Positive m-divisible non-crossing partitions and their cylic sieving

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Prologue

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24th International Conference on Formal Power Series and Algebraic Combinatorics, Nagoya, Hotel Lobby, 2012

Christian to Christian: *"Willst Du ein weiteres ‚Zyklisches-Sieben-Phänomen' beweisen?"*[1]

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Christian to Christian: *"Willst Du ein weiteres ‚Zyklisches-Sieben-Phänomen' beweisen?"*[1]

Christian to Christian: *"Sicher. Warum nicht?"*[2]

---

[1] *"Do you want to prove another cyclic sieving phenomenon?"*

[2] *"Sure. Why not?"*

m-divisible non-crossing partitions associated with reflection groups

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Let W be a finite real reflection group.

The *absolute length* (*reflection length*) $\ell_T(w)$ of an element $w \in W$ is defined by the smallest k such that

$$w = t_1 t_2 \cdots t_k,$$

where all t_i are reflections.

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where all t_i are reflections.

The *absolute order* (*reflection order*) $\leq_T$ is defined by

$$u \leq_T w \quad \text{if and only if } \ell_T(u) + \ell_T(u^{-1}w) = \ell_T(w).$$

m-divisible non-crossing partitions associated with reflection groups

Definition (Armstrong)

The m-divisible non-crossing partitions for a reflection group W are defined by

$$NC^{(m)}(W) = \big\{(w_0; w_1, \dots, w_m) : w_0 w_1 \cdots w_m = c \text{ and } \\ \ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m) = \ell_T(c)\big\},$$

where c is a Coxeter element in W.

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where c is a Coxeter element in W.

In particular,

$$NC^{(1)}(W) \cong NC(W),$$

the "ordinary" non-crossing partitions for W.

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Combinatorial realisation in type A (Armstrong)

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Example for $m = 3,\ W = A_6 (= S_7)$:

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EXAMPLE FOR $m = 3$, $W = A_6 (= S_7)$:
$w_0 = (4,5,6)$, $w_1 = (3,6)$, $w_2 = (1,7)$, and $w_3 = (1,2,6)$. Indeed,

$$(4,5,6)\,(3,6)\,(1,7)\,(1,2,6) = (1,2,3,4,5,6,7) = c$$

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$$\ell_T\big((4,5,6)\big) + \ell_T\big((3,6)\big) + \ell_T\big((1,7)\big) + \ell_T\big((1,2,6)\big) \\ = 2+1+1+2 = 6 = \ell_T\big((1,2,3,4,5,6,7)\big).$$

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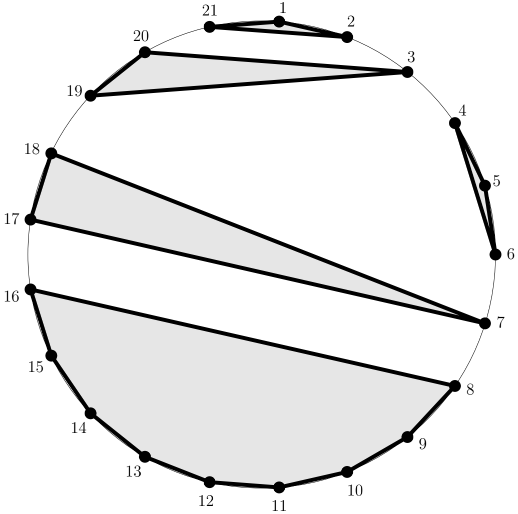

A 3-divisible non-crossing partition of type A_6

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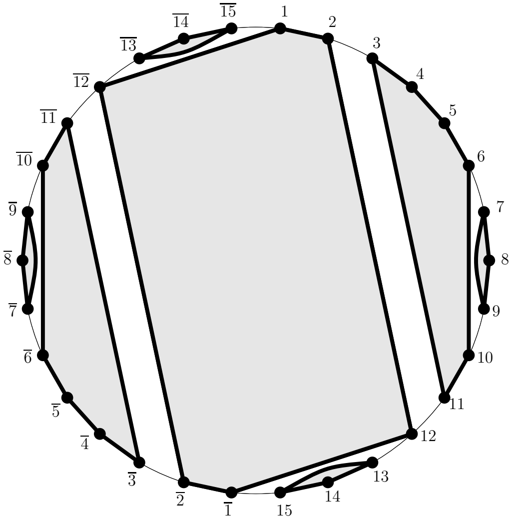

A 3-divisible non-crossing partition of type B_5

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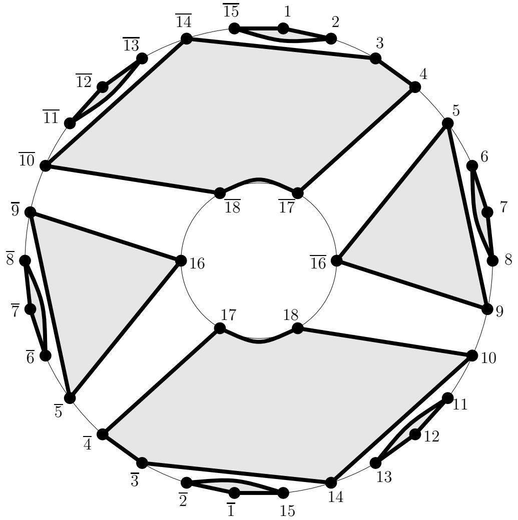

A 3-divisible non-crossing partition of type D_6

Positive m-divisible non-crossing partitions

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The positive m-clusters are those which do not contain any negative roots. They are enumerated by the *positive* Fuß–Catalan numbers

$$\mathrm{Cat}_{+}^{(m)}(W) := \prod_{i=1}^{n} \frac{mh + d_i - 2}{d_i}.$$

Here, $d_1 l d_2 \leq \cdots \leq d_n$ are the *degrees* of W, and $h = d_n$ is the *Coxeter number* of W.

