limit of a large N

Lie group

$$H = \sum_{1\le k\le N} \psi_k^\dagger \tau_2 i\partial_x \psi_k + \frac{\lambda}{2}\Big(\sum_{1\le k\le N} \psi_k^\dagger \psi_k\Big)^2$$

Integrable model controlled by its global symmetry $O(2N)$

Mass spectrum

Particle content: All fundamental representations.

Mass spectrum D_N:

$$\text{n-th tensor}: \quad m_n = m\,\frac{\sin\frac{\pi n}{2N-2}}{\sin\frac{\pi}{2N-2}}$$

$$\text{spinors:} \quad m_s = m_{\bar{s}} = \frac{m}{2\sin\frac{\pi}{2N-2}},$$

Scattering matrices, the mass spectrum, the Bethe Ansatz are known for all simple Lie groups: E. Ogievetski, N. Reshetikhin, P. W.

The scattering matrix is factorized into a product of two-particle scattering

$$S_{ab}(\theta), \quad \theta = \theta_a - \theta_b, \quad p_a(\theta \to \infty) \sim m_a \sinh\theta$$

Quantum integrable systems: Scattering matrices and TBA

The scattering matrix is factorized into a product of two-particle scattering

$$S_{ab}(\theta), \quad \theta = \theta_a - \theta_b, \quad p_a(\theta \to \infty) \sim m_a \sinh\theta$$

Thermodynamic Bethe-Ansatz equations for the "spectral curve" $K_{ab} = \frac{1}{2\pi i}\frac{d}{d\theta}\log S_{ab}$

$$\int K_{ab}(\theta_a - \theta_b)dp_b = m_a \sinh\theta_a, \qquad \int K_{ab}(\theta_a - \theta_b)\varepsilon_b = \mu_a - m_a \cosh\theta_a$$

Sum over particle content (along the Dynkin diagram 

Quantum integrable systems: Scattering matrices and TBA

The scattering matrix is factorized into a product of two-particle scattering

$$S_{ab}(\theta), \quad \theta = \theta_a - \theta_b, \quad p_a(\theta \to \infty) \sim m_a \sinh\theta$$

Thermodynamic Bethe-Ansatz equations for the "spectral curve" $K_{ab} = \frac{1}{2\pi i}\frac{d}{d\theta}\log S_{ab}$

$$\int K_{ab}(\theta_a - \theta_b)dp_b = m_a \sinh\theta_a, \qquad \int K_{ab}(\theta_a - \theta_b)\varepsilon_b = \mu_a - m_a \cosh\theta_a$$

Sum over particle content (along the Dynkin diagram 

Spectral curve $E =$ multivalued function (P) — spectral curve

$$P = \sum_a \int \sinh\theta\, dp_a, \qquad E - \mu N = \sum_a \int \cosh(\theta)\varepsilon_b d\theta$$

Scattering matrix D_N (in momentum space)

$$\hat{K}_{ab} = \begin{cases} \delta_{ab} + \frac{1}{2} e^{\frac{|k|}{2N-2}} \dfrac{\sinh \frac{(|a-b|-N+1)k}{2N-2} - \sinh \frac{(a+b-N+1)k}{2N-2}}{\sinh \frac{k}{2N-2} \cosh \frac{k}{2}}, & a, b \leq N-2 \\ \delta_{ab} + \frac{1}{4} e^{\frac{|k|}{2N-2}} \dfrac{\sinh \frac{(|N-1-b|-N+1)k}{2N-2} - \sinh \frac{b|k|}{2N-2}}{\sinh \frac{k}{2N-2} \cosh \frac{k}{2}}, & a > N-2, b \leq N-2 \\ \delta_{ab} + \frac{1}{4} e^{\frac{|k|}{2N-2}} \dfrac{\sinh \frac{(|N-1-a|-N+1)k}{2N-2} - \sinh \frac{a|k|}{2N-2}}{\sinh \frac{k}{2N-2} \cosh \frac{k}{2}}, & a \leq N-2, b > N-2 \\ \delta_{ab} - \frac{1}{4} e^{\frac{|k|}{2N-2}} \dfrac{\sinh \frac{k}{2}}{\sinh \frac{k}{2N-2} \cosh \frac{k}{2}} - \frac{1}{4} \dfrac{(-1)^{a+b} e^{\frac{|k|}{2N-2}}}{\cosh \frac{k}{2N-2}}, & a, b > N-2. \end{cases}$$