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One can give an intrinsic definition:

Definition

An m-divisible non-crossing partition $(w_0; w_1, \dots, w_m)$ in $NC^{(m)}(W)$ is positive, if and only if $w_0 w_1 \cdots w_{m-1}$ is not contained in any proper standard parabolic subgroup of W.

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Let $NC_+^{(m)}(W)$ denote the set of all positive m-divisible non-crossing partitions for W.
Trivial corollary:

$$|NC_+^{(m)}(W)| = \mathrm{Cat}_+^{(m)}(W).$$

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Buan, Reiten and Thomas then write:

> *"Other than that, there do not seem to be enumerative results known for these families."*

Enumeration of positive m-divisible non-crossing partitions

For "ordinary" m-divisible non-crossing partitions, closed-form enumeration results are known for:

- total number;
- number of those of given rank;
- number of those with given block sizes (in types A, B, D);
- number of chains;
- number of chains with elements at given ranks;
- number of chains with elements at given ranks and bottom element with given block sizes (in types A, B, D).

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Fact: Under Armstrong's map, the elements of $NC_+^{(m)}(A_{n-1})$ correspond to those m-divisible non-crossing partitions of $\{1, 2, \ldots, mn\}$ in which mn and 1 are in the same block.

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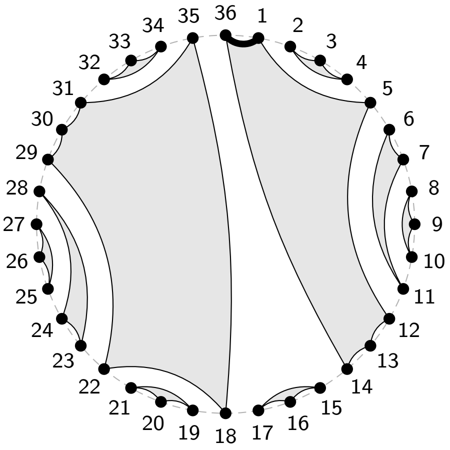

A positive 3-divisible non-crossing partition of type A_{12}

Enumeration in $NC_+^{(m)}(A_{n-1})$

Theorem

Let m, n be positive integers, The total number of positive m-divisible non-crossing partitions of $\{1, 2, \ldots, mn\}$ is given by

$$\frac{1}{n}\binom{(m+1)n-2}{n-1}.$$

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Theorem

Let m, n, l be positive integers, The number of multi-chains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{l-1}$ in the poset of positive m-divisible non-crossing partitions of $\{1, 2, \ldots, mn\}$ is given by

$$\frac{1+(l-1)(m-1)}{n-1}\binom{n-1+(l-1)(mn-1)}{n-2}.$$

Enumeration in $NC_+^{(m)}(A_{n-1})$

Theorem

Let m and n be positive integers, For non-negative integers $b_1, b_2, \dots, b_n$, the number of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ which have exactly b_i blocks of size mi, $i = 1, 2, \dots, n$, is given by

$$\frac{1}{mn-1}\binom{b_1+b_2+\cdots+b_n}{b_1, b_2, \dots, b_n}\binom{mn-1}{b_1+b_2+\cdots+b_n}$$

if $b_1 + 2b_2 + \cdots + nb_n = n$, and 0 otherwise.

Enumeration in $NC_+^{(m)}(A_{n-1})$

Theorem

Let m, n, l be positive integers, and let $s_1, s_2, \dots, s_l$ be non-negative integers with $s_1 + s_2 + \cdots + s_l = n - 1$. The number of multi-chains $\pi_1 \le \pi_2 \le \cdots \le \pi_{l-1}$ in the poset of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ with the property that $\mathrm{rk}(\pi_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \dots, l - 1$, is given by

$$\frac{mn - s_2 - s_3 - \cdots - s_l - 1}{(mn-1)n} \binom{n}{s_1} \binom{mn-1}{s_2} \cdots \binom{mn-1}{s_l}.$$

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$$\frac{mn - b_1 - b_2 - \cdots - b_n}{(mn-1)(b_1 + b_2 + \cdots + b_n)} \binom{b_1 + b_2 + \cdots + b_n}{b_1, b_2, \ldots, b_n} \times \binom{(l-1)(mn-1)}{b_1 + b_2 + \cdots + b_n - 1}$$

if $b_1 + 2b_2 + \cdots + nb_n = n$, and 0 otherwise.

Enumeration in $NC_+^{(m)}(A_{n-1})$

Theorem

Let m, n, l be positive integers, and let $s_1, s_2, \dots, s_l, b_1, b_2, \dots, b_n$ be non-negative integers with $s_1 + s_2 + \cdots + s_l = n - 1$. The number of multi-chains $\pi_1 \le \pi_2 \le \cdots \le \pi_{l-1}$ in the poset of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ with the property that $\operatorname{rk}(\pi_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \dots, l-1$, and that the number of blocks of size mi of π_1 is b_i, $i = 1, 2, \dots, n$, is given by

$$\frac{mn - b_1 - b_2 - \cdots - b_n}{(mn-1)(b_1 + b_2 + \cdots + b_n)} \binom{b_1 + b_2 + \cdots + b_n}{b_1, b_2, \dots, b_n} \times \binom{mn-1}{s_2} \cdots \binom{mn-1}{s_l}$$

if $b_1 + 2b_2 + \cdots + nb_n = n$ and $s_1 + b_1 + b_2 + \cdots + b_n = n$, and 0 otherwise.

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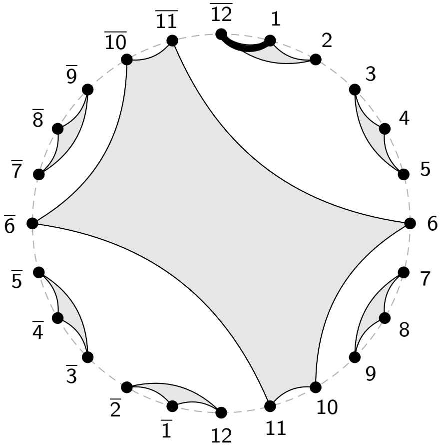
Fact: Under Armstrong's map, the elements of $NC_+^{(m)}(B_n)$ correspond to those m-divisible non-crossing partitions of $\{1, 2, \dots, mn, -1, -2, \dots, -mn\}$ which are invariant under rotation by 180°, and in which the block of 1 contains a negative element.

How do elements of $NC_+^{(m)}(B_n)$ look like?



A positive 3-divisible non-crossing partition of type B_4

Enumeration in $NC_{+}^{(m)}(B_n)$

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Theorem

Let m, n be positive integers, The total number of positive m-divisible non-crossing partitions of $\{1, 2, \ldots, mn, -1, -2, \ldots, -mn\}$ of type B is given by

$$\binom{(m+1)n-1}{n}.$$

Theorem

Let m, n be positive integers, The total number of positive m-divisible non-crossing partitions of $\{1, 2, \ldots, mn, -1, -2, \ldots, -mn\}$ of type B which have a zero block of size $2ma$ is given by

$$\binom{(m+1)n-a-2}{n-a}.$$

Enumeration in $NC_+^{(m)}(B_n)$

Theorem

Let m, n be positive integers. The number of positive m-divisible non-crossing partitions in $NC^{(m)}(B_n)$ with the property that the number of non-zero blocks of size mi of π_1 is $2b_i$, $i = 1, 2, \dots, n$, is given by

$$\binom{b_1 + b_2 + \cdots + b_n}{b_1, b_2, \dots, b_n} \binom{mn - 1}{b_1 + b_2 + \cdots + b_n}.$$

REMARK. We do not have results on chain enumeration.

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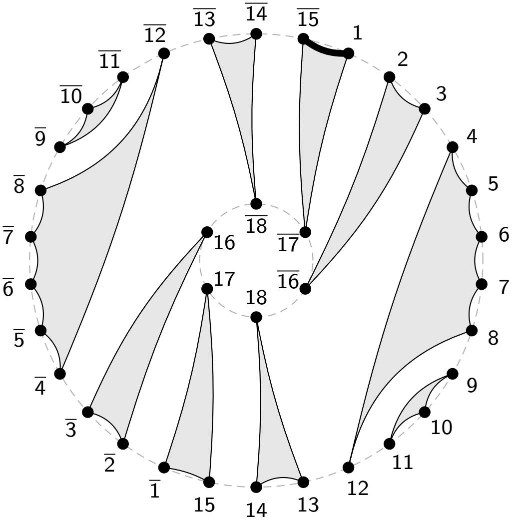
Fact: Under CK's map, the elements of $NC_+^{(m)}(D_n)$ correspond to those m-divisible non-crossing partitions on the annulus with $\{1, 2, \dots, m(n-1), -1, -2, \dots, -m(n-1)\}$ on the outer circle and $\{m(n-1)+1, \dots, mn, -m(n-1)-1, \dots, -mn\}$ on the inner circle which are invariant under rotation by 180°, satisfy the earlier mentioned and non-defined technical constraint, and in which the predecessor of 1 in its block is a negative element on the outer circle.

How do elements of $NC_+^{(m)}(D_n)$ look like?



A positive 3-divisible non-crossing partition of type D_6

Enumeration in $NC_{+}^{(m)}(D_n)$

Theorem

Let m and n be positive integers. The number of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn, -1, -2, \dots, -mn\}$ of type D equals

$$\frac{2m(n-1)+n-2}{n}\binom{(m+1)(n-1)-1}{n-1},$$

while the number of these partitions of which all blocks have size m equals

$$\frac{2m(n-1)-n}{n}\binom{m(n-1)-1}{n-1}.$$

A Fundamental Principle of Combinatorial Enumeration (2004ff)

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Every family of combinatorial objects satisfies the

cyclic sieving phenomenon!