Karowski and Thun, 1981

Ground State

The vacuum is filled by spinors: 

Ground State

The vacuum is filled by spinors: 

The TBA are reduced to:

$$\int_{-B}^{B} (K_{ss}+K_{s\bar{s}})(\theta-\theta')\,dp_s = m_s \sinh\theta, \quad 2\int_{-B}^{B} K_{as}(\theta-\theta')\,dp_s = m_a \cosh\theta, \quad \frac{N}{N_0} = \int_{-B}^{B} dp_s$$

$$K_{ss}+K_{s\bar{s}} = \frac{\tanh\frac{|k|}{2}}{2\left(1-e^{-\frac{|k|}{N-1}}\right)}, \qquad K_{as} = -\frac{\mathrm{e}^{\frac{\pi|k|}{2N-2}}}{2\cosh\frac{\pi k}{2}}$$

SINGULAR LARGE N LIMIT

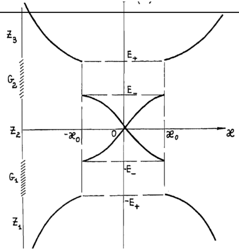
$$K_{ss}+K_{s\bar{s}}=\frac{\tanh\frac{k}{2}}{2\left(1-e^{-\frac{|k|}{N-1}}\right)}\underset{N\to\infty}{\longrightarrow}-\frac{N}{\pi^2}\log\coth\frac{\theta}{2}\qquad K_{as}=-\frac{\mathrm{e}^{\frac{\pi|k|}{2N-2}}}{2\cosh\frac{\pi k}{2}}\underset{N\to\infty}{\longrightarrow}-\frac{1}{2\pi\cosh\theta}$$

Singular large N limit

$$K_{ss}+K_{s\bar{s}}=\frac{\tanh\frac{k}{2}}{2\left(1-e^{-\frac{|k|}{N-1}}\right)}\underset{N\to\infty}{\longrightarrow}-\frac{N}{\pi^2}\log\coth\frac{\theta}{2}\qquad K_{as}=-\frac{\mathrm{e}^{\frac{\pi|k|}{2N-2}}}{2\cosh\frac{\pi k}{2}}\underset{N\to\infty}{\longrightarrow}-\frac{1}{2\pi\cosh\theta}$$

Integral equations degenerate to the Riemann-Hilbert problem

$$\int_{[-B,B]}\ln\coth\frac{\theta-\theta'}{2}\frac{dp_s}{\pi}=m\cosh\theta,\qquad \int_{[-B,B]}\ln\coth\frac{\theta-\theta'}{2}\varepsilon(\theta')\frac{d\theta'}{\pi}=m\cosh\theta-\frac{\mu}{2}$$



KdV spectral curve

$$dp=\frac{\varepsilon^2+\overline{\sigma^2}-E_+^2-E_-^2}{\sqrt{(\varepsilon^2-E_+^2)(\varepsilon^2-E_-^2)}}d\varepsilon$$

$$\frac{N}{N_0}=2k^{1/2}K(k),\quad E_\pm=\frac{\sigma_0}{2}(k^{-1/2}\pm k^{1/2})$$

Comments

Relation between Lie algebras and integrable equations

$A_N \Rightarrow NLS$

$D_N \Rightarrow KdV$

B_N, C_N ?

COMMENTS

Relation between Lie algebras and integrable equations

$A_N \Rightarrow NLS$

$D_N \Rightarrow KdV$

B_N, C_N ?

Quantum version of Krichever-Novikov algebro-geometric construct of periodic solution of soliton equation