Cyclic sieving (Reiner, Stanton, White)

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Ingredients:

— a set M of *combinatorial objects*,

— a *cyclic group* $C = \langle g \rangle$ acting on M,

— a *polynomial* $P(q)$ in q with non-negative integer coefficients.

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— a set M of *combinatorial objects*,

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Definition

The triple (M, C, P) exhibits the *cyclic sieving phenomenon* if

$$|\operatorname{Fix}_M(g^p)| = P\left(e^{2\pi i p/|C|}\right).$$

Cyclic sieving (Reiner, Stanton, White)

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$$M = \big\{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{1,3\},\{2,4\}\big\}$$

$$g : j \mapsto j+1 \pmod 4$$

$$P(q) = \begin{bmatrix}4\\2\end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$$

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$$g : j \mapsto j+1 \pmod 4$$

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$$|\operatorname{Fix}_M(g^0)| = 6 = P(1) = P\left(e^{2\pi i \cdot 0/4}\right),$$

Cyclic sieving (Reiner, Stanton, White)

EXAMPLE:

$$M = \big\{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{1,3\},\{2,4\}\big\}$$

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A Fundamental Principle of Combinatorial Enumeration (2004ff)

Every family of combinatorial objects satisfies the

cyclic sieving phenomenon!

A Fundamental Principle of Combinatorial Enumeration (2004ff)

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Corollary

The positive m-divisible non-crossing partitions satisfy the cyclic sieving phenomenon.

A cyclic action for m-divisible non-crossing partitions

A cyclic action for m-divisible non-crossing partitions

Let $K : NC^{(m)}(W) \to NC^{(m)}(W)$ be the map defined by

$$\begin{aligned}&(w_0; w_1, \dots, w_m)\\ &\qquad \mapsto \big((cw_m c^{-1}) w_0 (cw_m c^{-1})^{-1}; cw_m c^{-1}, w_1, w_2, \dots, w_{m-1}\big).\end{aligned}$$

It generates a cyclic group of order mh.

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Furthermore, let

$$\mathrm{Cat}^{(m)}(W; q) := \prod_{i=1}^{n} \frac{[mh + d_i]_q}{[d_i]_q},$$

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Theorem (with T. W. Müller)

The triple $(NC^{(m)}(W), \langle K\rangle, \mathrm{Cat}^{(m)}(W;q))$ exhibits the cyclic sieving phenomenon.

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Theorem (with T. W. Müller)

Let $NC^{(m;0)}(W)$ denote the subset of $NC^{(m)}(W)$ consisting of those elements for which $w_0 = id$. Then the triple $(NC^{(m;0)}(W), \langle K \rangle, \mathrm{Cat}^{(m-1)}(W; q))$ exhibits the cyclic sieving phenomenon.

A cyclic action for positive m-divisible non-crossing partitions?

A cyclic action for positive m-divisible non-crossing partitions?

Bad news:

The map $K : NC^{(m)}(W) \to NC^{(m)}(W)$ defined by

$$(w_0; w_1, \dots, w_m) \\ \mapsto \big((cw_mc^{-1})w_0(cw_mc^{-1})^{-1}; cw_mc^{-1}, w_1, w_2, \dots, w_{m-1}\big)$$

does not necessarily map positive m-divisible non-crossing partitions to positive ones!

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does not necessarily map positive m-divisible non-crossing partitions to positive ones!

Consequently: we have to modify the above action.

A cyclic action for positive m-divisible non-crossing partitions?

A cyclic action for positive m-divisible non-crossing partitions?

Let $K_+ : NC^{(m)}(W) \to NC^{(m)}(W)$ be the map defined by

$$(w_0; w_1, \dots, w_m) \mapsto \big((cw^R_{m-1}w_mc^{-1})w_0(cw^R_{m-1}w_mc^{-1})^{-1}; \; cw^R_{m-1}w_mc^{-1}, w_1, \dots, w^L_{m-1}\big),$$

where $w_{m-1} = w^L_{m-1}w^R_{m-1}$ is the factorisation of w_{m-1} into its "good" and its "bad" part.

Factorisation into "good" and "bad" part

A cyclic action for positive m-divisible non-crossing partitions?

Factorisation into "good" and "bad" part

Fix a reduced word $c = c_1 \cdots c_n$ for the Coxeter element c.
Define the *c-sorting word* $\mathrm{w}(c)$ for $w \in W$ to be the lexicographically first reduced word for w when written as a subword of c^∞.
Let $\mathrm{w}_\circ(c) = s_{k_1} \cdots s_{k_N}$ with $N = nh/2$ be the c-sorting word of the longest element $w_\circ \in W$.
The word $\mathrm{w}_\circ(c)$ induces a *reflection ordering* given by

$$T = \big\{ s_{k_1} <_c s_{k_1} s_{k_2} s_{k_1} <_c s_{k_1} s_{k_2} s_{k_3} s_{k_2} s_{k_1} <_c \dots \\ <_c s_{k_1} \dots s_{k_{N-1}} s_{k_N} s_{k_{N-1}} \dots s_{k_1} \big\}.$$

Associate to every element $w \in NC(W)$ a reduced T-word $\mathcal{T}_c(w)$ given by the lexicographically first subword of T that is a reduced T-word for w.
We decompose w as $w = w^L w^R$ where w^R is the part of $\mathcal{T}_c(w)$ within the last n reflections in T.

Cyclic sieving for positive m-divisible non-crossing partitions

Cyclic sieving for positive m-divisible non-crossing partitions

Let $K_+ : NC^{(m)}(W) \to NC^{(m)}(W)$ be the earlier defined map.

Furthermore, let

$$\mathrm{Cat}_+^{(m)}(W;q) := \prod_{i=1}^{n} \frac{[mh+d_i-2]_q}{[d_i]_q},$$

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Conjecture

The triple $(NC_+^{(m)}(W), \langle K_+ \rangle, \mathrm{Cat}_+^{(m)}(W;q))$ *exhibits the cyclic sieving phenomenon.*

Conjecture

Let $NC_+^{(m;0)}(W)$ *denote the subset of* $NC_+^{(m)}(W)$ *consisting of those elements for which* $w_0 = id$. *Then the triple* $(NC_+^{(m;0)}(W), \langle K_+ \rangle, \mathrm{Cat}_+^{(m-1)}(W;q))$ *exhibits the cyclic sieving phenomenon.*

Cyclic sieving for positive m-divisible non-crossing partitions

24th International Conference on Formal Power Series and Algebraic Combinatorics, Nagoya, Hotel Lobby, 2012

Christian to Christian: *"Willst Du ein weiteres ‚Zyklisches-Sieben-Phänomen' beweisen?"*[1]

Christian to Christian: *"Sicher. Warum nicht?"*[2]

Cyclic sieving for positive m-divisible non-crossing partitions

24th International Conference on Formal Power Series and Algebraic Combinatorics, Nagoya, Hotel Lobby, 2012

Christian to Christian: *"Willst Du ein weiteres ‚Zyklisches-Sieben-Phänomen' beweisen?"*[1]

Christian to Christian: *"Sicher. Warum nicht?"*[2]

To actually carry this out turned out to be slightly more involved than originally anticipated by Christian.

Cyclic sieving for positive m-divisible non-crossing partitions

Theorem

The triple $(NC_+^{(m)}(W), \langle K_+\rangle, \mathrm{Cat}_+^{(m)}(W;q))$ exhibits the cyclic sieving phenomenon.

Theorem

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Our proof is by a careful case-by-case verification. Along the way, we palso prove some **finer** cyclic sieving phenomena.

Realisation of the cyclic action in type A_{n-1}

Realisation of the cyclic action in type A_{n-1}

"In principle," under Armstrong's combinatorial realisation, the map K_+ becomes rotation by one unit, unless this would produce a non-positive m-divisible partition.

Realisation of the cyclic action in type A_{n-1}

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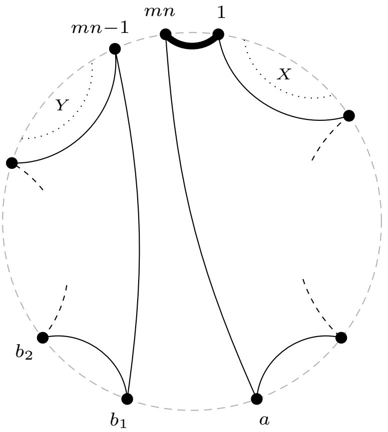
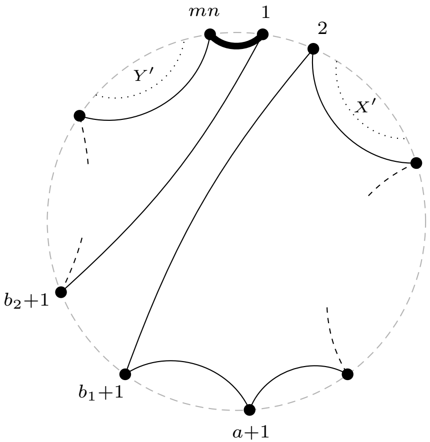


$\mapsto$



Cyclic sieving for positive m-divisible non-crossing partitions for type A_{n-1}

How do "pseudo-rotationally" invariant elements look like?

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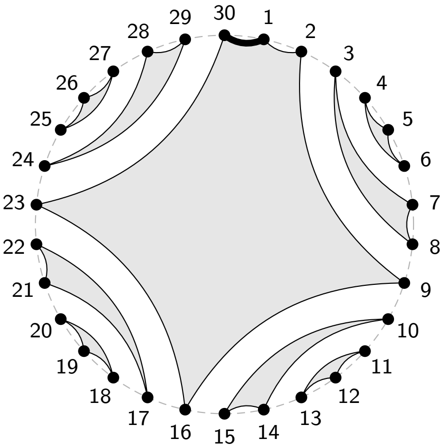

Cyclic sieving for positive m-divisible non-crossing partitions for type A_{n-1}

Theorem

Let m, n, r be positive integers with $r \geq 2$ and $r \mid (mn-2)$. Furthermore, let $b_1, b_2, \dots, b_n$ be non-negative integers. The number of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ which are invariant under the r-pseudo-rotation $K_+^{(mn-2)/r}$, the number of non-zero blocks of size mi being rb_i, $i = 1, 2, \dots, n$, the zero block having size $ma = mn - mr\sum_{j=1}^{n} jb_j$, is given by

$$\binom{b_1+b_2+\cdots+b_n}{b_1, b_2, \dots, b_n}\binom{(mn-2)/r}{b_1+b_2+\cdots+b_n}$$

if $b_1 + 2b_2 + \cdots + nb_n < n/r$, or if $r = 2$ and $b_1 + 2b_2 + \cdots + nb_n = n/2$, and 0 otherwise.

Cyclic sieving for positive m-divisible non-crossing partitions for type A_{n-1}

Theorem

Let C be the cyclic group of pseudo-rotations of an mn-gon generated by K_+.
Then the triple (M, C, P) exhibits the cyclic sieving phenomenon for the following choices of sets M and polynomials P:

1. $M = \widetilde{NC}_+^{(m)}(n)$, *and* $P(q) = \frac{1}{[n]_q} \begin{bmatrix} (m+1)n-2 \\ n-1 \end{bmatrix}_q$;
2. *M consists of all elements of $\widetilde{NC}_+^{(m)}(n)$ the block sizes of which are all equal to m, and* $P(q) = \frac{1}{[n]_q} \begin{bmatrix} mn-2 \\ n-1 \end{bmatrix}_q$;
3. *M consists of all elements of $\widetilde{NC}_+^{(m)}(n)$ which have rank s (or, equivalently, their number of blocks is $n - s$), and*

$$P(q) = \frac{1}{[n]_q} \begin{bmatrix} n \\ s \end{bmatrix}_q \begin{bmatrix} mn-2 \\ n-s-1 \end{bmatrix}_q ;$$

Cyclic sieving for positive m-divisible non-crossing partitions for type A_{n-1}

4. *M consists of all elements of $\widetilde{NC}_{+}^{(m)}(n)$ whose number of blocks of size mi is b_i, $i = 1, 2, \dots, n$, and*

$$P(q) = \frac{1}{[b_1 + b_2 + \cdots + b_n]_q} \begin{bmatrix} b_1 + b_2 + \cdots + b_n \\ b_1, b_2, \dots, b_n \end{bmatrix}_q \times \begin{bmatrix} mn - 2 \\ b_1 + b_2 + \cdots + b_n - 1 \end{bmatrix}_q .$$

Realisation of the cyclic action in type B_n

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"In principle," under Armstrong's combinatorial realisation, the map K_+ becomes rotation by one unit, unless this would produce a non-positive m-divisible partition.

Realisation of the cyclic action in type B_n

Cyclic sieving for positive m-divisible non-crossing partitions for type B_n

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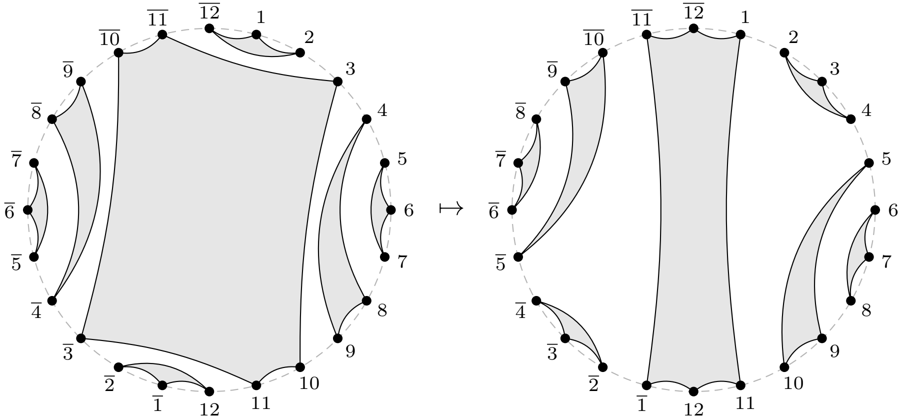

Cyclic sieving for positive m-divisible non-crossing partitions for type B_n

How do "pseudo-rotationally" invariant elements look like?

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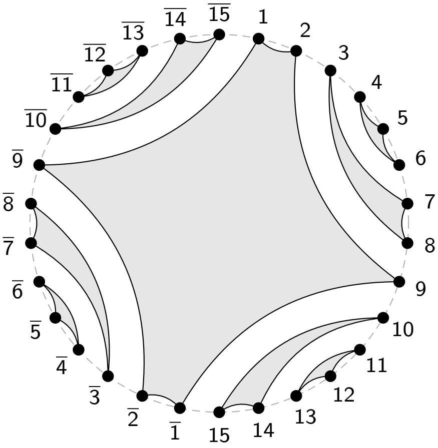

Cyclic sieving for positive m-divisible non-crossing partitions for type B_n

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Let m, n, a, r be positive integers with $r \mid (mn-1)$. Furthermore, let $b_1, b_2, \dots, b_n$ be non-negative integers. The number of positive m-divisible non-crossing partitions of $\{1, 2, \dots, mn, -1, -2, \dots, -mn\}$ of type B which are invariant under the $2r$-pseudo-rotation $K_+^{(mn-1)/r}$, where the number of non-zero blocks of size mi is $2rb_i$, $i = 1, 2, \dots, n$, the zero block having size $2ma = 2mn - 2mr \sum_{j=1}^{n} jb_j > 0$, is given by

$$\binom{b_1 + b_2 + \cdots + b_n}{b_1, b_2, \dots, b_n} \binom{(mn-1)/r}{b_1 + b_2 + \cdots + b_n}.$$

Cyclic sieving for positive m-divisible non-crossing partitions for type B_n

Theorem

Let C be the cyclic group of pseudo-rotations of the $2mn$-gon consisting of the elements $\{1, 2, \dots, mn, -1, -2, \dots, -mn\}$ generated by K_+, viewed as a group of order $2mn - 2$.
Then the triple (M, P, C) exhibits the cyclic sieving phenomenon for the following choices of sets M and polynomials P:

1. $M = \widetilde{NC}_+^{(m)}(B_n)$, *and* $P(q) = \begin{bmatrix} (m+1)n-1 \\ n \end{bmatrix}_{q^2}$;
2. *M consists of the elements of $\widetilde{NC}_+^{(m)}(B_n)$ all of whose blocks have size m, and* $P(q) = \begin{bmatrix} mn-1 \\ n \end{bmatrix}_{q^2}$;
3. *M consists of all elements of $\widetilde{NC}_+^{(m)}(B_n)$ which have rank s (or, equivalently, their number of non-zero blocks is $2(n-s)$), and*

$$P(q) = \begin{bmatrix} n \\ s \end{bmatrix} \begin{bmatrix} mn-1 \\ n-s \end{bmatrix};$$

Cyclic sieving for positive m-divisible non-crossing partitions for type B_n

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$$P(q) = \begin{bmatrix} b_1 + b_2 + \cdots + b_n \\ b_1, b_2, \dots, b_n \end{bmatrix}_{q^2} \begin{bmatrix} mn - 1 \\ b_1 + b_2 + \cdots + b_n \end{bmatrix}_{q^2}.$$

Realisation of the cyclic action in type D_n

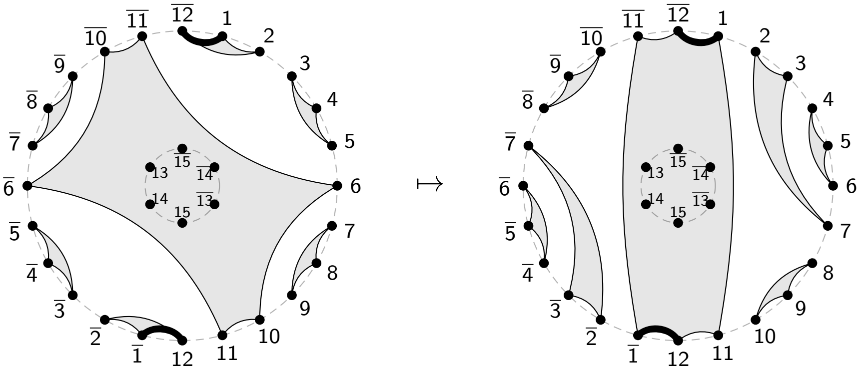
Realisation of the cyclic action in type D_n

"In principle," under CK's combinatorial realisation, the map K_+ becomes rotation by one unit, unless this would produce a non-positive m-divisible partition.

Realisation of the cyclic action in type D_n

Cyclic sieving for positive m-divisible non-crossing partitions for type D_n

Realisation of the cyclic action in type D_n



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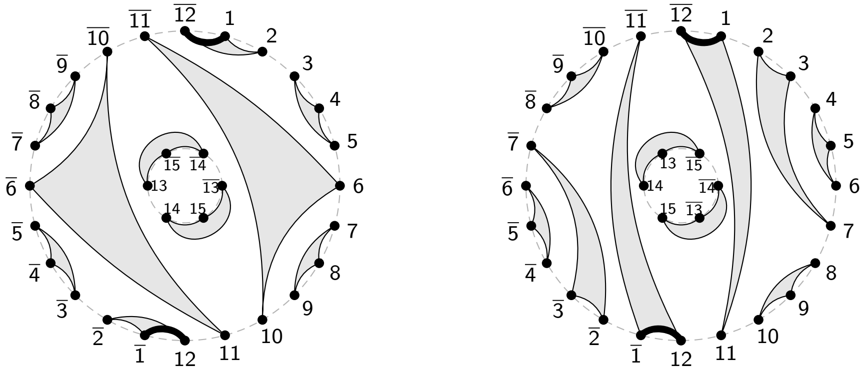

Cyclic sieving for positive m-divisible non-crossing partitions for type D_n

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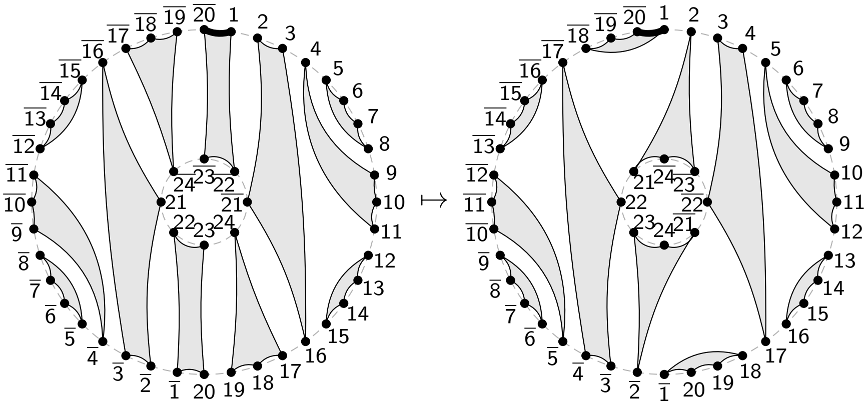

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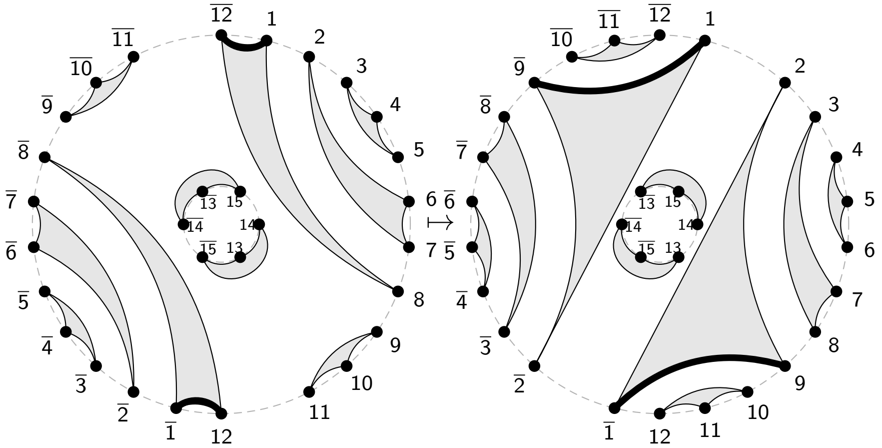
Realisation of the cyclic action in type D_n



Cyclic sieving for positive m-divisible non-crossing partitions for type D_n

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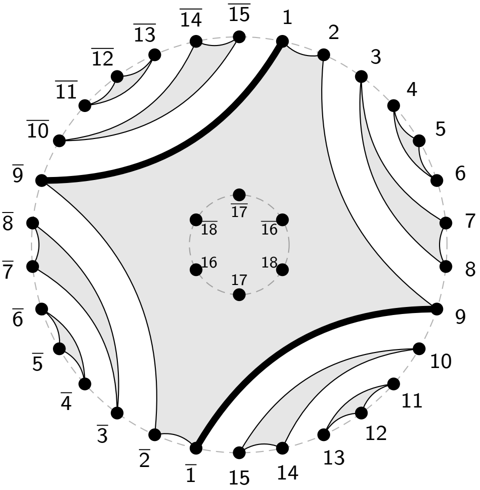

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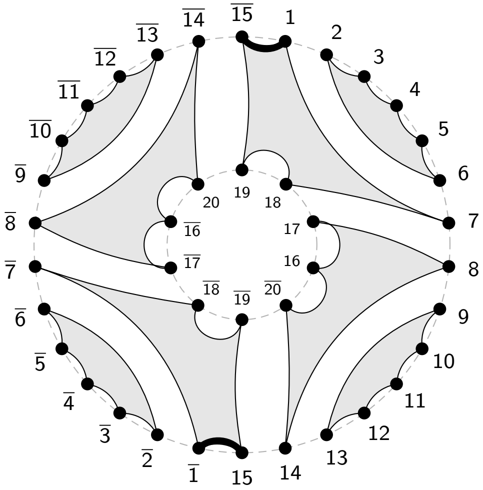

Cyclic sieving for positive m-divisible non-crossing partitions for type D_n

In this case, we contented ourselves just proving the relevant enumeration formulae, since things get quite involved. Probably one can do more if one is braver ...

Cyclic sieving for positive m-divisible non-crossing partitions for type D_n

Theorem

Let C be the cyclic group of pseudo-rotations of the annulus with $\{1,2,\dots,m(n-1),-1,-2,\dots,-m(n-1)\}$ on the outer circle and $\{m(n-1)+1,\dots,mn,-(m(n-1)+1),\dots,-mn\}$ on the inner circle generated by K_+, viewed as a group of order $2m(n-1)-2$.
Then the triple (M,P,C) exhibits the cyclic sieving phenomenon for the following choices of sets M and polynomials P:

1. $M=\widetilde{NC}_+^{(m)}(D_n)$, *and*
   $P(q)=\frac{[2m(n-1)+n-2]_q}{[n]_q}\begin{bmatrix}(m+1)(n-1)-1\\ n-1\end{bmatrix}_{q^2}$;
2. *M consists of the elements of $\widetilde{NC}_+^{(m)}(D_n)$ all of whose blocks have size m, and* $P(q)=\frac{[2m(n-1)-n]_q}{[n]_q}\begin{bmatrix}m(n-1)-1\\ n-1\end{bmatrix}_{q^2}$.

Proof method

Proof method

1. Careful combinatorial decomposition of the non-crossing objects;
2. generating function calculus;
3. Lagrange inversion formula.

Cyclic sieving for positive m-divisible non-crossing partitions for the exceptional types

The (positive) m-divisible non-crossing partitions

$$(w_0; w_1, \dots, w_m)$$

for the exceptional types become "sparse" for large m.
This allows one to reduce the occurring enumeration problems to finite problems.

"Other than that, there do not seem to be enumerative results known for these families."